Title: Testing microscopic discretization

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Abstract: What can we say about the spectra of a collection of microscopic variables when only their coarse-grained sums are experimentally accessible? In this paper, using the tools and methodology from the study of quantum nonlocality, we develop a mathematical theory of the macroscopic fluctuations generated by ensembles of independent microscopic discrete systems. We provide algorithms to decide which multivariate gaussian distributions can be approximated by sums of finitely-valued random vectors. We study non-trivial cases where the microscopic variables have an unbounded range, as well as asymptotic scenarios with infinitely many macroscopic variables. From a foundational point of view, our results imply that bipartite gaussian states of light cannot be understood as beams of independent d-dimensional particle pairs. It is also shown that the classical description of certain macroscopic optical experiments, as opposed to the quantum one, requires variables with infinite cardinality spectra.

TESTING MICROSCOPIC DISCRETIZATION

MN[1], David Pérez-García[2] and Ignacio Villanueva[2]

[1]Institute for Quantum Optics and Quantum Information (IQOQI), Vienna, Austria.
[2]Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain

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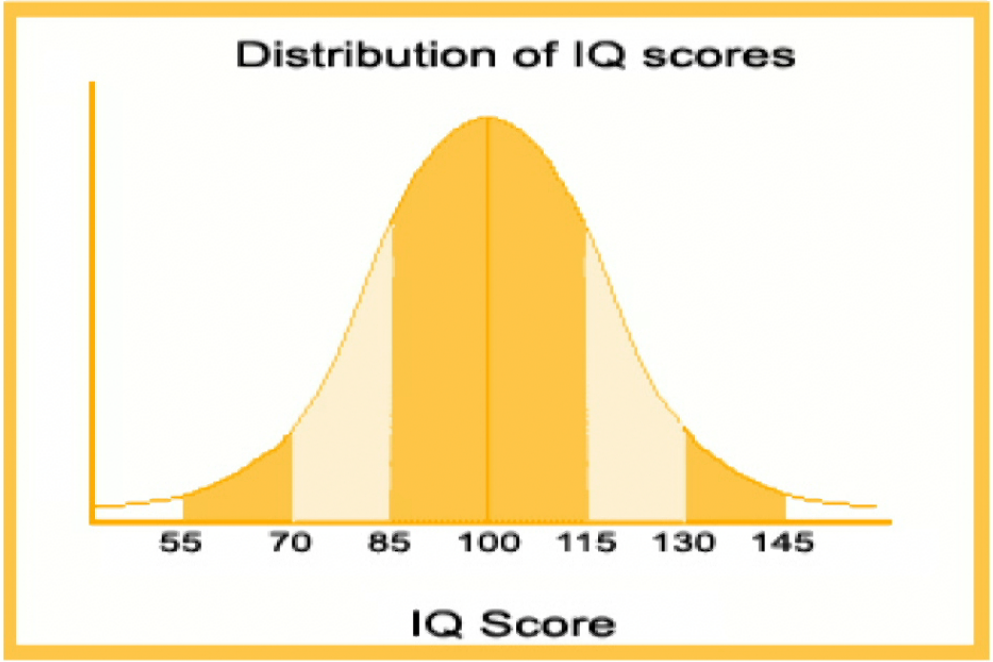

Distribution of IQ scores
55
70
85
100
115
130
145
IQ Score

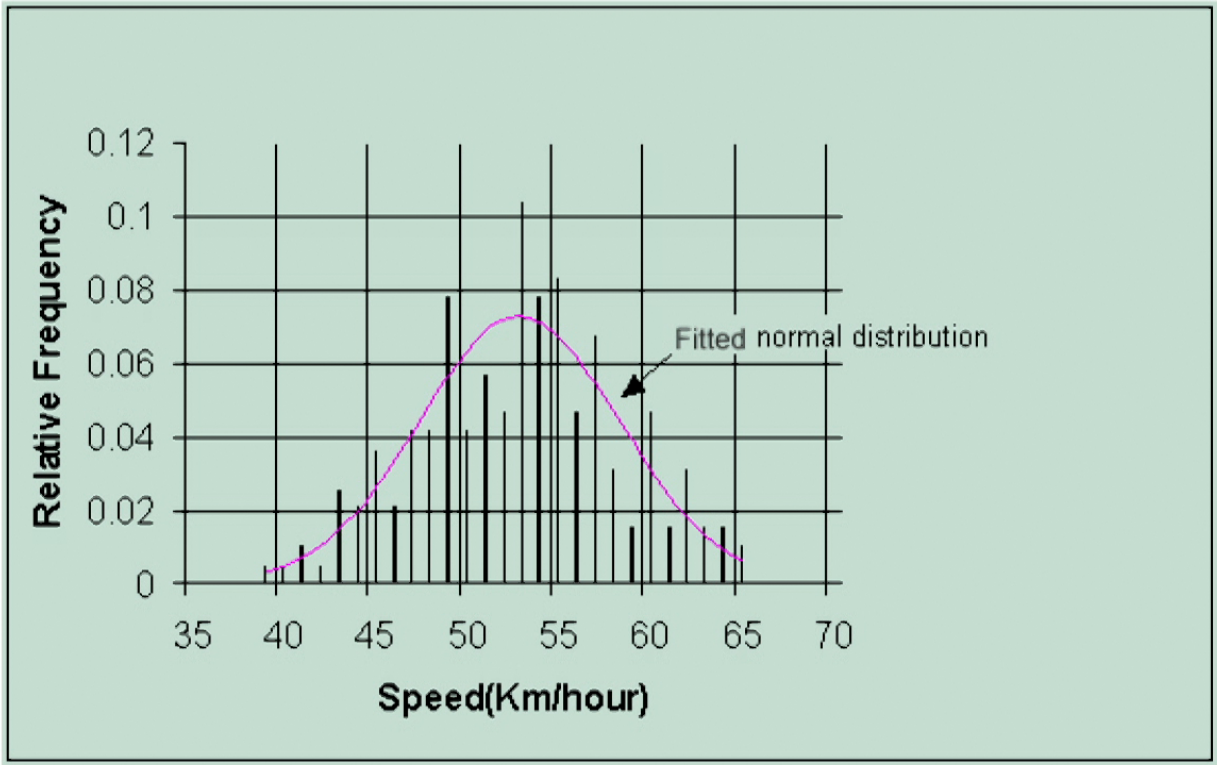


Figure 6(b) : Relative frequency distribution of speed superimposed with the fitted normal distribution.

Many physical quantities follow a gaussian distribution

Central Limit Theorem

$$
\begin{array}{cccc}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
 & \vdots & & \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)})
\end{array}
$$

$$
\begin{array}{cccc}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

$$
\begin{array}{llll}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

Multivariate gaussian distribution

$$
\begin{array}{cccc}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

i.i.d.?

Multivariate gaussian distribution

$$
\begin{array}{llll}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

i.i.d.?

Multivariate gaussian distribution

$$
\begin{array}{cccc}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \text{i.} \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

Multivariate gaussian distribution

$$
\begin{array}{llll}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \approx \text{i.} \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

Multivariate gaussian distribution

Microscopic

Macroscopic

$$(x_1, x_2, ..., x_n) \longrightarrow (X_1, X_2, ..., X_n)$$

Macroscopic Microscopic

$$(X_1, X_2, ..., X_n) \longrightarrow (x_1, x_2, ..., x_n)?$$

Macroscopic

Microscopic

$$(X_1, X_2, ..., X_n) \longrightarrow (x_1, x_2, ..., x_n)?$$

Discrete models

$$x_k \in \{\lambda_a^k : a = 1, ..., d\}$$

Discrete microscopic model

Macroscopic model

$$(x_1, x_2, ..., x_n) \longrightarrow (X_1, X_2, ..., X_n)$$

Discrete microscopic model

$$(x_1, x_2, \ldots, x_n) \longrightarrow (X_1, X_2, \ldots, X_n)$$

Macroscopic model

Restriction $\longrightarrow$ Restriction?

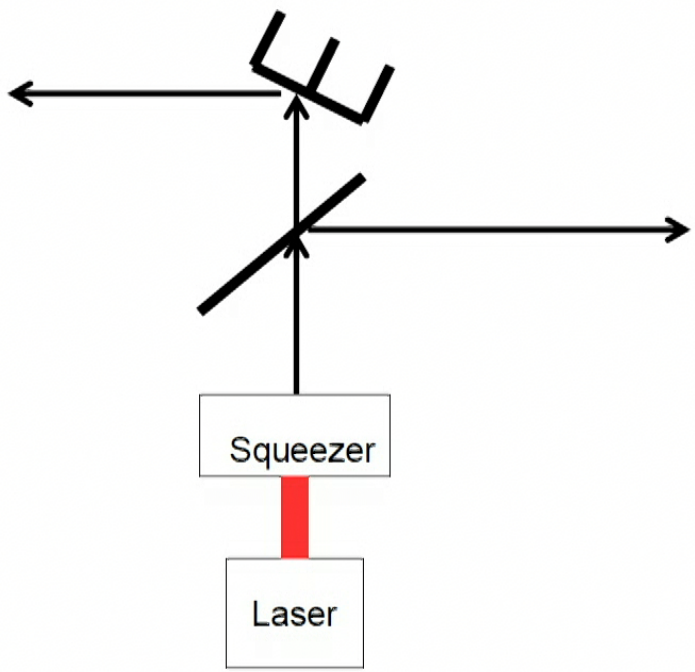
Why discrete microscopic models?

Fundamental Physics

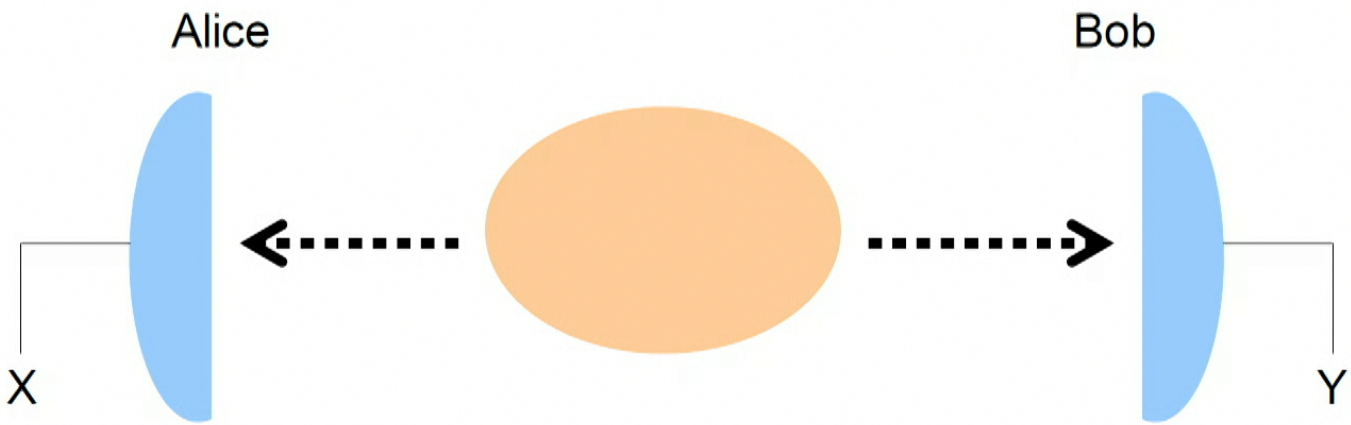
Certain physical quantities, like time and space,are usually assumed to be continuous, but could actually be discrete (quantized).

Does this quantization show at the macroscopic scale?

A first example: entangled gaussian states of light

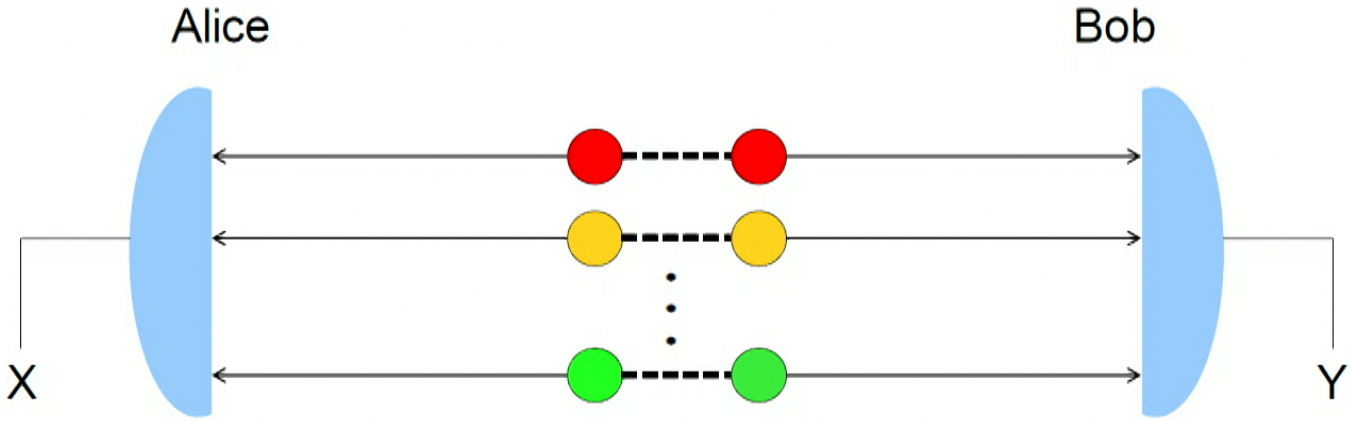


A first example: entangled gaussian states of light

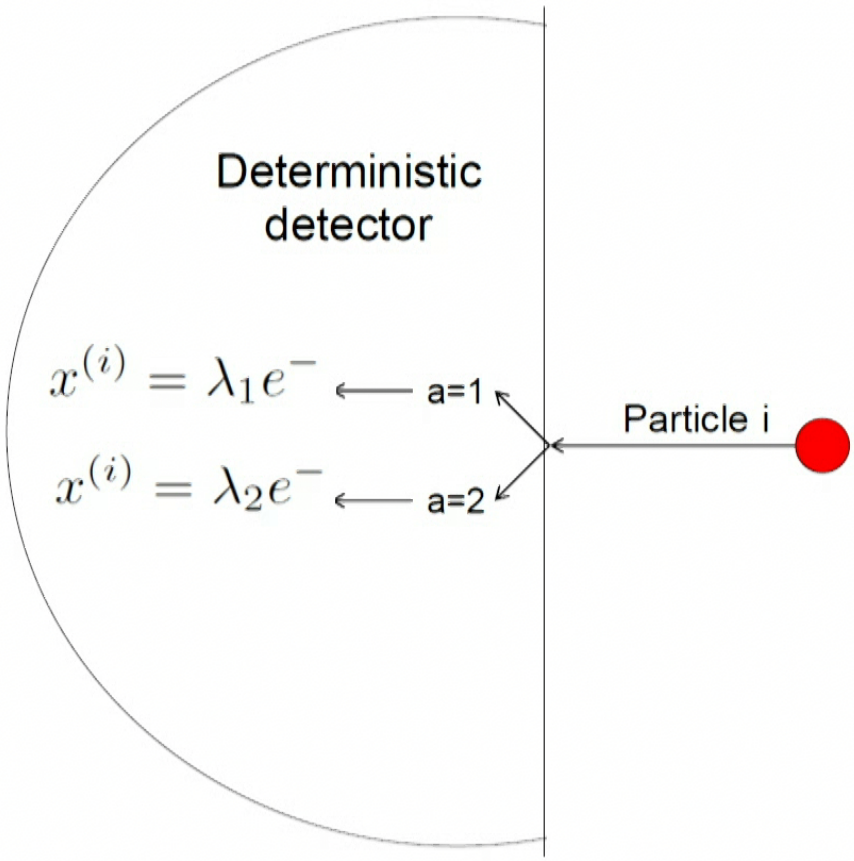


$$P(X,Y),\ \text{gaussian}$$

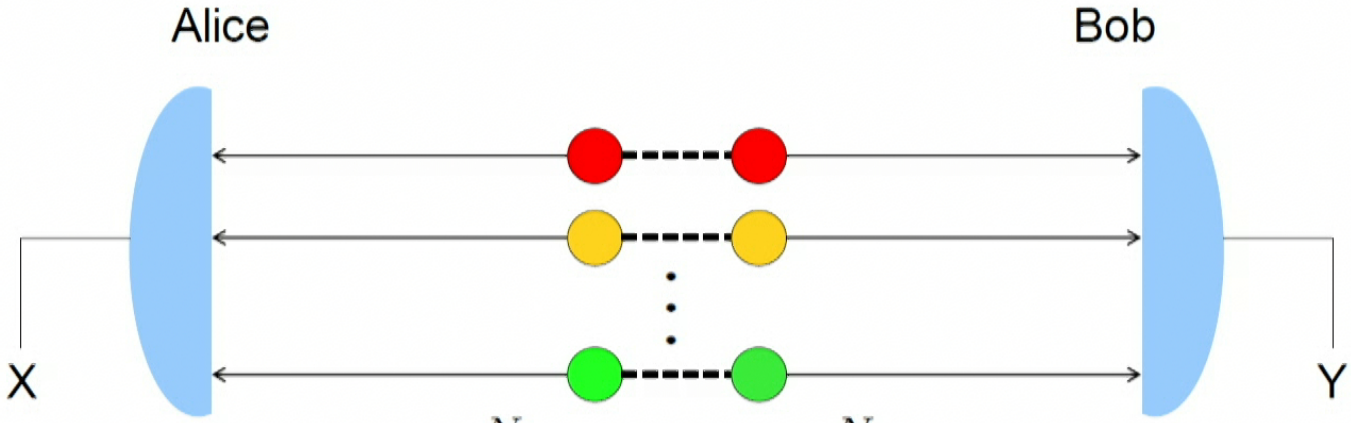
A first example: entangled gaussian states of light



Microscopic dichotomic model



Microscopic dichotomic model



$$X = \sum_{i=1}^{N} x^{(i)}, Y = \sum_{i=1}^{N} y^{(i)}$$

Microscopic dichotomic model

$$P(X,Y),\ \text{gaussian}$$

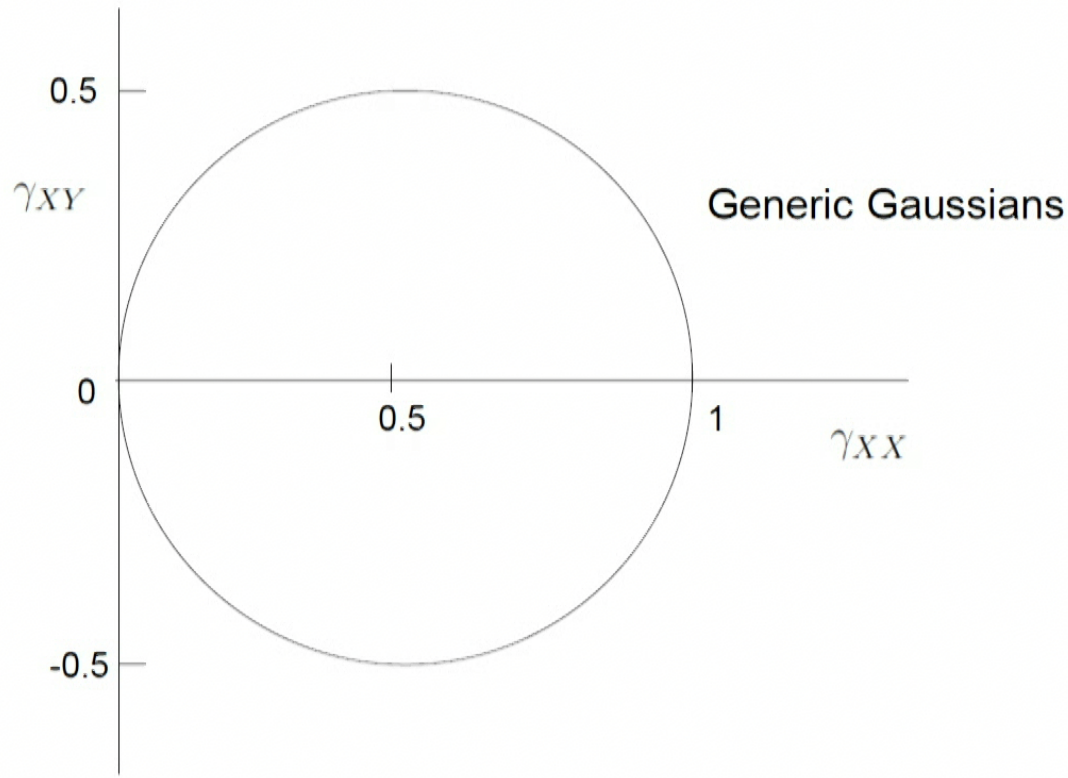
Displacement vector V $\rightarrow$

$$V_X = \langle X \rangle$$
$$V_Y = \langle Y \rangle$$

Covariance matrix γ $\rightarrow$

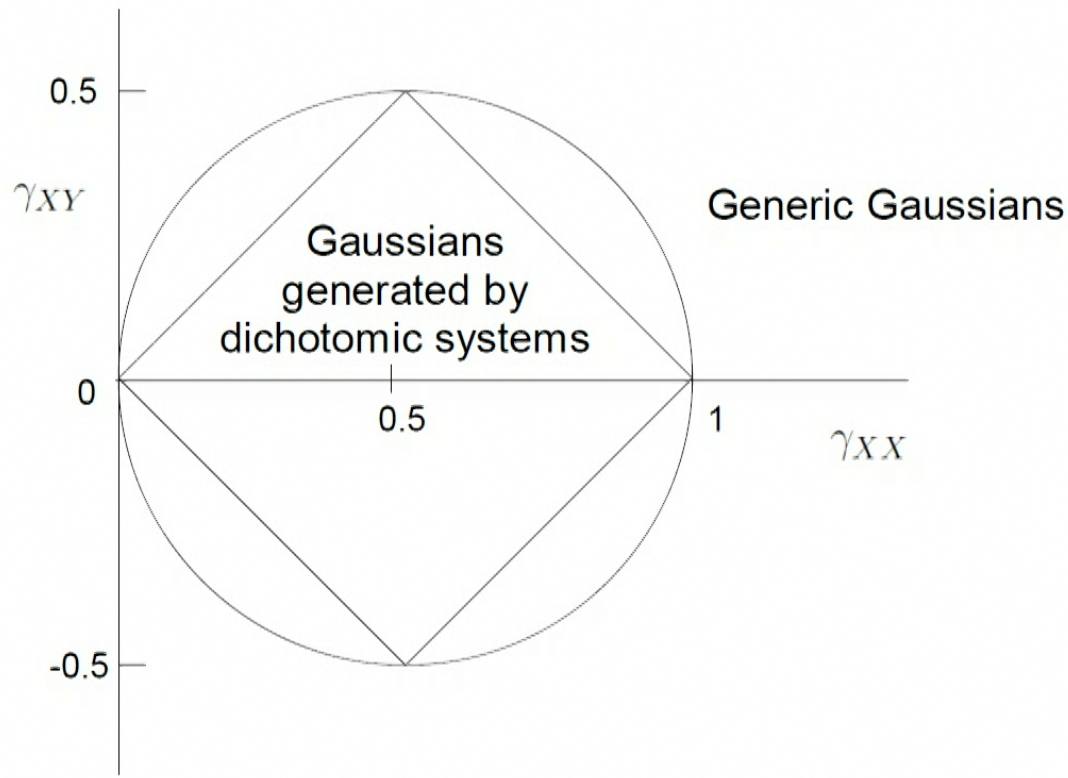
$$\gamma_{XX} \equiv \langle X^2 \rangle - \langle X \rangle^2$$
$$\gamma_{XY} \equiv \langle XY \rangle - \langle X \rangle \langle Y \rangle$$
$$\gamma_{YY} \equiv \langle Y^2 \rangle - \langle Y \rangle^2$$

Predictions of MDM



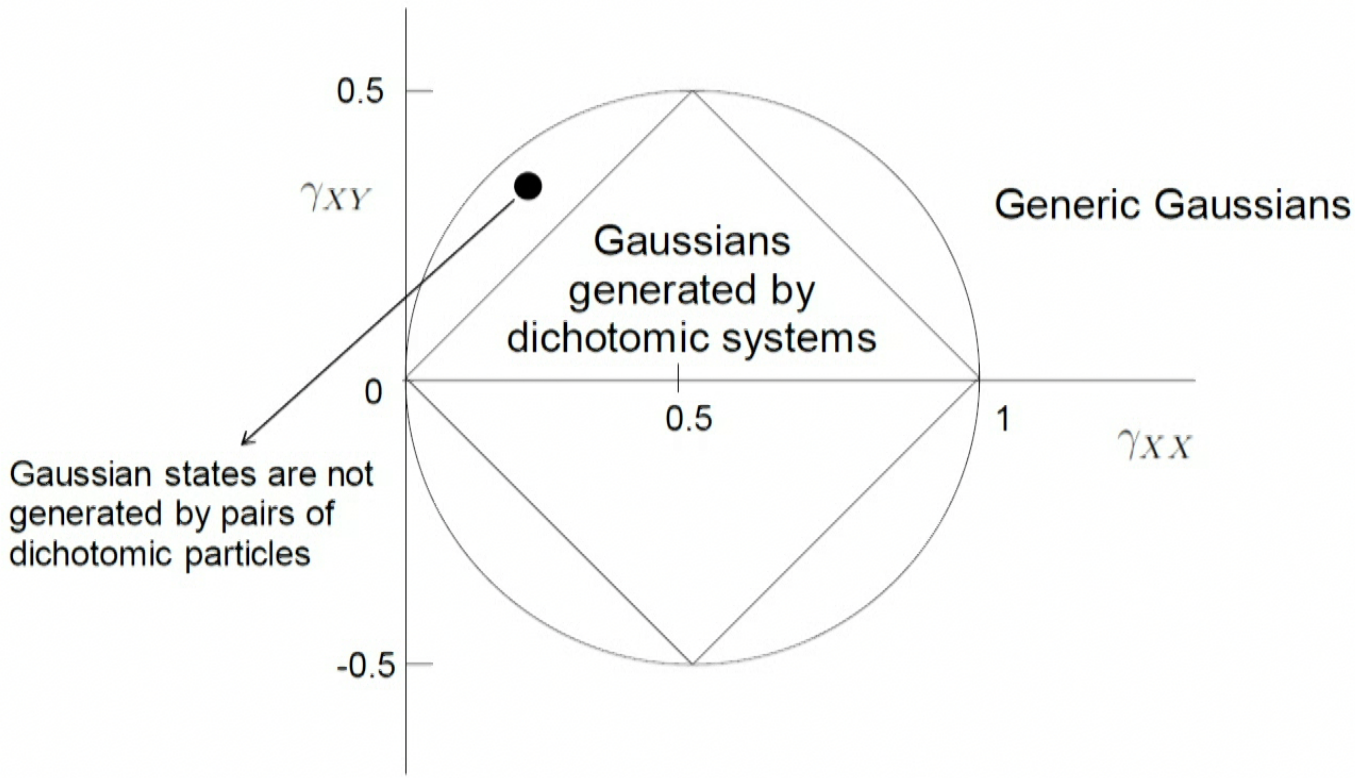
$$\gamma_{XX} + \gamma_{YY} = 1$$

Predictions of MDM



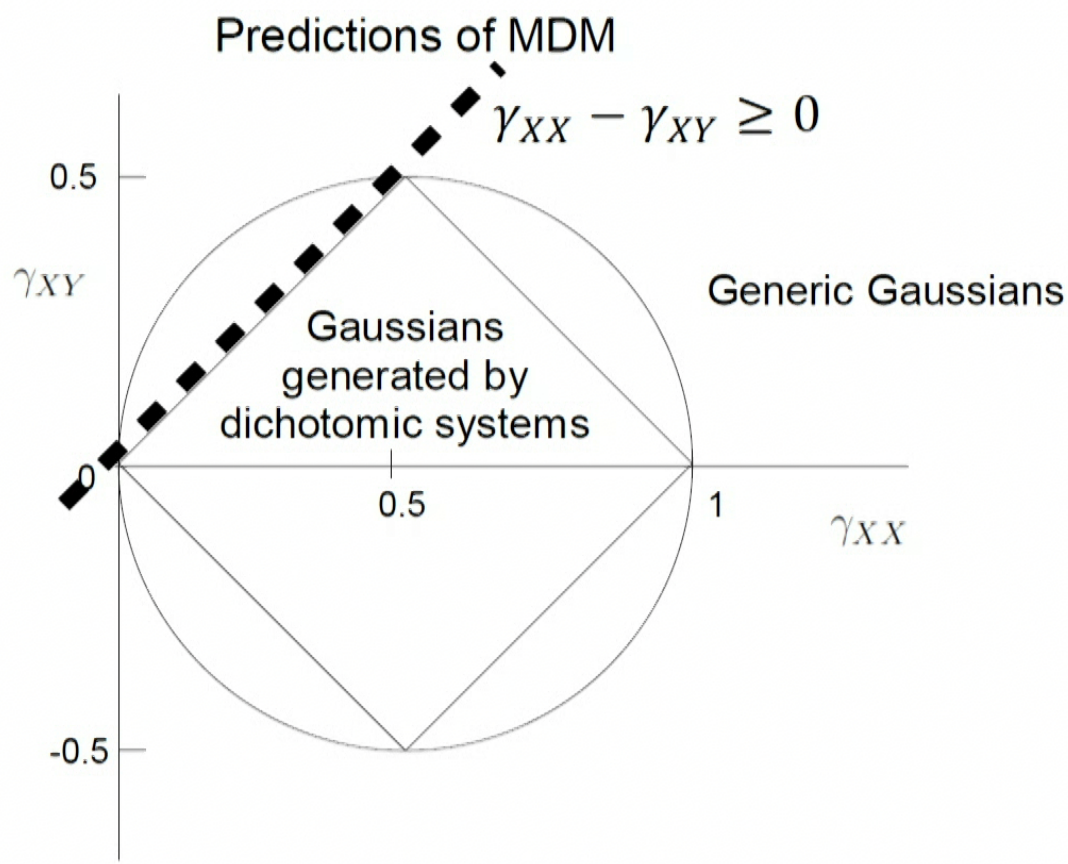
$$\gamma_{XX} + \gamma_{YY} = 1$$

Predictions of MDM



$$\gamma_{XX} + \gamma_{YY} = 1$$

Why?



$$\gamma_{XX} + \gamma_{YY} = 1$$

$$\gamma_{XX} - \gamma_{XY} \geq 0$$

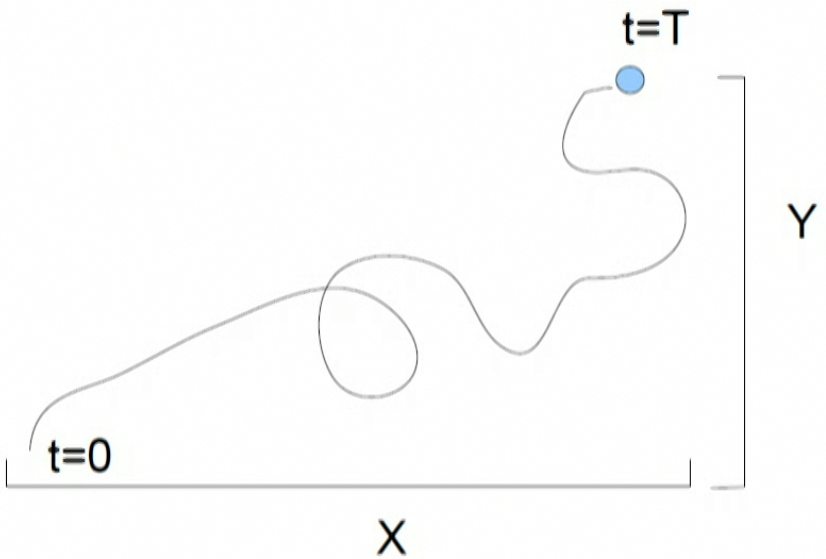
$$p_{AB}^{(i)}(a,b)$$

$$\gamma_{XX} \propto \sum_{i=1}^{N} (\lambda_1^{(i)} - \lambda_2^{(i)})^2 p_A^{(i)}(1)(1 - p_A^{(i)}(1))$$

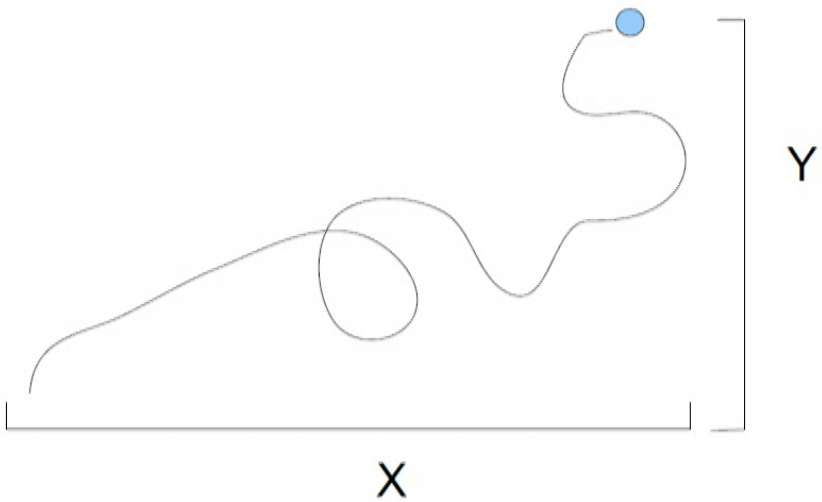
$$\gamma_{XY} \propto \sum_{i=1}^{N} (\lambda_1^{(i)} - \lambda_2^{(i)})^2 [p_{AB}^{(i)}(1,1) - p_A^{(i)}(1) p_B^{(i)}(1)]$$

$$\gamma_{XX} - \gamma_{XY} \propto$$
$$\propto p_A(1)[1 - p_A(1)] - p_{AB}(1,1) + p_A(1)p_B(1) \geq$$
$$\geq p_{AB}(1,1)[1 - p_A(1)] - p_{AB}(1,1) + p_A(1)p_B(1) =$$
$$= p_A(1)[p_B(1) - p_{AB}(1,1)] \geq 0.$$

Application: Brownian motion

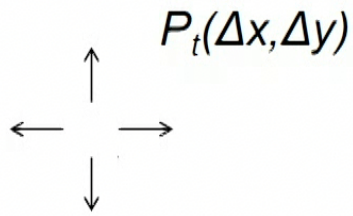


Application: Brownian motion



P(X,Y), gaussian $\longrightarrow$ Covariance matrix γ.

Application: Brownian motion



Microscopic model

1) At each time t, the probability of moving along each of the four diagonals is independent of the trajectory of the particle.
2) $P_t(\Delta x,\Delta y)$ is periodic in t, with period much smaller than T.

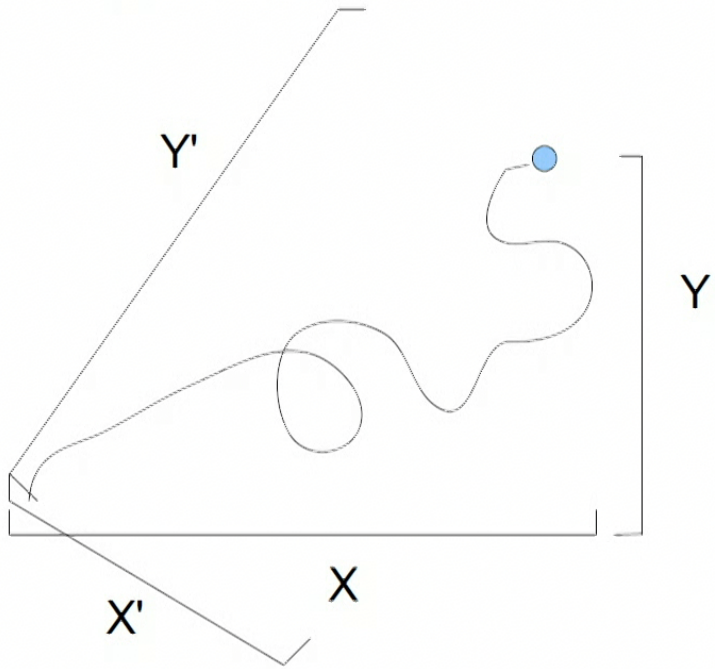
Application: Brownian motion

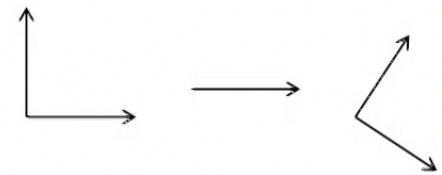
$$\mathrm{tr}(\gamma W_{i,j}) \geq 0, \text{ for } i,j = 0,1$$

$$\gamma_{XX} - |\gamma_{XY}|, \gamma_{YY} - |\gamma_{XY}| \geq 0$$

$$\boxed{W_{ij} = \mathbb{I} + (-1)^i \sigma_x + (-1)^k \sigma_z}$$

Application: Brownian motion





Macroscopic isotropy
and parity

Application: Brownian motion

$$\mathrm{tr}(U\gamma U^T W_{i,j}) \geq 0, \forall U \in O(2)$$

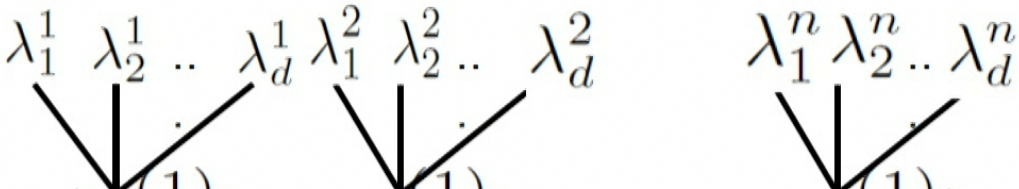
$$W_{ij} = \mathbb{I} + (-1)^i \sigma_x + (-1)^k \sigma_z$$

Application: Brownian motion

$$\det(\gamma) \geq \frac{1}{8}\mathrm{tr}(\gamma)^2$$

The model can only be applied to macroscopic systems with an uncertainty relation

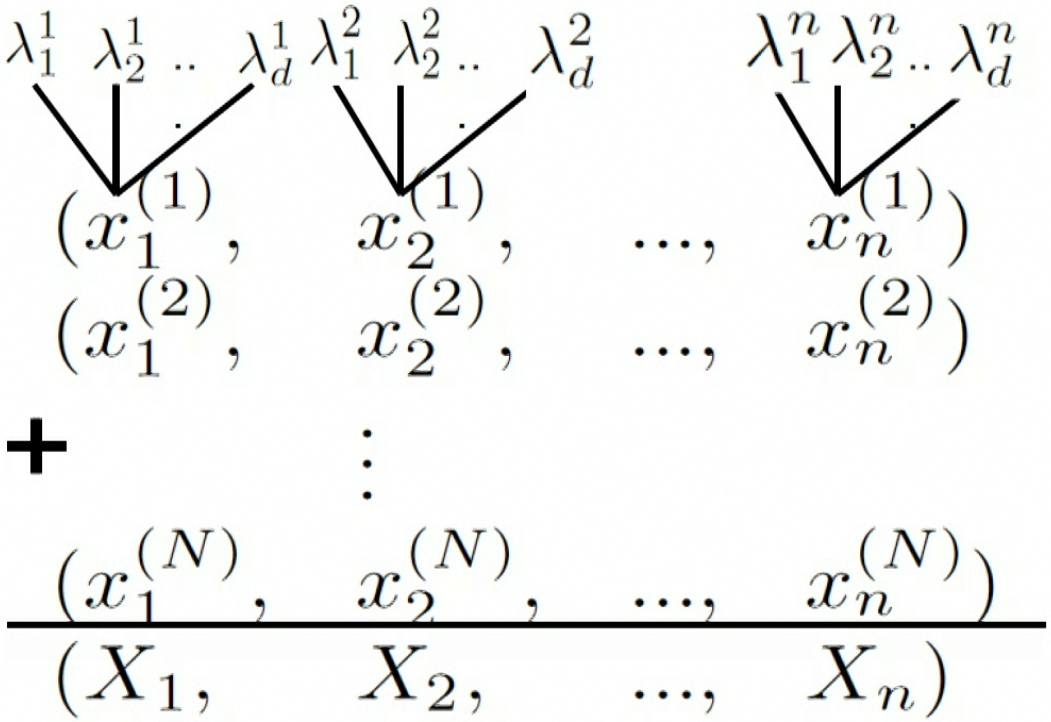
Fixed output structure



$$
\begin{array}{llll}
(x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\
(x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\
+ & \vdots & & \\
(x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\
\hline
(X_1, & X_2, & \ldots, & X_n)
\end{array}
$$

Generalization

Fixed output structure

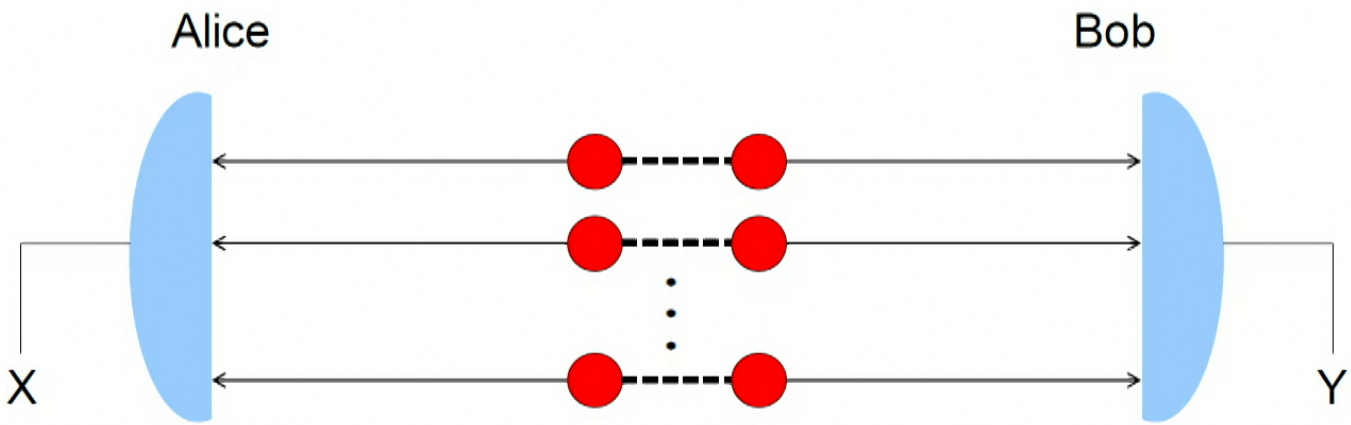


The values

$$\{\lambda_a^k \,:\, k = 1, \ldots, n; a = 1, \ldots, d\}$$

are known up to a proportionality constant

E.g.: entangled gaussian states of light



$$\{\lambda_a \propto a - (d+1)/2, a = 1, ..., d\}$$

Microscopic spin model

Gaussian distribution

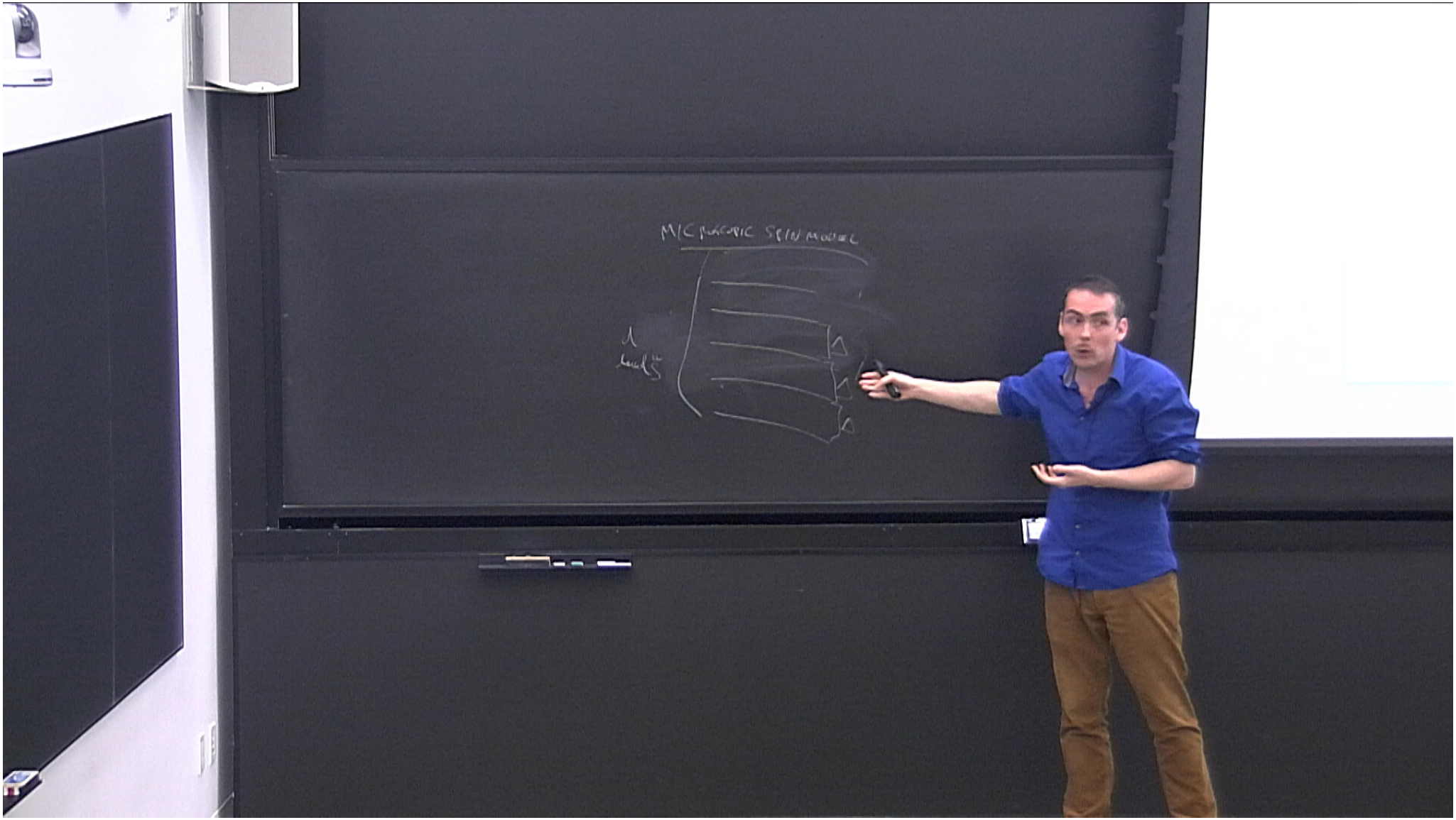
$$P(X_1, ..., X_n) dX_1 ... dX_n$$

Gaussian distribution

$$P(X_1, ..., X_n) dX_1 ... dX_n$$

Displacement vector

$$V_i = \langle X_i \rangle$$


MICROSCOPIC SPIN MODEL

Gaussian distribution

$$P(X_1, ..., X_n) dX_1 ... dX_n$$

Covariance matrix

$$\gamma_{ij} = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle$$

Gaussian distribution

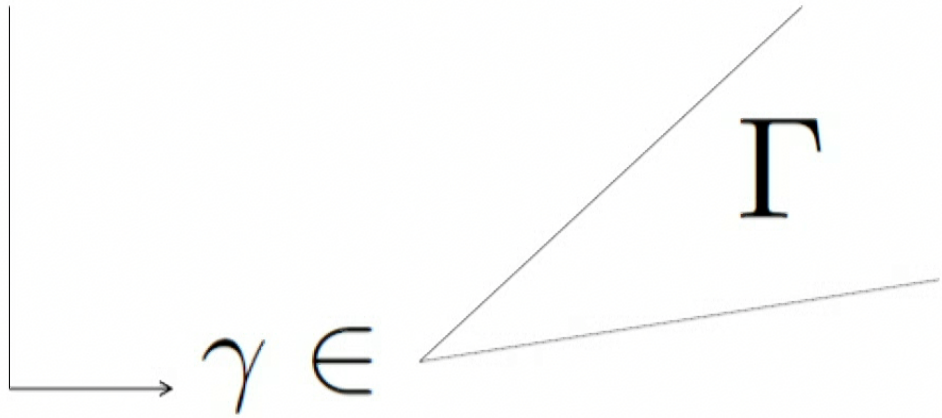
$$P(X_1, ..., X_n) dX_1 ... dX_n$$

Covariance matrix

$$\gamma_{ij} = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle$$

Non-linear term!!!

$$\{\lambda_a^k \,:\, k = 1, ..., n; a = 1, ..., d\}$$



Γ is generated by a finite number of rays

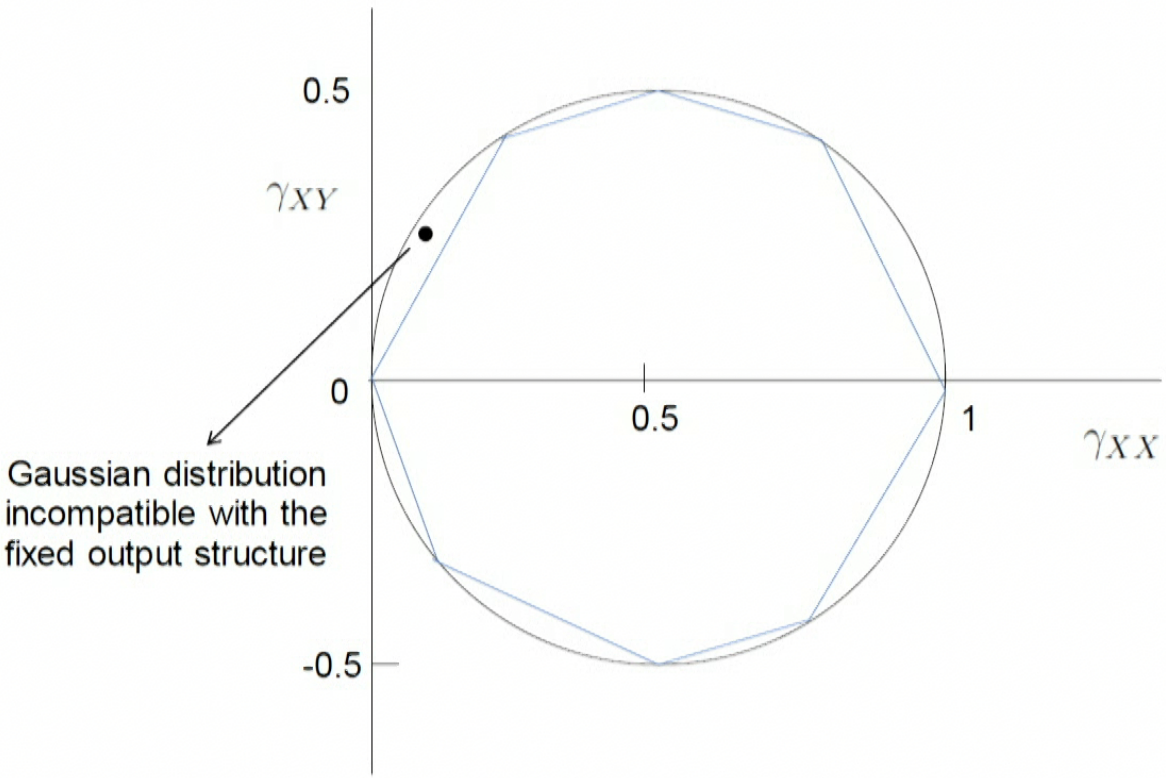
Fixed output structure

Given

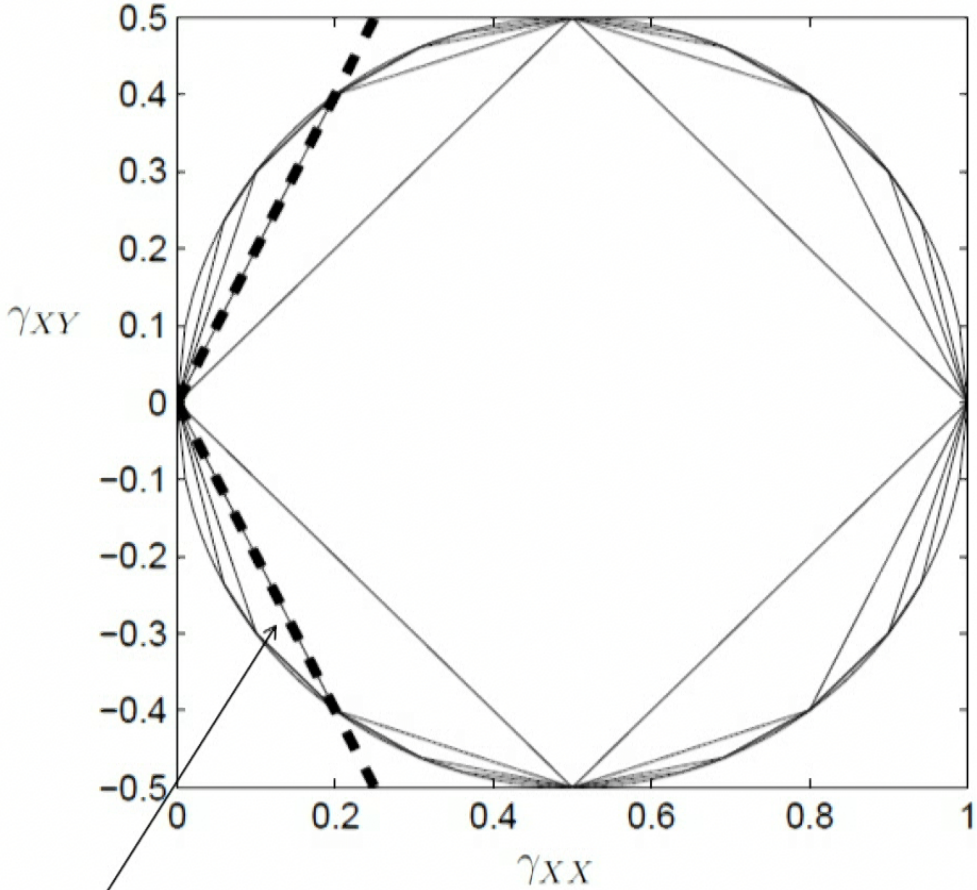
$$\{\lambda_a^k \,:\, k = 1, \ldots, n; a = 1, \ldots, d\}$$

the problem of deciding if a covariance matrix can be generated by this microscopic model can be formulated as a linear program

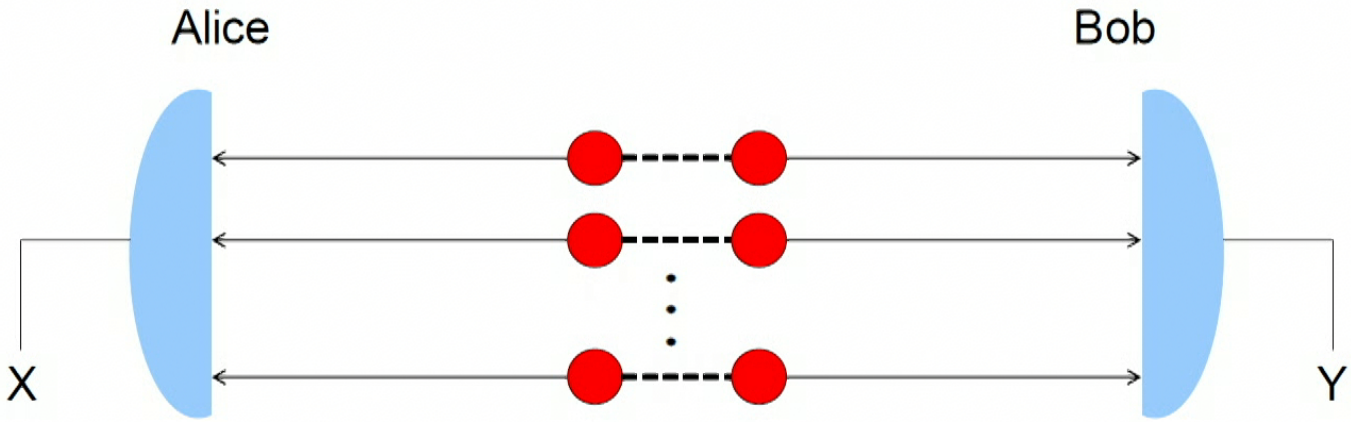
For any microscopic model with finite dimension d, there always exist bipartite gaussian distributions that cannot be approximated.



$$\gamma_{XX} + \gamma_{YY} = 1$$

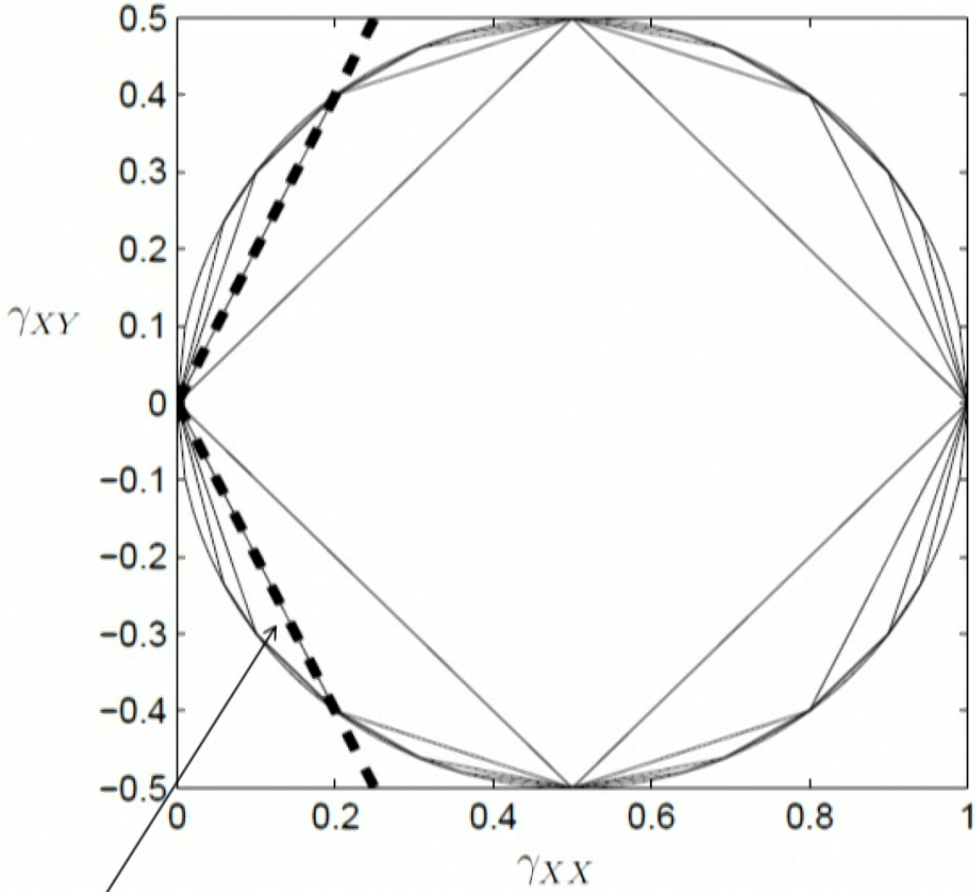


$$|\gamma_{XY}| \leq (d-1)\gamma_{XX}$$

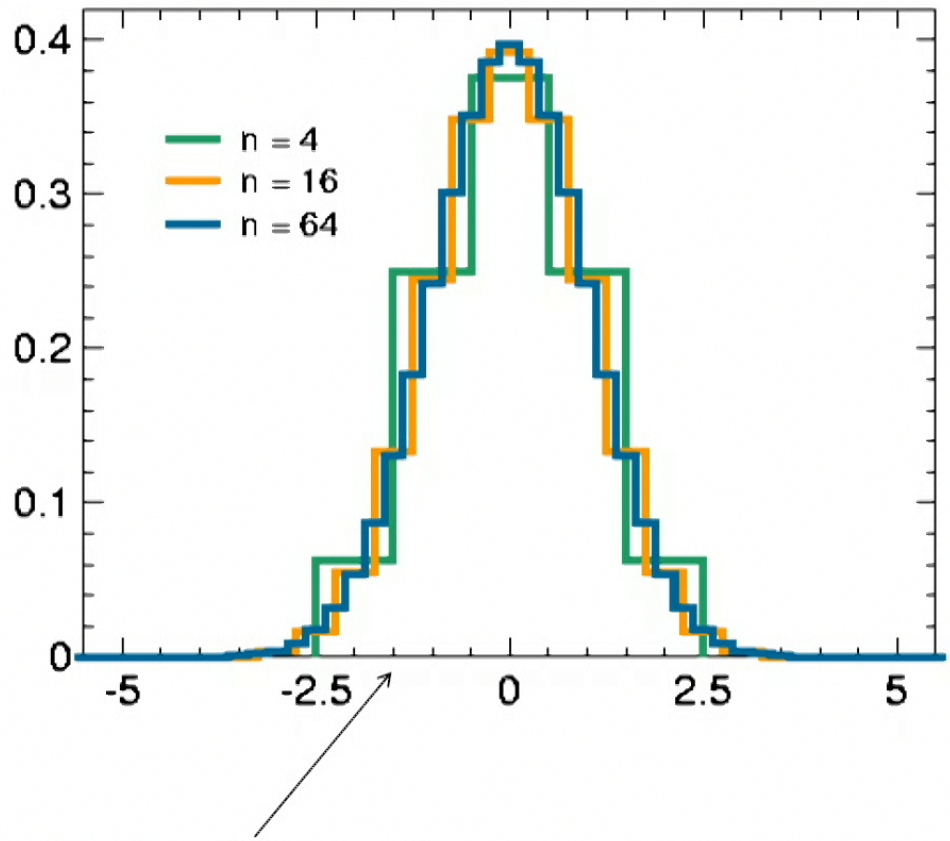


$$\{\lambda_a \propto a - (d+1)/2, a = 1, ..., d\}$$

Given enough energy, for each value of d, one can create a bipartite gaussian state of light whose statistics refute the microscopic spin model



$$|\gamma_{XY}| \leq (d-1)\gamma_{XX}$$ Irrelevant for high d



Discretization of the continuous variable

$$\{\lambda_a^k = f(a) : a \in \mathbb{N}\},$$

f rational, non-constant

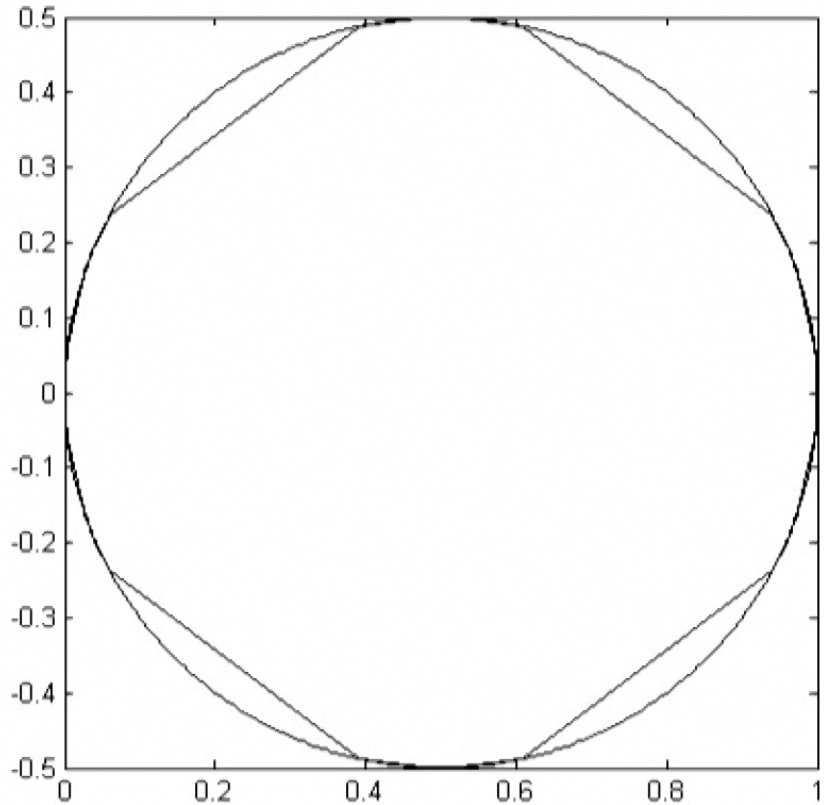
$$\Downarrow$$

All n-variate covariance matrices can be recovered

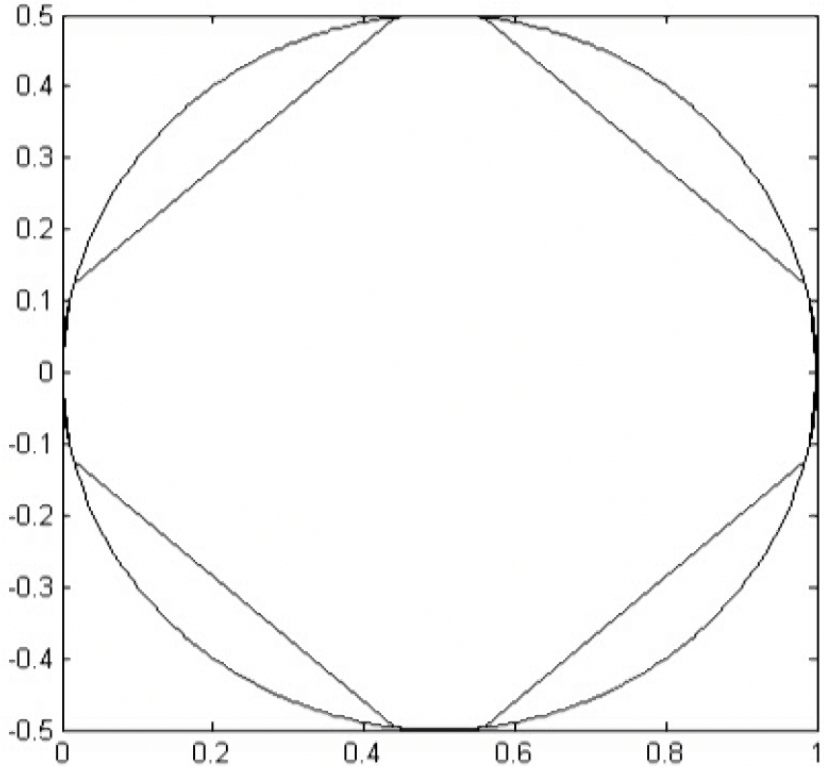
Does any infinite set of microscopic outputs allow to generate all gaussian distributions?

No!

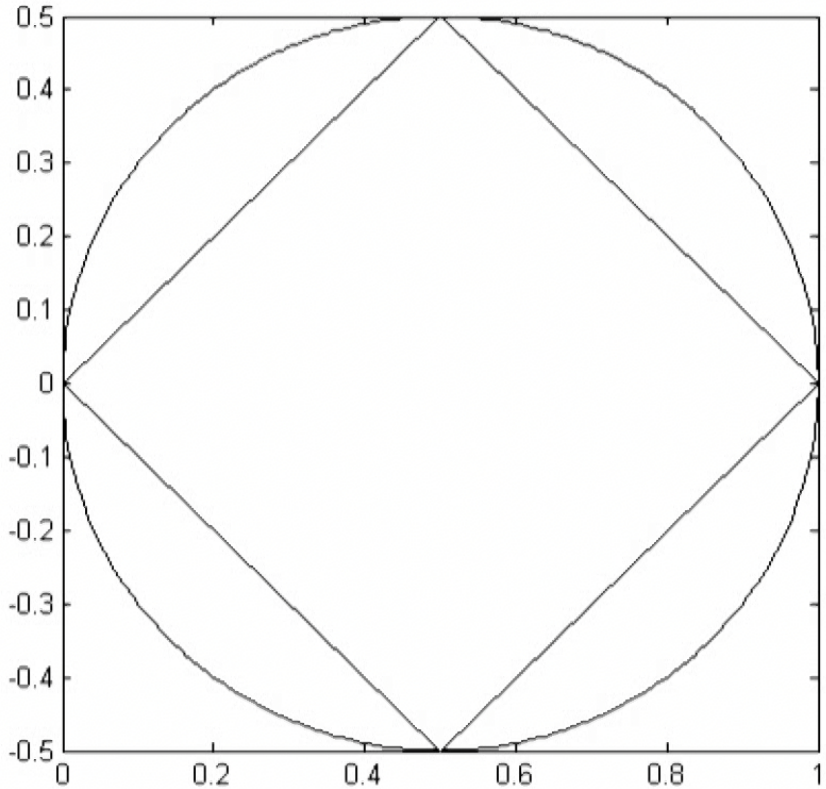
Consider $\{\lambda_a = \mu^a : a \in \mathbb{Z}\}$



$\mu=1/5$



$\mu=1/9$

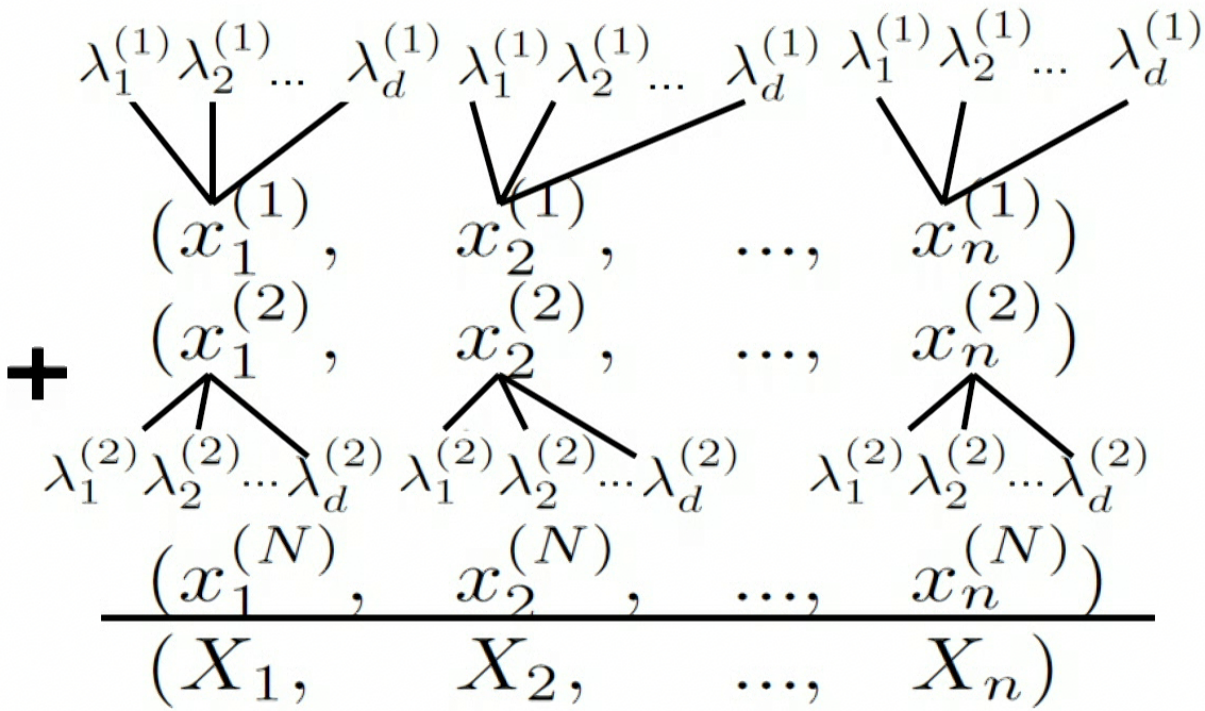


$\mu \rightarrow 0$

Free output structure

One can recover all gaussian distributions with d=2

Free output structure



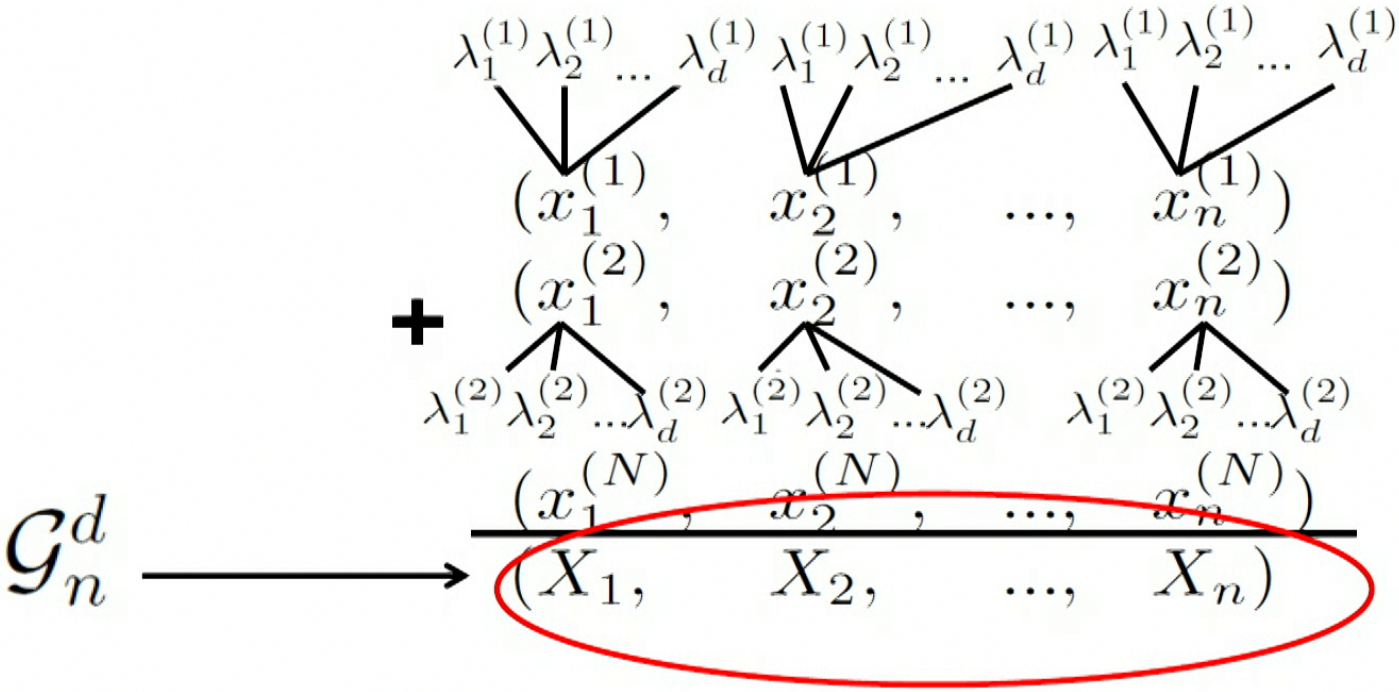
Free output structure

E.g.: time measurements

$$X_k = X(t_k)$$

$$+ \quad \begin{array}{cccc} t_1 & t_2 & & t_n \\ (x_1^{(1)}, & x_2^{(1)}, & \ldots, & x_n^{(1)}) \\ (x_1^{(2)}, & x_2^{(2)}, & \ldots, & x_n^{(2)}) \\ & \vdots & & \\ (x_1^{(N)}, & x_2^{(N)}, & \ldots, & x_n^{(N)}) \\ \hline (X_1, & X_2, & \ldots, & X_n) \end{array}$$

Free output structure



Theorem

$$\mathcal{G}_n^{d=n+1} = \mathcal{G}$$

$$\mathcal{G}_n^{d=n} \neq \mathcal{G}$$

Theorem

Construction

$$\mathcal{G}_n^{d=n+1} = \mathcal{G}$$

$$\mathcal{G}_n^{d=n} \neq \mathcal{G}$$

Theorem

$$\mathcal{G}_n^{d=n+1} = \mathcal{G}$$

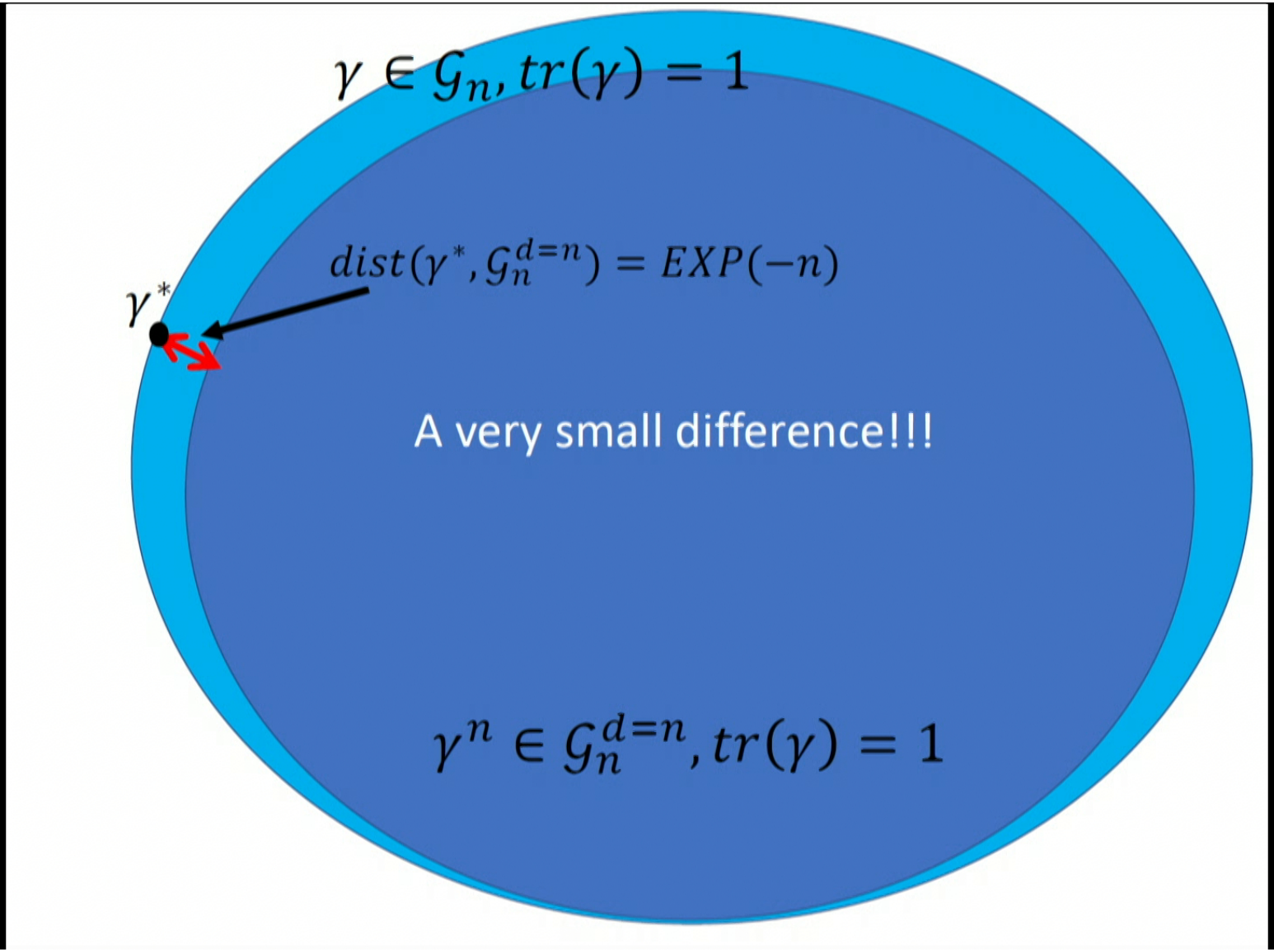
Witness

$$\mathcal{G}_n^{d=n} \neq \mathcal{G}$$

$$\gamma \in \mathcal{G}_n^n$$

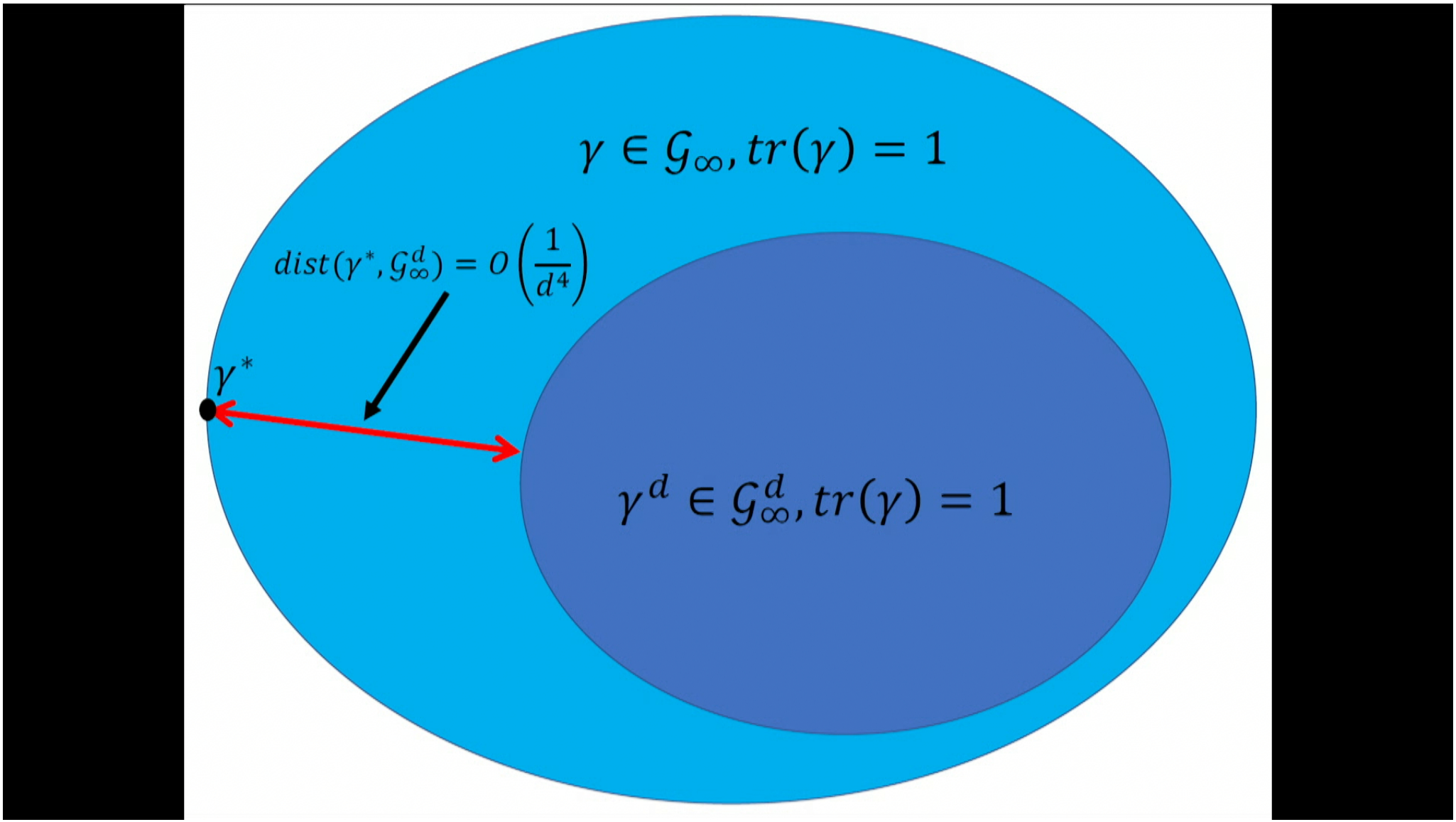
$$\downarrow$$

$$B(\gamma) \equiv \left(1 - \frac{3}{n(4^n - 1)}\right) \sum_{i=1}^{n} \gamma_{ii} - \frac{3}{4^n - 1} \sum_{i,j=1}^{n} 2^{i+j-2} \gamma_{ij} \geq 0$$


$\gamma \in \mathcal{G}_n, tr(\gamma) = 1$
$dist(\gamma^*, \mathcal{G}_n^{d=n}) = EXP(-n)$
γ^*
A very small difference!!!
$\gamma^n \in \mathcal{G}_n^{d=n}, tr(\gamma) = 1$

Surprising?

n→ ∞
VS
∞
d


$\gamma \in \mathcal{G}_\infty, tr(\gamma) = 1$
$dist(\gamma^*, \mathcal{G}_\infty^d) = O\left(\frac{1}{d^4}\right)$
γ^*
$\gamma^d \in \mathcal{G}_\infty^d, tr(\gamma) = 1$

Deciding if $\gamma \in \mathcal{G}_n^d$

can be cast as an SDP

$$\gamma \in \mathcal{G}_n^d \longleftrightarrow \gamma = \sum_{P \in \mathcal{P}_n^d} P Z_P P^T$$

$$\gamma \in \mathcal{G}_n^d \longleftrightarrow \gamma = \sum_{P \in \mathcal{P}_n^d} P Z_P P^T$$

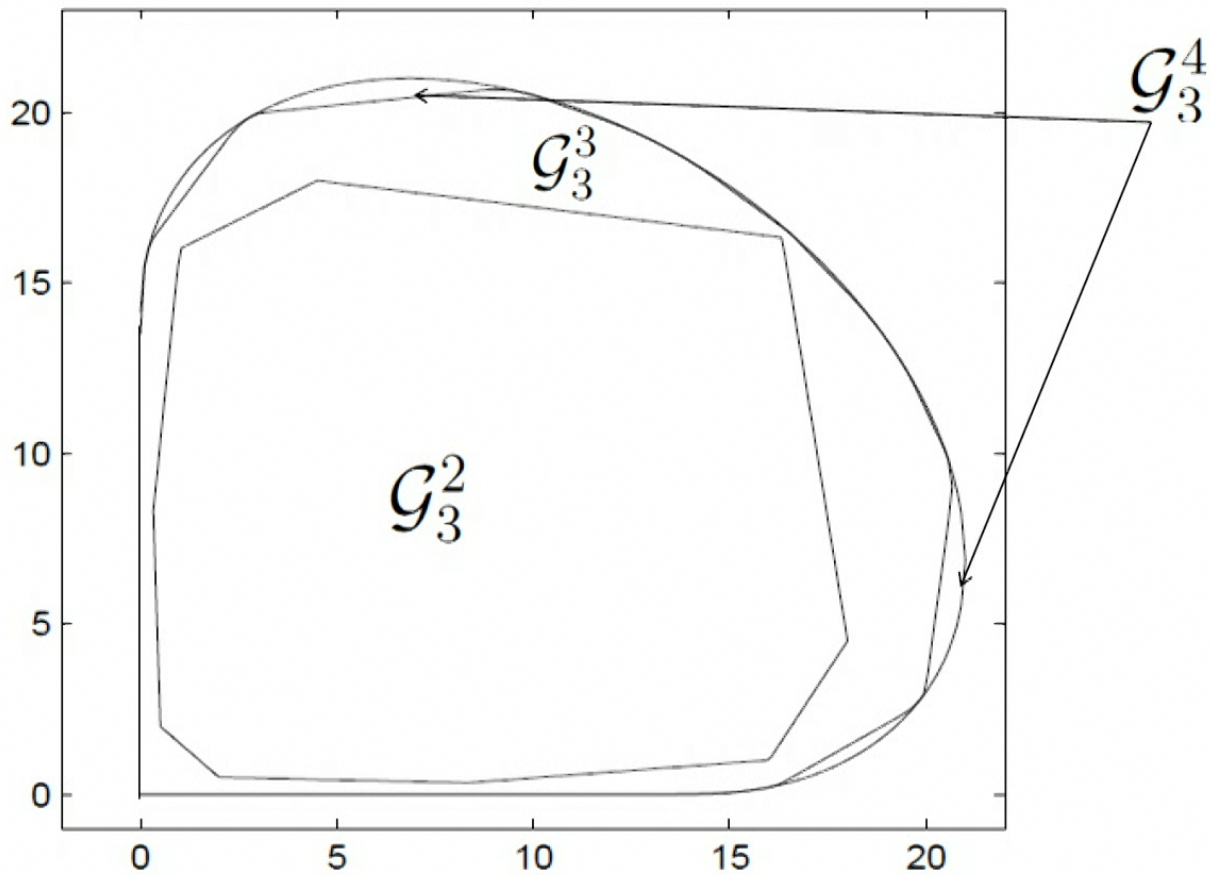
$$Z_P \in M_{d-1 \times d-1}, Z_P \geq 0$$

$$\gamma \in \mathcal{G}_n^d \longleftrightarrow \gamma = \sum_{P \in \mathcal{P}_n^d} P Z_P P^T$$

$$\mathcal{P}_n^d \subset M_{n \times d-1}$$

$$\mathcal{P}_n^{n+1} = \{\mathbb{I}_n\}$$

$$\mathcal{P}_3^3 = \boxed{\begin{array}{c} P_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, P_2^i = \begin{pmatrix} 1 & 0 \\ i & 0 \\ 0 & 1 \end{pmatrix}, \\ P_3^{i,j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ i & j \end{pmatrix} \quad i,j = \pm 1 \end{array}}$$



A slice of the set of tri-variate Gaussian distributions

Drawback

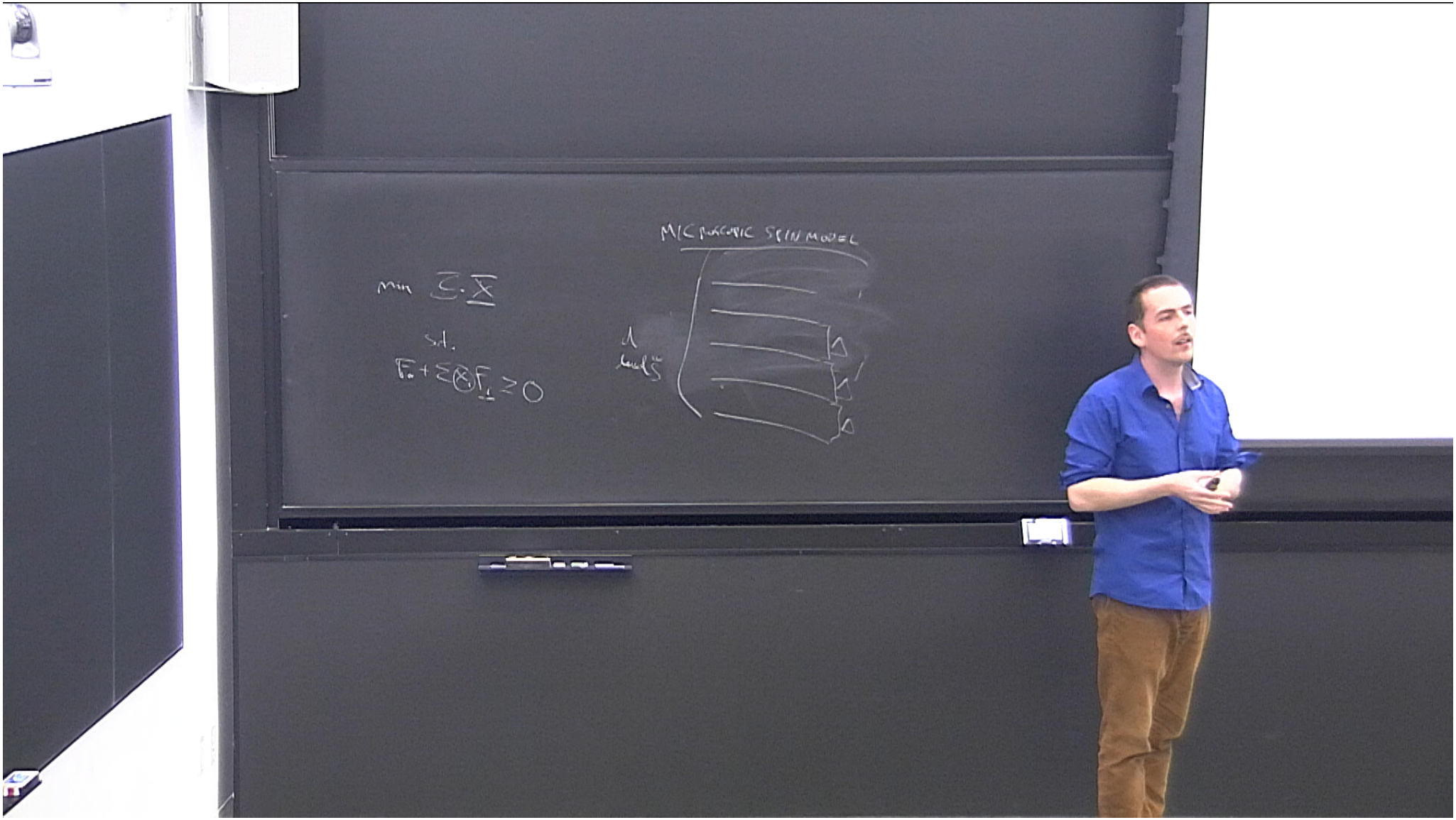
$\left|\mathcal{P}_n^d\right|$ grows exponentially with n

Deciding if $\gamma \in \mathcal{G}_n^d$

can be cast as an SDP

… but, with a normal computer, one can only study scenarios with a small number of macroscopic variables

Are there more efficient methods to characterize $\mathcal{G}_n^d$?


MICROSCOPIC SPIN MODEL
min C·X
s.t.
F₀ + Σ ⊗ F⊥ ≥ 0

Are there more efficient methods to characterize $\mathcal{G}_n^d$?

Most likely, not.

$\gamma, n \times n$ matrix, $\mathrm{tr}(\gamma) = 1$

decide if $\mathrm{dist}(\gamma, \mathcal{G}_2^n) < \frac{1}{q(n)}$

$q(n)$, polinomial

γ $n \times n$ matrix, $\mathrm{tr}(\gamma) = 1$

NP-hard

decide if $\mathrm{dist}(\gamma, \mathcal{G}_2^n) < \frac{1}{q(n)}$

$q(n)$, polinomial

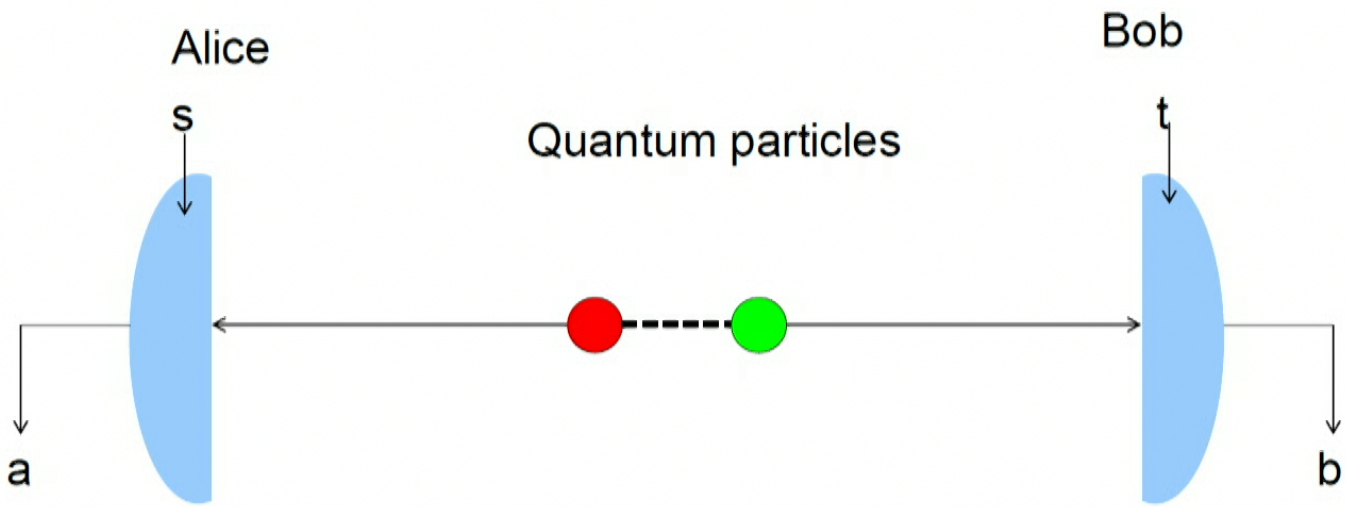
Problem complexity

QPRATIO

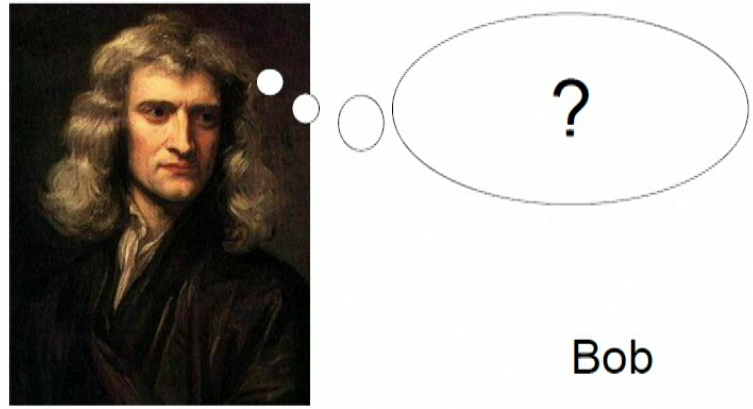
$$A_{ij} \Rightarrow \max_{\{1,0,-1\}^n} \frac{\sum_{i,j=1}^{n} A_{ij} x_i x_j}{\sum_{i=1}^{n} x_i^2}$$

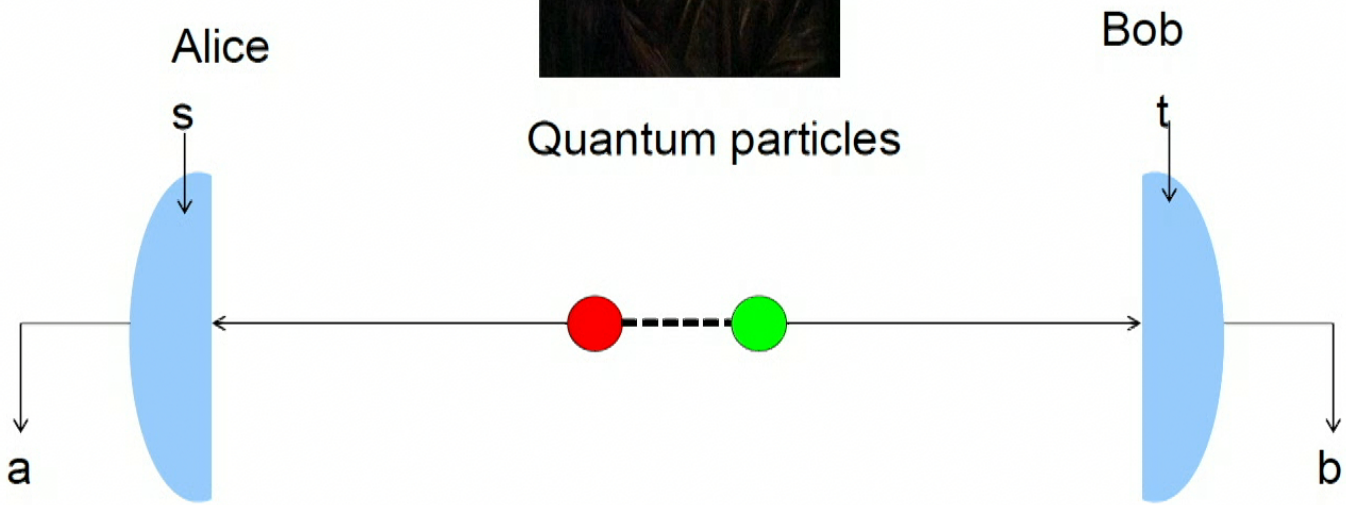
A. Bhaskara, M. Charikar, R. Manokaran, A, Vijayaraghava, On Quadratic Programming with a Ratio Objective, ICALP2012, arXiv:1101.1710.

Connection with quantum non-locality

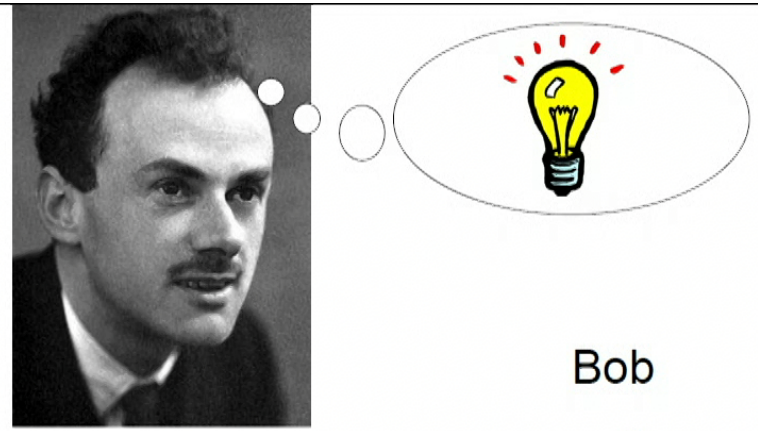


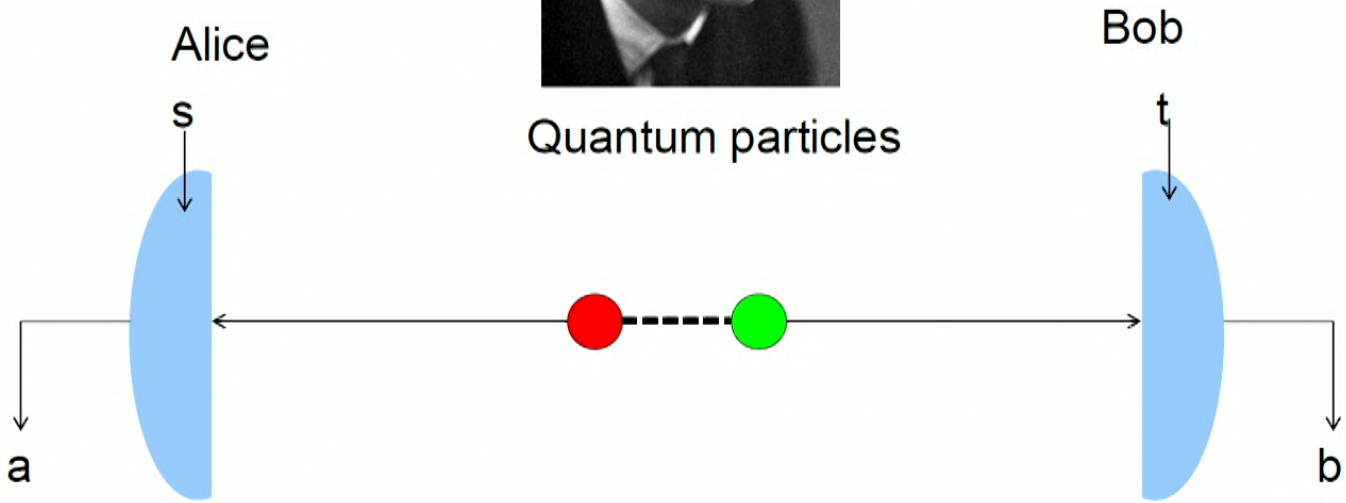
$P(a,b|s,t)$, non-local



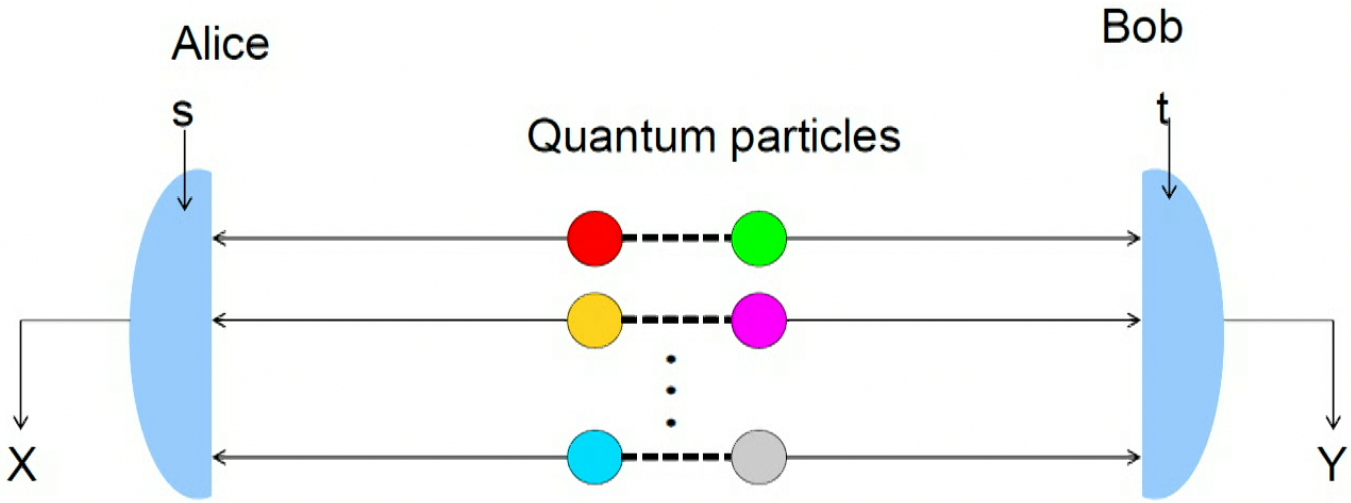


$P(a,b|s,t)$, non-local

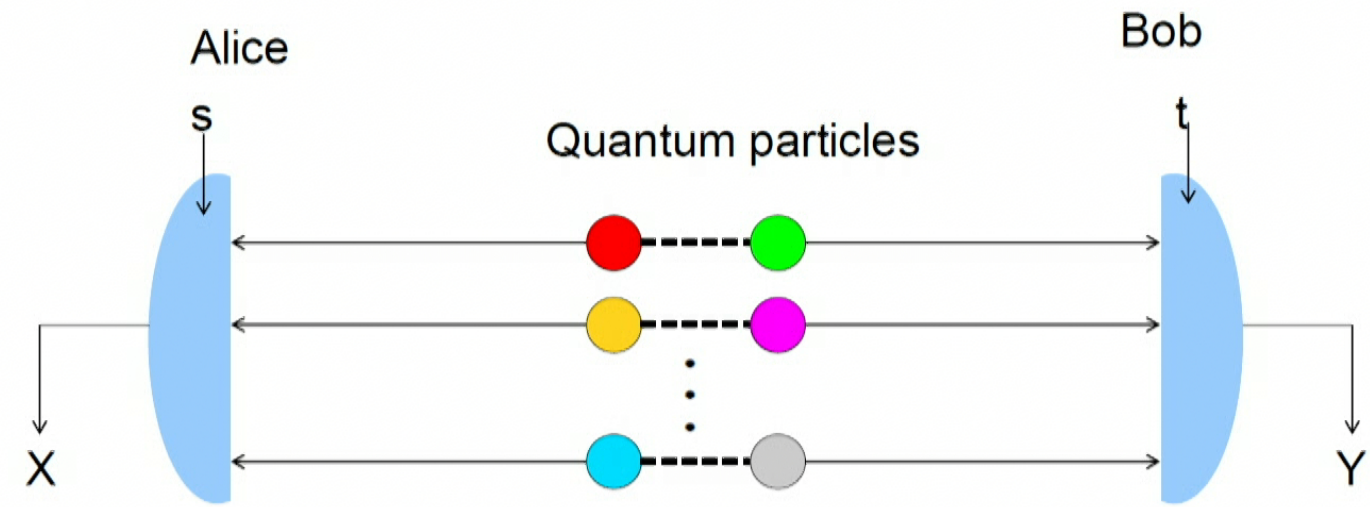




$P(a,b|s,t)$, non-local

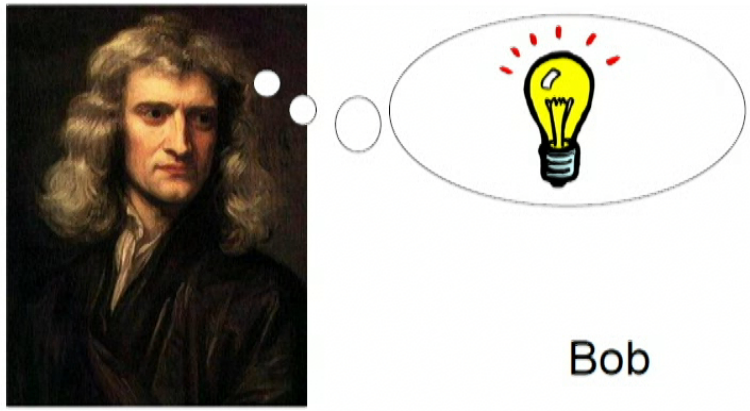


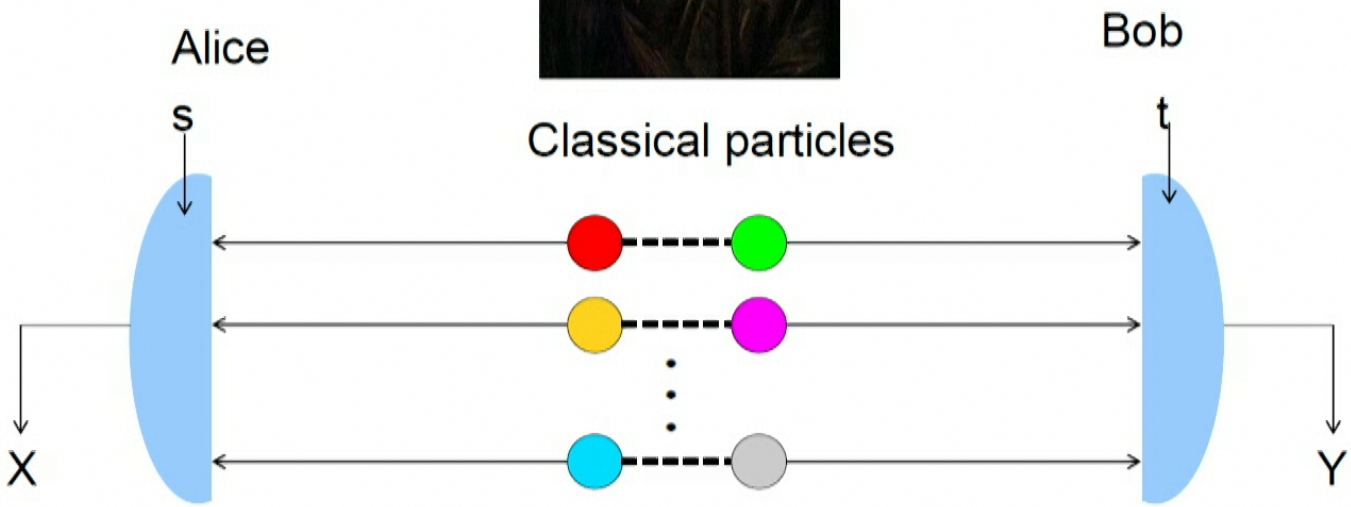
$P_Q(X,Y|s,t)$, gaussian



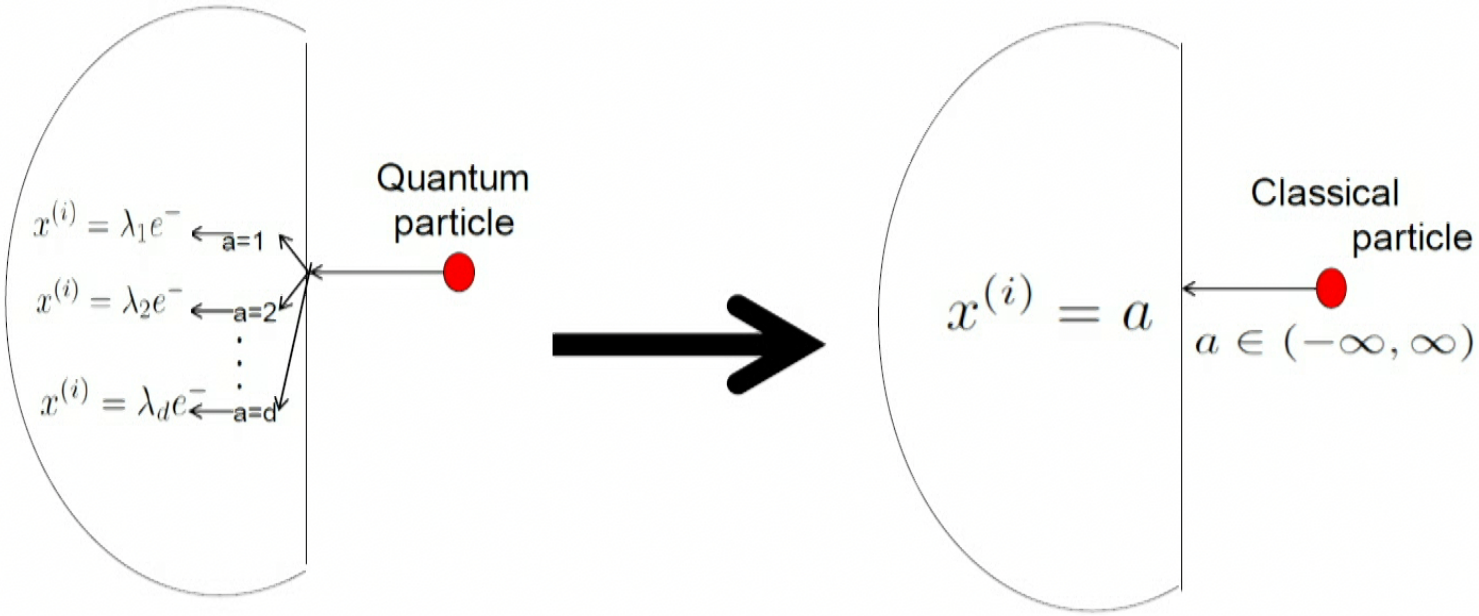
Macroscopic Locality → $P_Q(X,Y|s,t)$, local

MN and H. Wunderlich, Proc.Roy.Soc.Lond.A 466, 881-890 (2009)



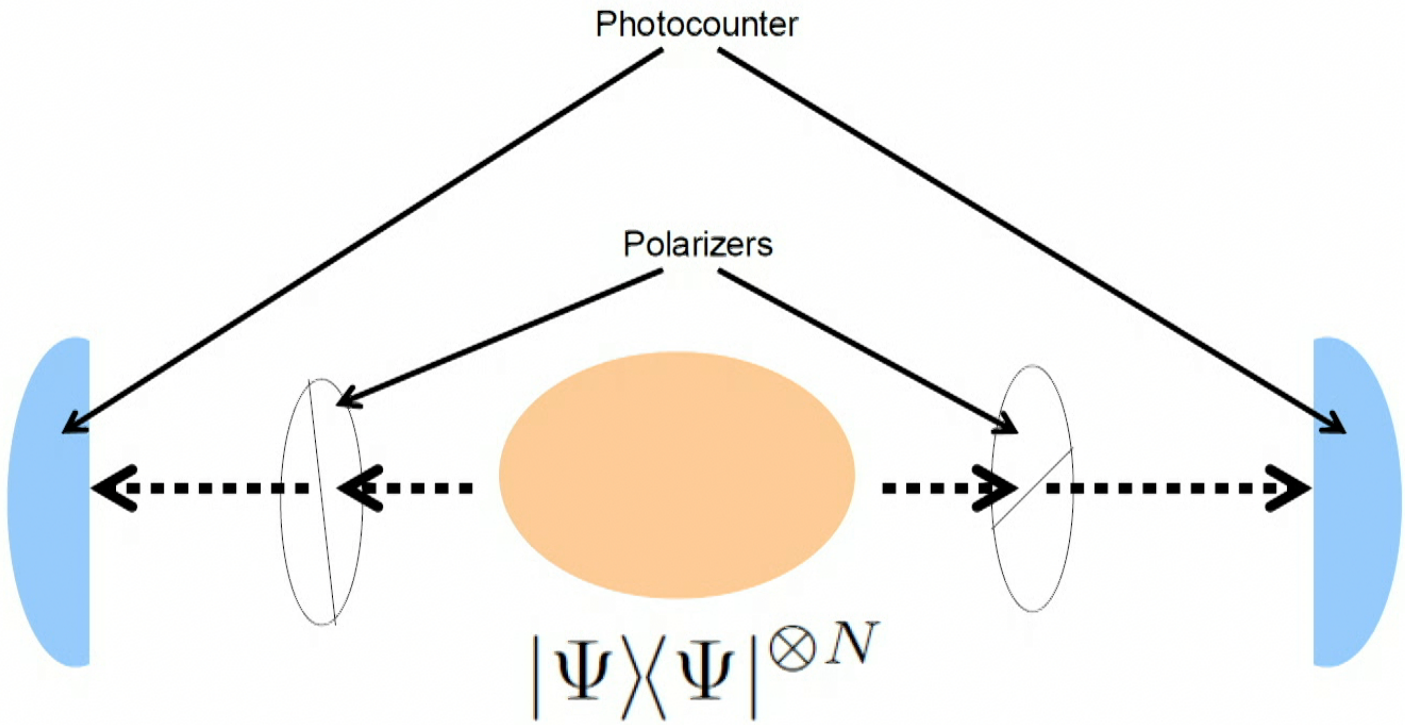


$$P_C(X,Y|s,t)=P_Q(X,Y|s,t)$$


$x^{(i)} = \lambda_1 e^-$
a=1
$x^{(i)} = \lambda_2 e^-$
a=2
$x^{(i)} = \lambda_d e^-$
a=d
Quantum particle
$x^{(i)} = a$
Classical particle
$a \in (-\infty, \infty)$

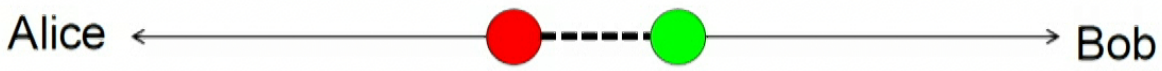
$$d_Q < \infty \rightarrow d_C < \infty?$$

No!!!

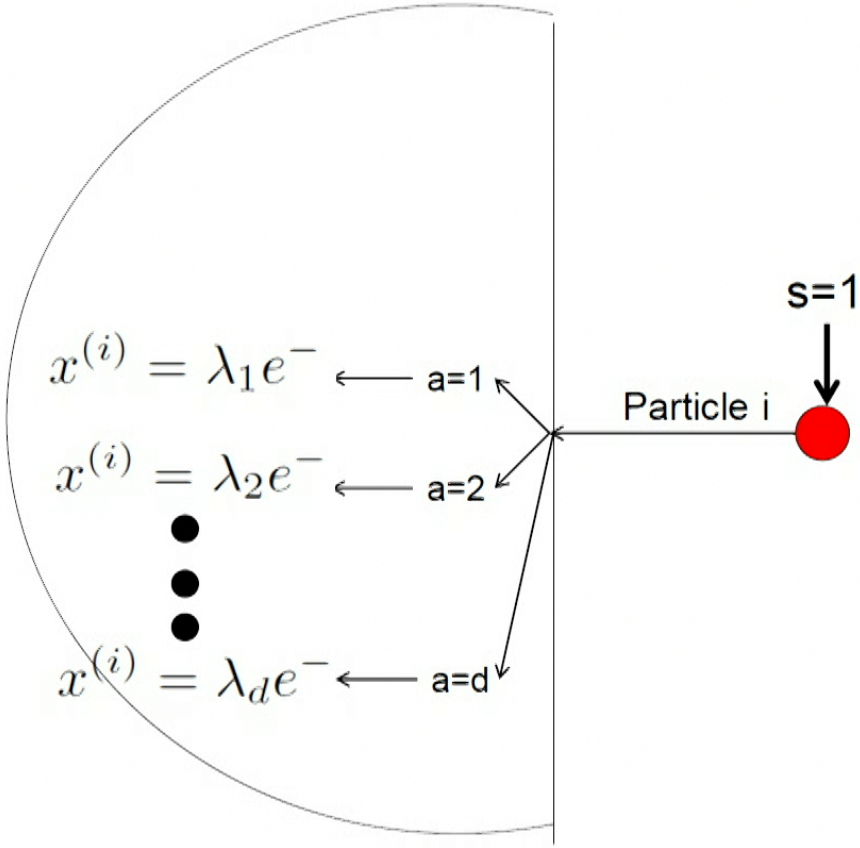


$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

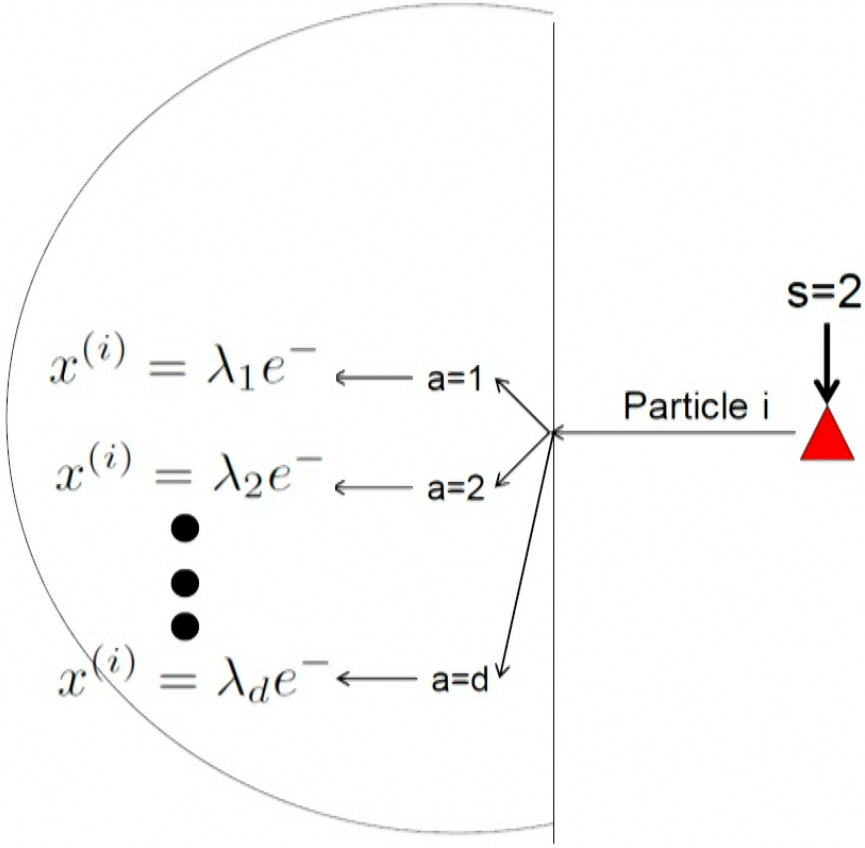
Assumptions of the classical particle model



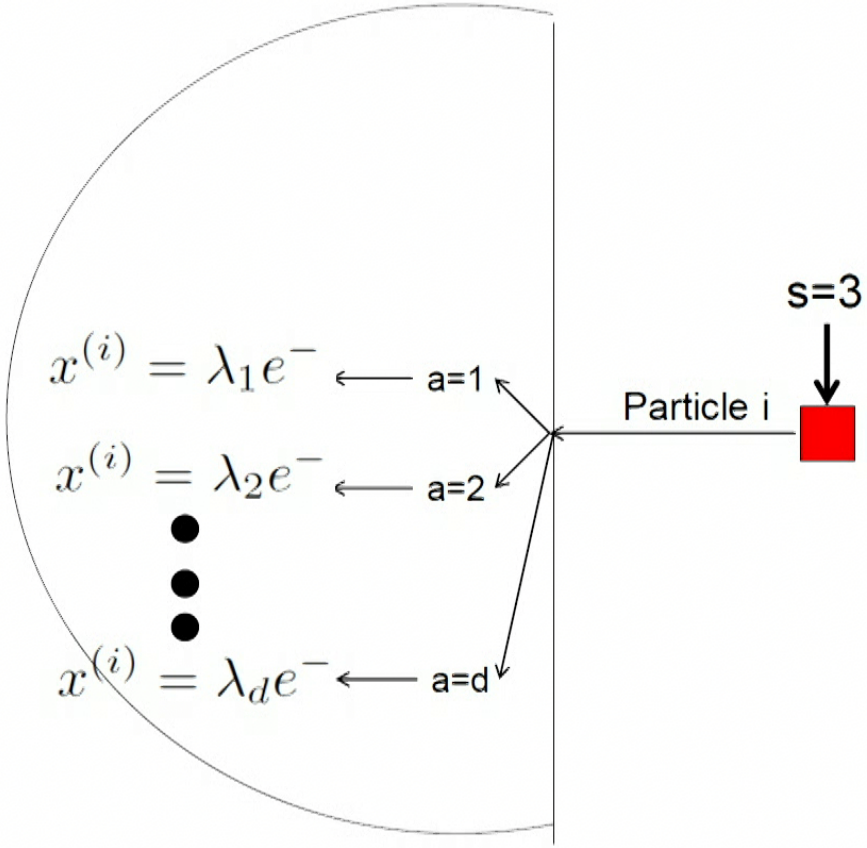
Alice's particle and Bob's particle can be of different nature



Effect of interactions

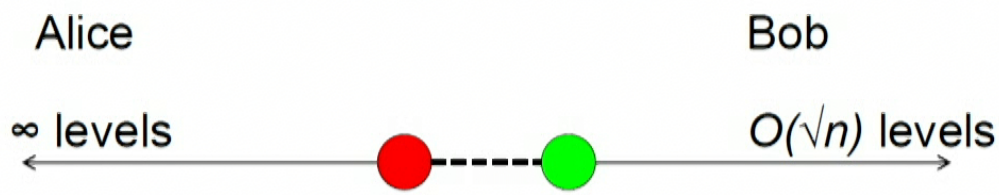


Effect of interactions



Effect of interactions

There exists distributions $P_Q(X,Y|s,t)$, with s,t=1,2,...,n which cannot be approximated by classical systems of the form



$$d_Q < \infty \not\to d_C < \infty$$

Conclusion

The analysis of macroscopic data provides scale-independent information about the structure of underlying microscopic discrete systems

Although this work is focused on foundational physics, we hope that some of these results will find an application in computer modelling, signal analysis, or even biology and sociology.

There exist quantum microscopic two-level systems which generate macroscopic correlations impossible to reproduce by any finite-dimensional classical model

THE END