Title: Hamiltonian and Lagrangian perspectives on elliptic cohomology

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Abstract: <p><span style="font-size:9.5pt;font-family:"Times New Roman",serif;mso-fareast-font-family:
Calibri;mso-fareast-theme-font:minor-latin;mso-ansi-language:EN-CA;mso-fareast-language:
EN-CA;mso-bidi-language:AR-SA">The physics proof of the Atiyah-Singer index theorem relates the Hamiltonian and Lagrangian approaches to quantization of N=1 supersymmetric mechanics. Similar ideas applied to the N=(0,1) supersymmetric sigma model construct two versions of elliptic cohomology: elliptic cohomology at the Tate curve over the integers and the universal elliptic cohomology theory over the complex numbers. Quantization procedures give analytic constructions of wrong-way maps in these cohomology theories. Relating these to the Ando-Hopkins-Strickland-Rezk string orientation of topological modular points to intricate torsion invariants associated with these sigma models. </p>

Elliptic Cohomology from the Lagrangian & Hamiltonian perspectives

M smooth mfld. Two field theories related to homotopy theory:

① $N=1$ SUSY mechanics

$$\begin{tikzcd}
\left\{ \begin{array}{l} \phi : \mathbb{R}^{1|1} \longrightarrow M \\ S(\phi) = \int_{\mathbb{R}^{1|1}} \langle \partial_t \phi, D\phi \rangle \end{array} \right\} \arrow[rr, "\substack{M \text{ spin} \\ \text{Hamiltonian/ canonical}}", "\text{quantization}"'] \arrow[dr, "\substack{\text{Lagrangian/} \\ \text{path integral} \\ \text{partition f'n}}"', "M \text{ oriented}"'] \arrow[rr, phantom, shift right=6ex, "\circlearrowleft"] & & \left\{ \begin{array}{l} \Gamma(\$)_\sigma \\ e^{-t\not{D}^2 + \theta \not{D}} \end{array} \right\} \arrow[dl, "\mathrm{sTr}(e^{-t\not{D}^2})"] \\
& \mathbb{C} &
\end{tikzcd}$$

compatibility is the following.

Thm [Atiyah-Singer]

$$\mathrm{sTr}(e^{-t\not{D}^2}) = \mathrm{Ind}(\not{D}) = \int \hat{A}(M)$$

(2) $N=(0,1)$ $d=2$ sigma model

$w_1(M) = w_2(M) = 0$

$\Updownarrow$

M Spin

"spinor bundle on loop space near constant loops"

$$
\begin{tikzcd}
\left\{ \begin{array}{l} \phi : \mathbb{R}^{2|1} \to M \\ S(\phi) = \int_{\mathbb{R}^{2|1}} \langle \partial_z \phi, D\phi \rangle \end{array} \right\} \arrow[r, "\text{Hamiltonian}", "\text{version of quantization}"'] \arrow[d, "\substack{\text{path integral} \\ \text{for partition f'n.}}"'] & \left\{ \begin{array}{c} \Gamma(\$_{LM}) \\ \circlearrowleft \\ q^{L_0} \bar{q}^{\bar{L}_0} + \theta G_0 \end{array} \right\} \arrow[d, "sTr(q^{L_0} \bar{q}^{\bar{L}_0})"] \\
MF \arrow[r, "q\text{-expansion}"] & \mathbb{C}((q))
\end{tikzcd}
$$

$p_1(M) = 0$
(rational string structure)

MF ↖ (weakly) modular form.

Thm [Witten-Zagier]

The square commutes.

$$sTr(q^{L_0} \bar{q}^{\bar{L}_0}) = \int Wit(M) \simeq \int \exp\left(\sum_{k \geq 2} E_{2k} \, ph_k(M) \right)$$

Thm [Witten–Zagier]

The square commutes.

$$\mathrm{sTr}\left(q^{L_0}\,\bar{q}^{\bar{L}_0}\right) = \int \mathrm{Wit}(M) \simeq \int \exp\Big(\sum_{k\geq 1} E_{2k}\, ph_k(M)\Big)$$

$$
\begin{tikzcd}
\left(S(\phi) = \int_{\mathbb{R}^{1|1}} \langle \partial_t \phi, D\phi \rangle \right) \arrow[r, "\text{quantization}"] \arrow[dr, "\substack{\text{Lagrangian/} \\ \text{path integral} \\ \text{partition f/n}}"'] & \left\{ e^{-t\cancel{D}^2 + \theta \cancel{D}} \right\} \arrow[d, "\mathrm{sTr}\left(e^{-t\cancel{D}^2}\right)"] \\
& \mathbb{C}
\end{tikzcd}
$$

M oriented

homotopical rephasing:

$$
\begin{tikzcd}
\left\{ \begin{matrix} \text{Spin} \\ \text{mflds} \end{matrix} \right\} \arrow[r, "\text{Index}"] \arrow[dr, "\int \hat{A}(-)"'] & \mathbb{Z} \cong K(pt) \arrow[d, "ch"] \\
& \mathbb{C} \cong H^{ev}_{\mathbb{C}}(pt)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left\{ \begin{array}{c} \phi : \mathbb{R}^{2|1} \to M \\ S(\phi) = \int_{\mathbb{R}^{2|1}} \langle \partial_z \phi, D\phi \rangle \end{array} \right\} \arrow[rr, "\text{Hamiltonian}"', "\text{version of quantization}"] \arrow[d, "\substack{\text{path integral}\\ \text{for partition f'n.}}"'] & & \left\{ \begin{array}{c} \mathcal{H}(\mathcal{S}_{LM}) \\ \circlearrowleft \\ q^{L_0}\bar{q}^{\bar{L}_0} + \theta G_0 \end{array} \right\} \arrow[d, "sTr(q^{L_0}\bar{q}^{\bar{L}_0})"] \\
MF \arrow[rr, "q\text{-expansion}"] & & \mathbb{C}((q)) \\
\{\text{string mflds}\} \arrow[rr] \arrow[d, "\int wit(-)"'] & & \mathbb{Z}((q)) \cong K(pt)((q)) \arrow[d] \\
MF \cong H_{MF}(pt) \arrow[rr] & & \mathbb{C}((q)) \cong H_{\mathbb{C}}(pt)((q))
\end{tikzcd}
$$

$\uparrow$ (weakly) modular form.

$p_1(M) = 0$

(rational string structure)

The square commutes.

$sTr(q^{L_0}\bar{q}^{\bar{L}_0}) = \int$

For K-theory, we have a family version of this story:

<u>Thm</u> [Bismut, Atiyah-Singer] $M \to X$ family of even dim'l spin mflds,

$$\Big[\int \hat{A}(M/X)\Big] = \Big[ch(\underbrace{\mathrm{Ind}(\not{D}_{M/X})}_{\text{"index bundle"}})\Big] \in H^{ev}_c(X)$$

Equivalently:

$$\begin{tikzcd} Mspin(X) \arrow[rr, "\mathrm{Ind}"] \arrow[dr, "\int \hat{A}(*/X)"'] & \arrow[d, phantom, "\circlearrowleft"] & K(X) \arrow[dl, "ch"] \\ & H^{ev}_c(X) & \end{tikzcd}$$

(one) translation into SUSY mechanics:

<u>Observation</u>: $e^{-t\not{D}^2 + \theta\not{D}}$ is a super semigroup representation of the category of super paths

(t,θ) (t',θ')

<u>Obj</u> $= *$

<u>Mor</u> $= \mathbb{R}^{1|1}_{\geq 0}$

There is a category of constant super paths in M, $sP_0(M)$

<u>Prop</u>: $\mathrm{Rep}^{f.d.}(sP_0(M)) \simeq \left\{ \begin{matrix} (V,\mathbb{A}) \text{ super v.b.} \\ \text{w/ super conn} \end{matrix} \right\}$

$$\begin{matrix} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ e^{-t\mathbb{A}^2+\theta\mathbb{A}} \in \Omega(M;V) & \longleftarrow & (V,\mathbb{A}) \end{matrix}$$

Gives us:

$$\begin{tikzcd}
\left\{ \begin{matrix} X\text{-family of} \\ \text{spin mflds} \end{matrix} \right\} \arrow[rr, "\substack{\text{Hamiltonian} \\ \text{quantization}}", "+\text{ choose cutoff}"'] \arrow[d, "\substack{\text{fiberwise} \\ \text{path integral} \\ \text{partitu f'n}}"'] \arrow[dr, phantom, "\Searrow \text{homotopy}"] & & \mathrm{Rep}(s\,P_0(M)) \arrow[dll, "\text{char}"] \\
\Omega^{ev}_{cl}(X) \cong C^\infty(\mathcal{L}_0^{1|1}(X)) & &
\end{tikzcd}$$

homotopy depends on the cut off;
can be written as an integral
over high energy states

2d version: Have a diagram:

X-family of string mflds $\longrightarrow$

$$\begin{tikzcd}
MString(X) \arrow[r] \arrow[rrr, bend left] \arrow[dr, bend right] & TMF(X) \arrow[dr, dashed] \arrow[rr] \arrow[d] & & K(X)((q)) \arrow[dd, "ch"] \\
& H_{MF}(X) & K_{MF}(X) \arrow[ur] \arrow[l] & \\
& H_{MF}(X) \arrow[rr, "q\text{-expansion}"'] & & H_{\mathbb{C}}(X)((q))
\end{tikzcd}$$

$K_{MF} :=$ htpy pullback.

There is a
super

Prop: B

e^{-t}

2d version: Have a diagram:

$$\begin{tikzcd}
X\text{-family of string mflds} \arrow[r, bend left] & MString(X) \arrow[dr] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& & TMF(X) \arrow[d, dashed] \arrow[rr] \arrow[dd, bend right] & & K(X)((q)) \arrow[dd, "ch"] \\
& & K_{MF}(X) \arrow[rru] \arrow[d] & & \\
& & H_{MF}(X) \arrow[rr, "q\text{-expansion}"'] & & H_{\mathbb{C}}(X)((q))
\end{tikzcd}$$

$K_{MF} :=$ htpy pullback

Point of K_{MF} is a c.hom. that combines $K(-)((q))$ & $H_{MF}(-)$ in the simplest way possible to analyze the pair of quantization maps

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Translation to field theories:

Observation: $q^{L_0}\bar{q}^{\bar{L}_0 + \theta G_0}$ is a representation of a cat. of super annuli

Can build a cat $sAnn(X)$ of constant super annuli in X.

<u>Thm</u> $PER(sAnn_0(X)) \simeq \left\{ \bigoplus_{k>-N} (V_k, A_k) \text{ sequences of super v.b w/ super conn on } X \right\}$

$\bigoplus_k q^k e^{-2im(\tau) A_k^2 + \theta A_k}$

:

a representation

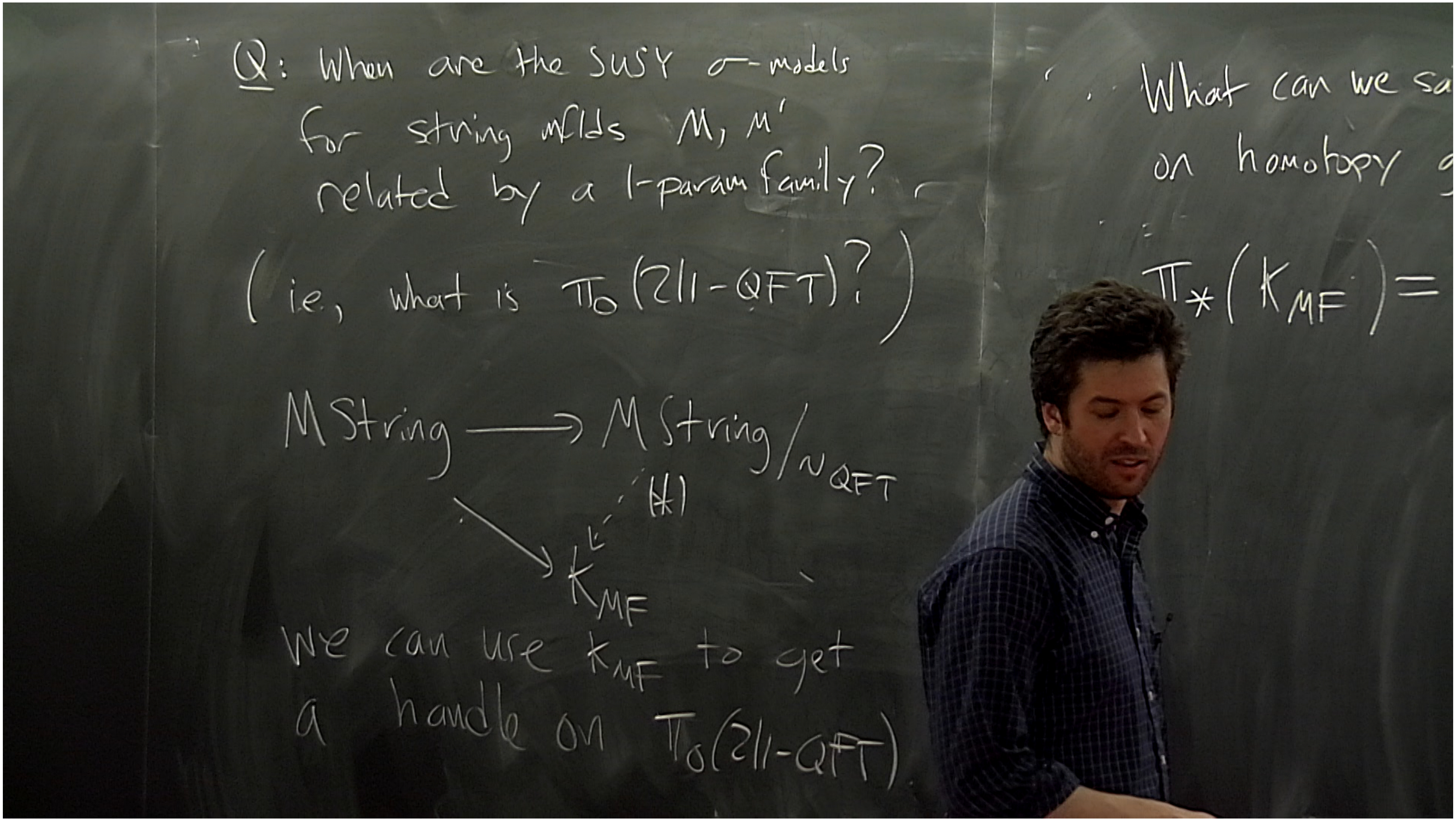
annuli

$$
\begin{tikzcd}
\left\{\begin{matrix} X\text{-family of} \\ \text{string mflds} \end{matrix}\right\} \arrow[rr, "\substack{\text{Hamiltonian}\\ \text{quantization}}", "+\text{ choose cutoffs}"'] \arrow[d, "\substack{\text{path integral}\\ \text{for partition}\\ \text{f'n}}"'] & & PER(sAnn_0(M)) \arrow[d, "char"] \\
\Omega(X, MF) \cong \mathcal{O}(\mathcal{L}^{2|1}_0(X)) \arrow[rr] \arrow[urr, phantom, "\Nearrow \text{homotopy}"] & & \mathcal{O}(\widetilde{\mathcal{L}}^{2|1}_0(X)) \cong \Omega(X)((q))
\end{tikzcd}
$$

homotopy depends on cutoffs <u>and</u> a choice of rational string str. on the family.

<u>Explicitly</u>: $M \to X$, $p_1(T(M/X)) = dH$

$H \in \Omega^3(M)$ is the rational string str.


Q: When are the SUSY σ-models
for string mflds M, M'
related by a 1-param family?
(i.e., what is π_0(2|1-QFT)?)
MString → MString/$\sim_{QFT}$
(*)
K_{MF}
We can use K_{MF} to get
a handle on π_0(2|1-QFT)
What can we sa
on homotopy g
$\pi_*(K_{MF}) =$

What can we say about (*) on homotopy groups?

$$\pi_*(K_{MF}) = \begin{cases} MF^{\mathbb{Z}} & * = \text{even} \\ \mathbb{C}((q)) / \mathbb{Z}((q)) + MF & * = \text{odd} \end{cases}$$

integral modular forms

different partition f'ns ⇒ 2|1-Euc field theories can't be deformed into each other

choosing trivializations of anomalies

<u>Fact 1:</u> $M\mathrm{String} \to TMF \to K_{MF}$

& $\pi_{odd} TMF$ finite and known

<u>Fact 2:</u> <u>Thm</u> [Bunke-Naumann]

There is an injection

$\pi_{4k-1} TMF \to \pi_{4k-1} KO_{MF}$ on 3-torsion

<u>Conj</u> [Segal, Stolz-Teichner]

∃ htpy equiv

$2|1\text{-}QFT \simeq TMF$