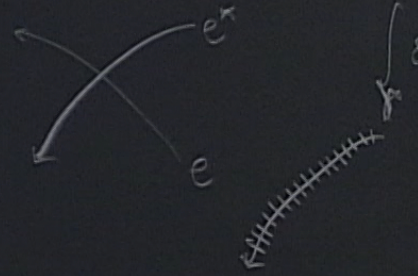
Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 11

Date: Apr 03, 2017 10:15 AM

URL: http://pirsa.org/17040015

Abstract:

Length Operator



$\delta^{ij} e^i_a e^j_b \dot\gamma^a \dot\gamma^b \, ds$ ordinary length

$\sum_{n=0}^{\infty} \sqrt{\sum_K E^K_{e_n} E^K_{e_n}}$ finite sum

$$\hat{l}^2_{e^*} |j\ m\ n\rangle := \underbrace{\left(\sum_K \hat{E}^K_{e^*} \hat{E}^K_{e}\right)}_{\text{what is this?}} |j\ m\ n\rangle$$

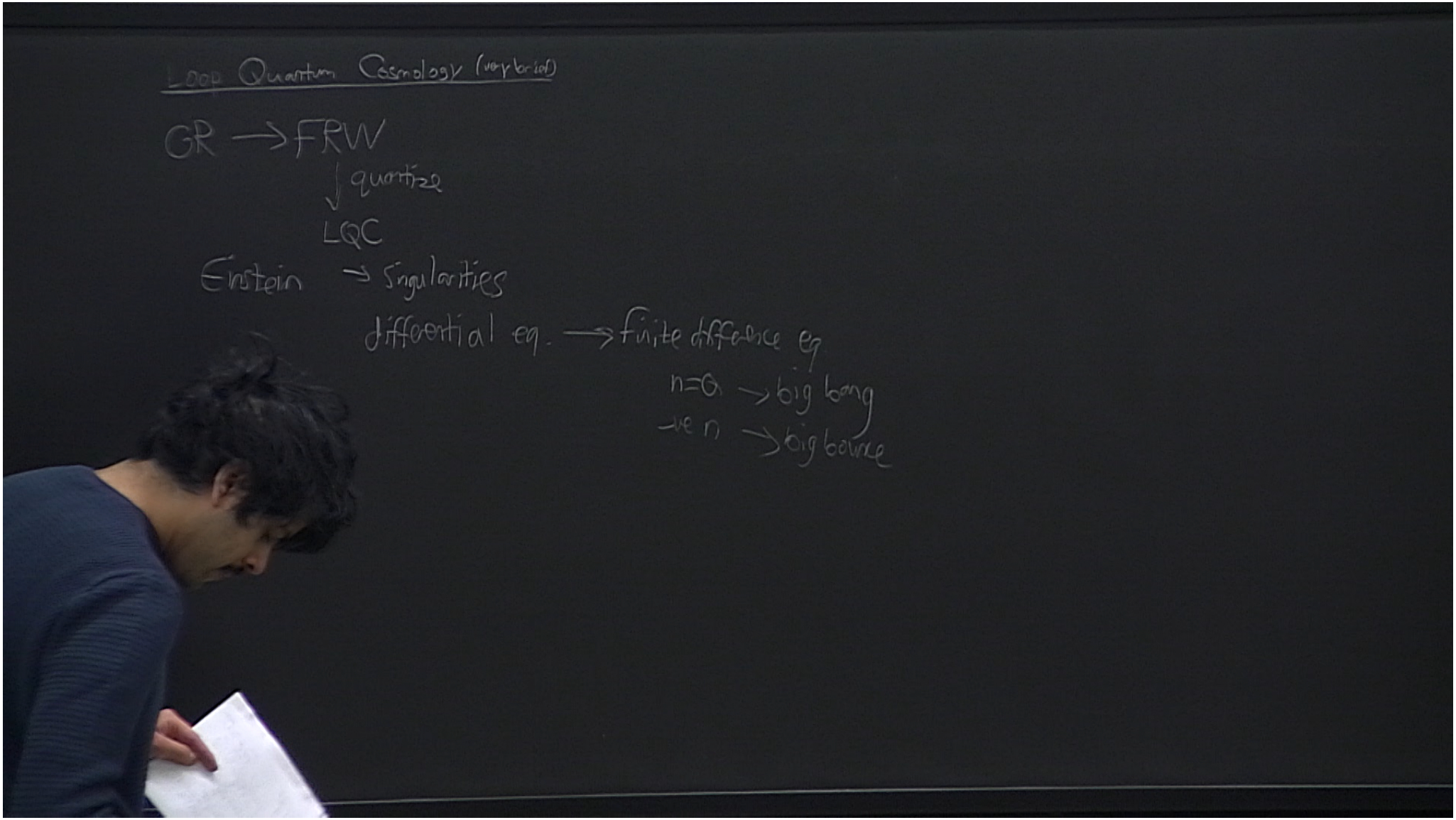
$$\hat{E}^K_e |j\ m\ n\rangle = i\hbar \sum_p |j\ m\ p\rangle D^j_{pn}(T^K)$$

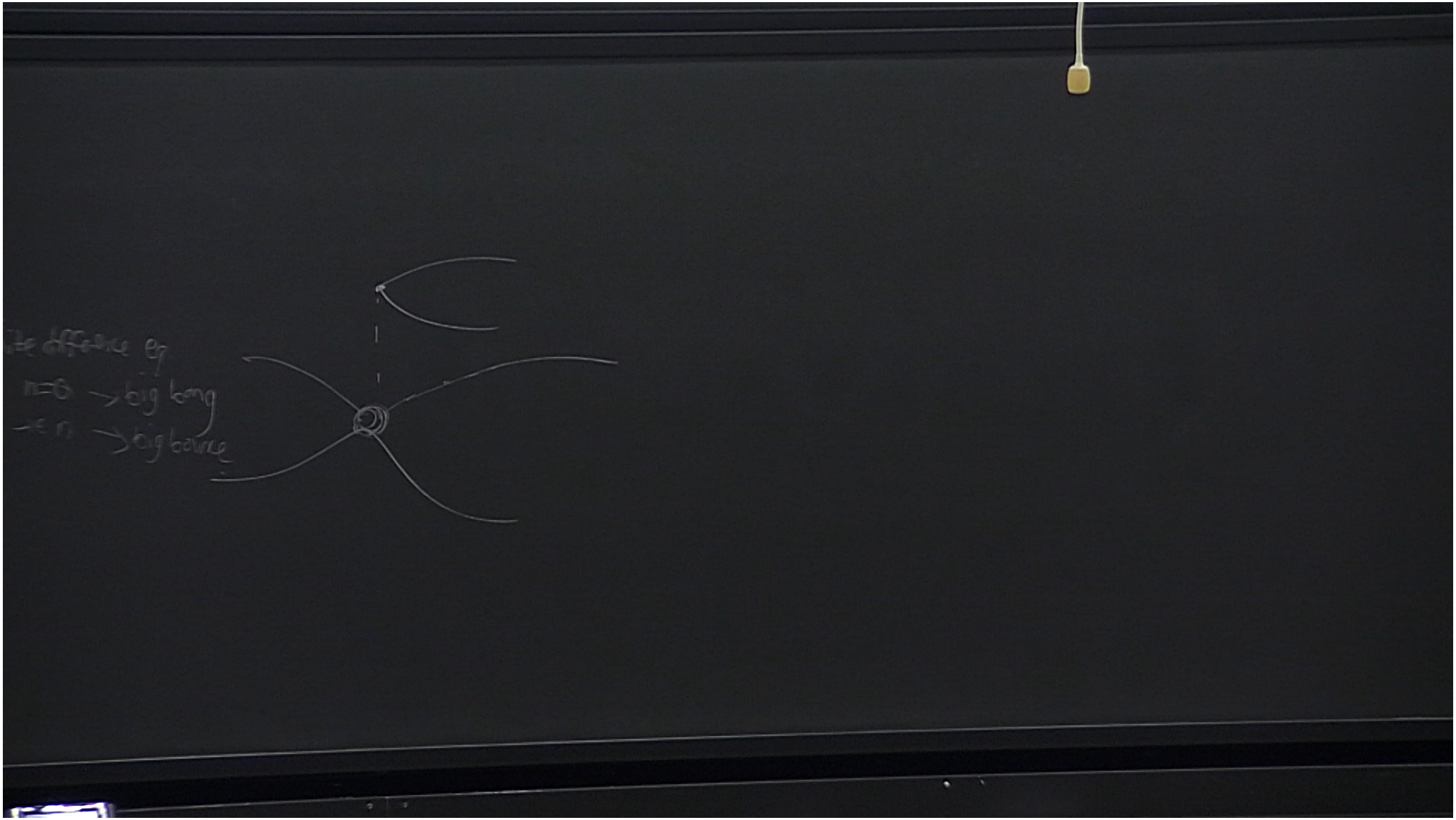
$$\hat{l}^2_{e^*} |j\ m\ n\rangle = -\hbar^2 \sum_{pq} |j\ m\ q\rangle \underbrace{\sum_K D^j_{qp}(T^K) D^j_{pn}(T^K)}_{C^j_{qn}}$$

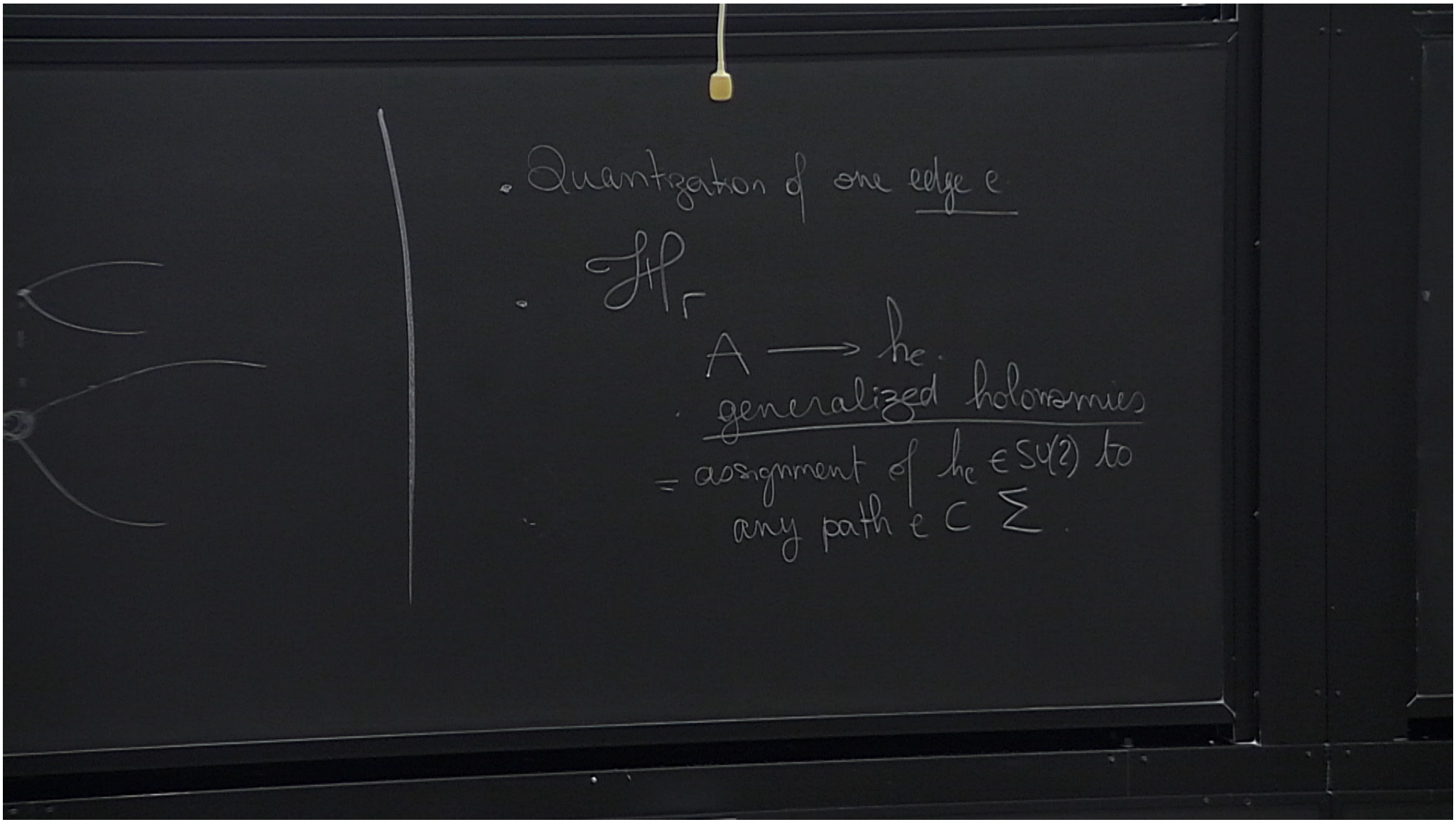
Schur $C = \lambda \mathbb{1}$

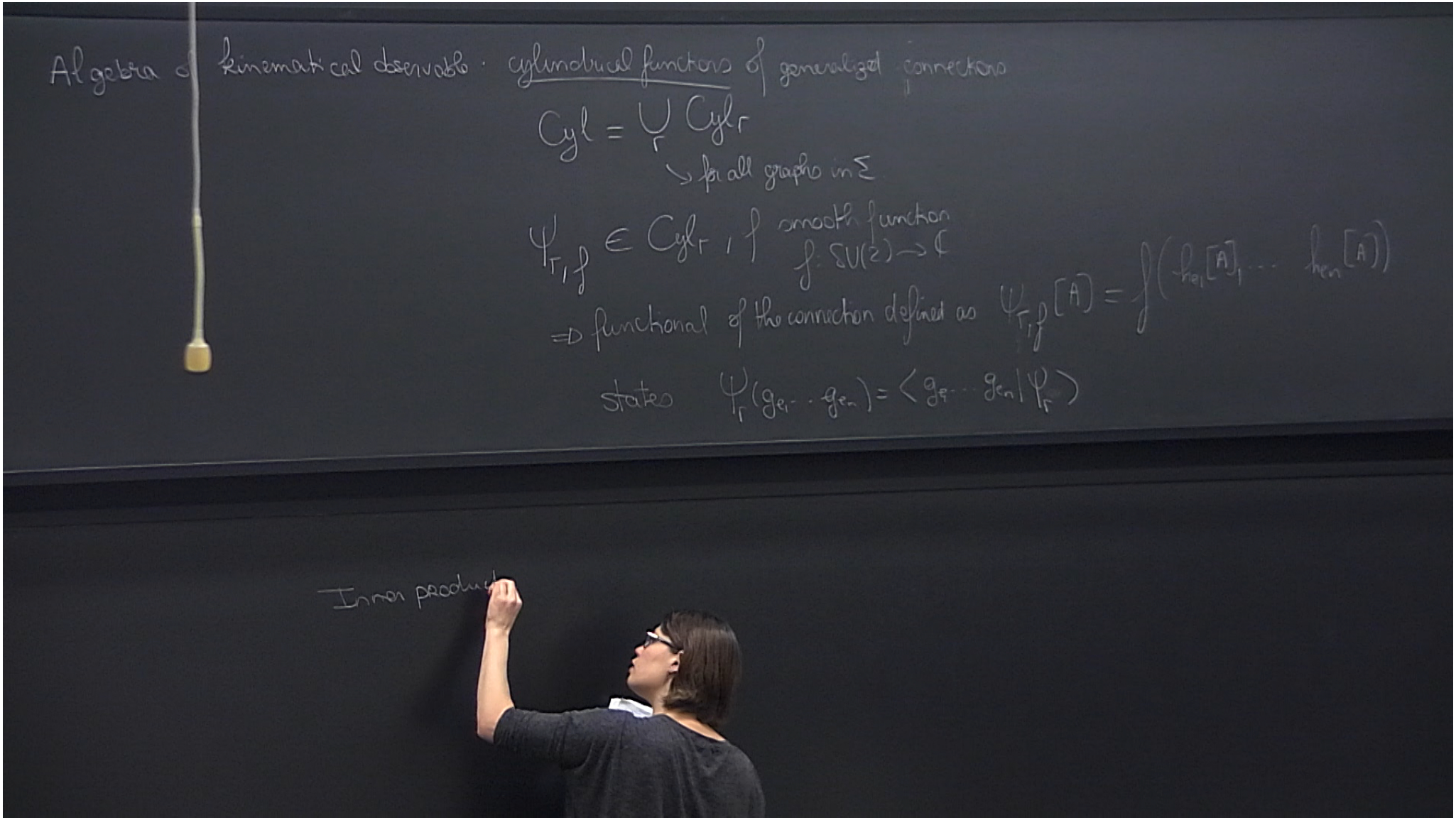
$$\hat{l}^2_e |j\ m\ n\rangle = \hbar^2 j(j+1) |j\ m\ n\rangle \quad \lambda = -j(j+1)$$

$$= (l^{(3)}_P)^2 j(j+1)$$


Loop Quantum Cosmology (very brief)
GR → FRW
↓ quantize
LQC
Einstein → singularities
differential eq. → finite difference eq.
n=0 → big bang
-ve n → big bounce


big bang
big bounce

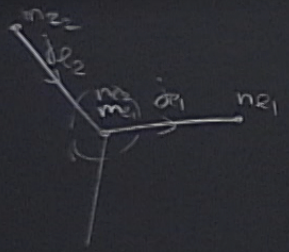

• Quantization of one edge e
• $\mathcal{H}_\Gamma$
$A \longrightarrow h_e$
generalized holonomies
= assignment of $h_e \in SU(2)$ to
any path $e \subset \Sigma$

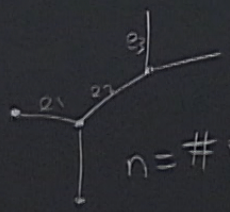

Algebra of kinematical observable · cylindrical functions of generalized connections
$Cyl = \bigcup_\Gamma Cyl_\Gamma$
→ for all graphs in Σ
$\Psi_{\Gamma,f} \in Cyl_\Gamma$, f smooth function $f: SU(2) \to \mathbb{C}$
⇒ functional of the connection defined as $\Psi_{\Gamma,f}[A] = f(h_{e_1}[A], \dots h_{e_n}[A])$
states $\Psi_\Gamma(g_{e_1} \cdots g_{e_n}) = \langle g_{e_1} \cdots g_{e_n} | \Psi_\Gamma \rangle$
Inner product

states $\Psi_\Gamma(g_{e_1} \cdots g_{e_n}) = \langle g_{e_1} \cdots g_{e_n} | \Psi_\Gamma \rangle$

Inner product on the tensor product $\mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e$

$$\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{G^n} \overline{\Psi_{\Gamma,1}(g_{e_1} \cdots g_{e_n})}\, \Psi_{\Gamma,2}(g_{e_1} \cdots g_{e_n})\, dg_{e_1} \cdots dg_{e_n}$$





$n = \#$ of edges

Orthonormal basis for $\mathcal{H}_\Gamma$ can be obtained by tensoring the basis function $|j\ m\ n\rangle$ for each edge

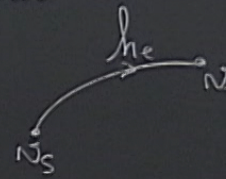
$$|\{j\ m\ n\}_\Gamma\rangle = |j_{e_1}, m_{e_1}, n_{e_1}\rangle \otimes \cdots \otimes |j_{e_n}, m_{e_n}, n_{e_n}\rangle$$

$$\Rightarrow \mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e = \bigotimes_e L^2(G) \cong L^2(G^n)$$

Quantization of the Gauss constraint $G \sim \partial E + [A, E]$

↳ indirect way: action of the Gauss constraint is simple

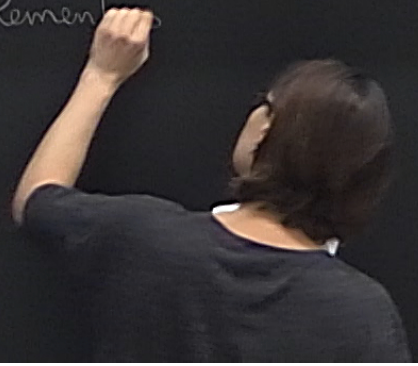
- Gauge transfo of a holonomy h_e



$h_e \longrightarrow g(N(t))\, h_e\, g^{-1}(N(0))$

- States $\langle h_e | \Psi_e \rangle = \Psi_e(h_e)$
  ↳ transform as $\Psi_e\big(g(N(t))\, h_e\, \bar{g}^{-1}(N(0))\big) = \langle h_e | L_{g^{-1}(N(t))}\, R_{\bar{g}^{-1}(N(0))} | \Psi_e \rangle$

- Implement



- States $\langle h_e | \psi_e \rangle = \psi_e(h_e)$
  L_1 transform as $\psi_e\left(g(v(t))\, h_e\, \vec{g}^{-1}(v(s))\right) = \langle h_e | L_{g^{-1}(v(t))}\, R_{\vec{g}(v(s))} | \psi_e \rangle$

- Implement of projector on the subspace of states satisfying the Gauss const by averaging over the group $SU(2)$ for both edge ends

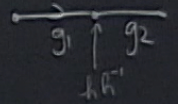
$$\delta(\hat{G})\,|\psi_e\rangle = \int_{G\times G} L_{g_1} R_{g_2} |\psi_e\rangle\, dg_1\, dg_2$$

$$\int_{G\times G} L_{g_1} R_{g_2}\, \psi_e(h)\, dg_1\, dg_2 = \int_{G\times G} \psi_e\left(\overline{g_1^{-1} h\, g_2}\right) dg_1\, dg_2 = \text{const}\ (\text{independent of } h)$$

$\Rightarrow \mathcal{H}_\Gamma = \bigotimes_e \mathcal{H}_e = \bigotimes \mathcal{G}(G) \simeq \mathcal{G}(G')$

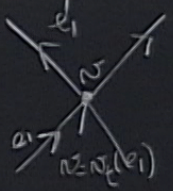
. Solution is non-trivial if considering several edges meeting at one vertex.

<u>2 edges</u> $\psi(g_1, g_2) \Rightarrow$ invariant function of the type $\psi(g_1, g_2)$

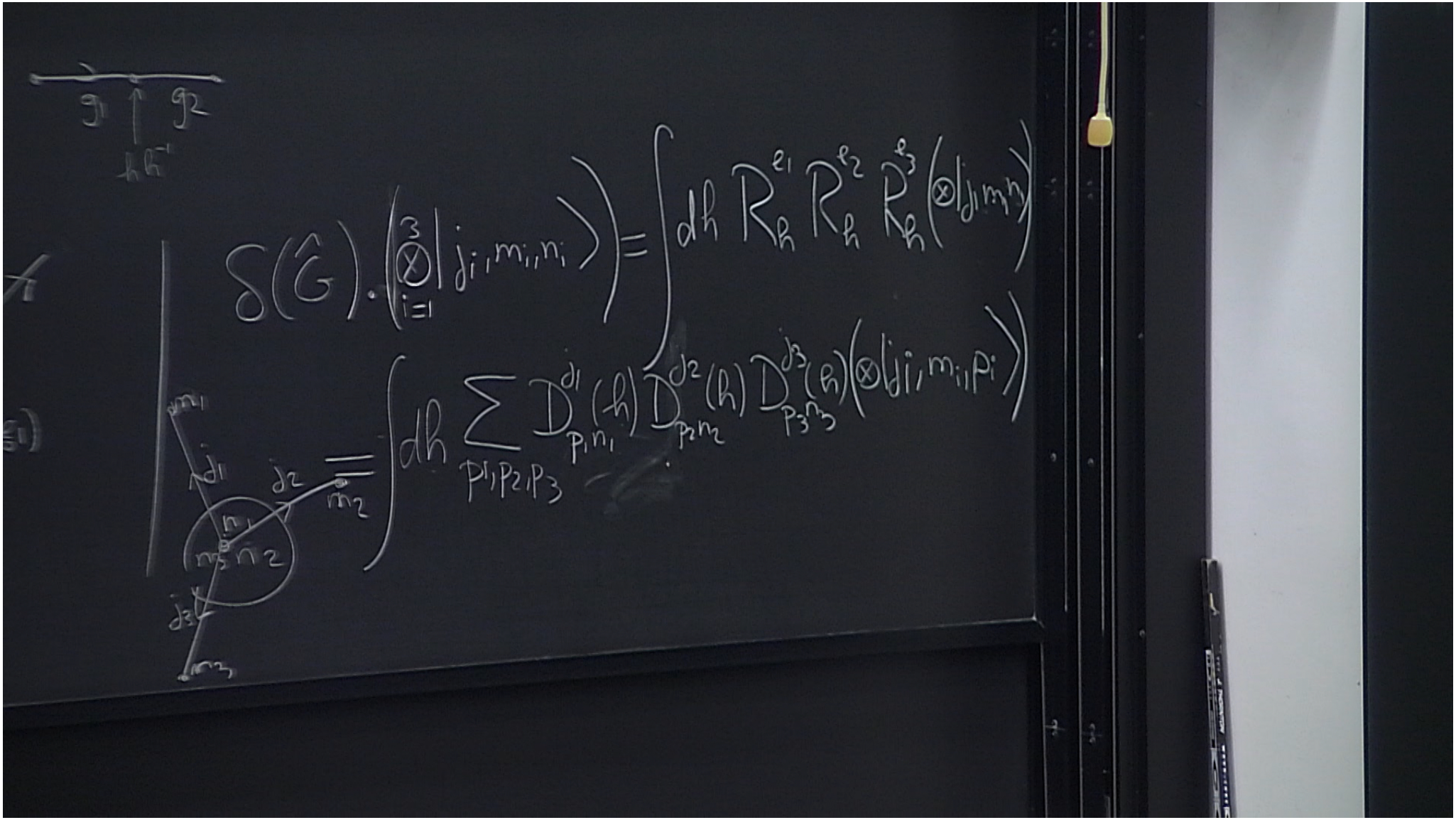


3 valent vertex v with 3 outgoing edges e_1, e_2, e_3

(*) for a vertex v $\quad \delta(\hat{G}_v) = \int \left(\bigotimes_{e/v=v_t(e)} L_h^e\right) \otimes \left(\bigotimes_{e'/v=v_s(e')} R_h^{e'}\right)$



$\delta(G).(\otimes$


$\hat{G}$
$\delta(\hat{G}) \cdot \left(\bigotimes_{i=1}^{3} |j_i, m_i, n_i\rangle \right) = \int dh \, R^{j_1}_{n_1} R^{j_2}_{n_2} R^{j_3}_{n_3} \left(\bigotimes |j_i, m_i, n_i\rangle \right)$
$= \int dh \sum_{p_1 p_2 p_3} D^{j_1}_{p_1 n_1}(h) D^{j_2}_{p_2 m_2}(h) D^{j_3}_{p_3 m_3}(h) \left(\bigotimes |j_i, m_i, p_i\rangle \right)$
m_1
j_1
j_2
m_2
n_1
m_3 n_2
j_3
m_3

From recoupling theory $$\int_{SU(2)} dh\, D^{j_1}_{m_1 n_1}(h)\, D^{j_2}_{m_2 n_2}(h)\, D^{j_3}_{m_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

Wigner 3jm symbol $$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3+1}} \underbrace{\langle j_1 m_1, j_2 m_2 | j_3, -m_3 \rangle}_{\text{Clebsh-Gordan}}$$

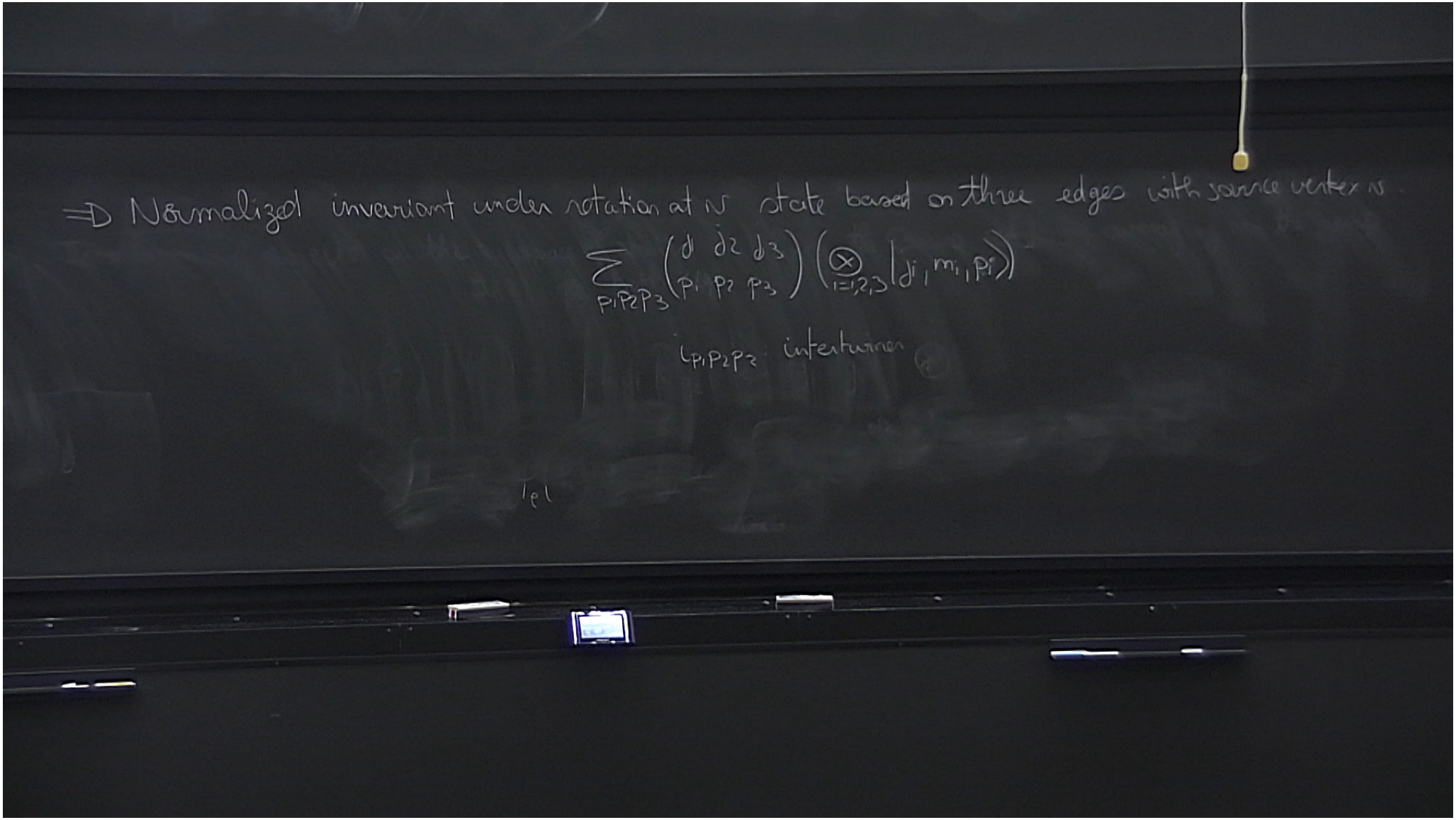
$$\Rightarrow \hat{\delta}(\hat{G}_3)\left(\bigotimes |j_i m_i h_i\rangle\right) = \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i m_i, p_i\rangle\right)$$

$= \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}$

$\frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3+1}} \underbrace{\langle j_1 m_1 \, j_2 m_2 \,|\, j_3 \, -m_3 \rangle}_{\text{Clebsh-Gordan.}}$

$\left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$

$\sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1$


⇒ Normalized invariant under rotation at N state based on three edges with source vertex N
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i\rangle \right)$
$\iota_{p_1 p_2 p_3}$: intertwiner

an at N state based on three edges with source vertex N.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1,2,3} |j_i, m_i, p_i^\epsilon\rangle \right)$$

p_3

$\iota_{p_1 p_2 p_3}$: intertwiner

