Title: W infinity - Higher spins and integrability

Date: Mar 31, 2017 11:00 AM

URL: http://pirsa.org/17030095

Abstract: <p>I will discuss the chiral algebra W_infty which is obtained from the Virasoro algebra by adding fields of spin 3, 4, ... Via a non-local non-linear map one can show that it is equivalent to Tsymbaliuk's Yangian of affine u(1). In this way we find an infinite number of commuting conserved charges. Diagonalizing these, the representation theory reduces to combinatorial study of plane partitions, 3-dimensional generalization of the Young diagrams. Tsymbaliuk's presentation can be derived from RTT relations using Maulik-Okounkov's free boson R-matrix.</p>

W algebras

$\mathcal{W}$-algebas: extensions of the Virasoro algebra by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV, KP)
- (old) matrix models
- instanton partition functions and AGT
- holographic dual description of 3d higher spin theories
- quantum Hall effect
- topological strings (toric CY)
- higher spin square (Gaberdiel, Gopakumar)
- combinatorics: plane partitions ($\equiv$ 3d Young diagrams)

Zamolodchikov $\mathcal{W}_3$ algebra

As in illustration, the $\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

We need to find the OPE of W with itself such that the resulting chiral algebra is 'associative'.

The result:

$$
\begin{aligned}
W(z)W(w) \;\sim\; & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
& + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) \\
& + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right)
\end{aligned}
$$

Λ is a quasiprimary 'composite' (spin 4) field,

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).$$

The algebra is non-linear, not a Lie algebra in the usual sense (in fact linearity should not be expected for spins ≥ 3).

Construction of $\mathcal{W}_N$-algebras

We will be interested in specific class of algebras $\mathcal{W}_N$, which have generators of spin $2, 3, \ldots, N$. These algebras are naturally associated to Lie algebras $\mathfrak{sl}(N)$. We can construct them in various ways:

- from affine Lie algebras - Casimir algebra, coset construction, Drinfeld-Sokolov reduction ($\mathfrak{su}(N) \to \hat{\mathfrak{su}}(N) \to \mathcal{W}_N$)
- from free fields - Miura transformation (oper)
- classically ($c \to \infty$) Poisson brackets in algebra of pseudodifferential operators
- directly - bootstrap (CFT associativity conditions)

Digression: higher spin algebra hs(λ)

- so far we defined a family $\mathcal{W}_N$ with discrete parameter N; our main object of study in the following is the algebra $\mathcal{W}_\infty[\lambda]$ continuously interpolating between these
- there is an analogous construction of interpolating algebra in the case of $\mathfrak{gl}(N)/\mathrm{End}(V_N)$
- $\mathfrak{sl}(2) \subset \mathfrak{gl}(N)$ (principal) and decompose adjoint $\rightarrow$ spins $0, 1, \ldots, N-1$

$$T^{l_1}_{m_1} * T^{l_2}_{m_2} = \sum C^{l_1 l_2 l}_{m_1 m_2}(N)\, T^{l}_{m_1+m_2}$$

- the structure constants can be chosen to be rational in N
- we can thus take generators of spin $0, 1, \ldots, \infty$ and define a new associative algebra $hs(\lambda)$ using these rational functions as its structure constants (analytic continuation, $N \rightarrow \lambda$)
- specializing λ to an integer N (and truncating spins $\geq N$), we get back $\mathfrak{gl}(N)$

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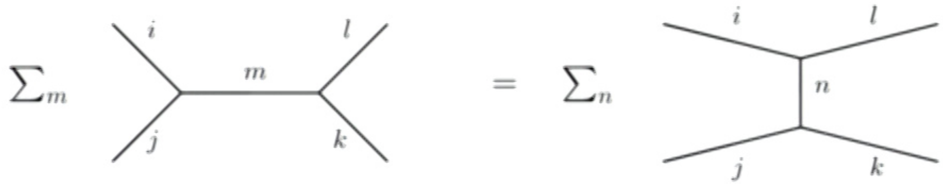
- explicit formula for multiplication using 3j and 6j symbols (Pope, Shen, Romans)
- there is also a one-line algebraic definition of hs(λ)

$$hs(\lambda) = \frac{\mathcal{U}(\mathfrak{sl}(2))}{(X^2 + Y^2 + Z^2 - (\lambda^2 - 1))}$$

- this definition also makes clear the geometric meaning of $hs(\lambda)$ as the quantum sphere S^2
- expansion at $\lambda = \infty$: $hs(\lambda)$ is deformation quantization ($\hbar \sim \frac{1}{\lambda}$) of algebra of functions on S^2
- truncation to $gl(N)$ for $\lambda \in \mathbb{N}$ goes under the name of fuzzy sphere (quantization - integral phase space volume)

$\mathcal{W}_\infty$ from bootstrap

- Gaberdiel-Gopakumar: use the associativity conditions to construct $\mathcal{W}_\infty$ (spins $2, 3, 4, \ldots$)



- equivalently (Jacobi identity for mode operators)

$$[[A_j, B_k], C_l] + [[B_k, C_l], A_j] + [[C_l, A_j], B_k] = 0$$

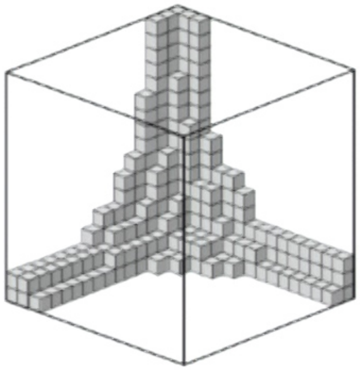
- two-parametric family of solutions: central charge c and rank parameter λ (Gaberdiel/Gopakumar)
- choosing $\lambda = N \to$ truncation of $\mathcal{W}_\infty$ to $\mathcal{W}_N$, so $\mathcal{W}_\infty$ is interpolating algebra for $\mathcal{W}_N$ series (cf. $\mathfrak{gl}(N)$ vs $hs(\lambda)$)
- working in basis of fields from Miura transformation, one can find explicit expressions for OPE (quadratic nonlinearity)

The resulting OPEs are very complicated

$$
\begin{aligned}
U_3(z)U_5(w) \quad \sim \quad & \frac{1}{(z-w)^7}\Big(\frac{1}{2}\alpha(n-3)(n-2)(n-1)n\left(4\alpha^2\left(\alpha^2(n(5n-9)+1)-3n+4\right)+1\right)\mathbb{1}\Big) \\
& +\frac{1}{(z-w)^6}\Big(\frac{1}{6}(n-3)(n-2)(n-1)\left(6\alpha^4 n(2n-3)+\alpha^2(10-9n)+1\right)U_1(w)\Big) \\
& +\frac{1}{(z-w)^5}\Big(-\alpha(n-3)(n-2)(n-1)\left(-4\alpha^2+3\alpha^2 n-1\right)(U_1U_1)(w) \\
& +\alpha(n-3)(n-2)\left(4\alpha^2 n^2-4\alpha^2 n-n-2\right)U_2(w) \\
& -\frac{1}{2}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1'(w)\Big) \\
& +\frac{1}{(z-w)^4}\Big(-\alpha(n-3)(n-2)(n-1)\left(\alpha^2(3n-4)-1\right)(U_1'U_1)(w) \\
& -\frac{1}{2}(n-3)(n-2)\left(2\alpha^2(n-1)-1\right)(U_1U_2)(w) \\
& +(n-3)\left(\alpha^2\left(n^2+2\right)-3\right)U_3(w) \\
& -\frac{1}{4}\alpha^2(n-3)(n-2)(n-1)\left(4\alpha^2 n(2n-3)-3n+2\right)U_1''(w) \\
& +\alpha(n-3)(n-2)\left(\alpha^2(n-1)n-1\right)U_2'(w)\Big) \\
& +\cdots
\end{aligned}
$$

Topological vertex

- characters of other special $\mathcal{W}_{1+\infty}$ reps are counting plane partitions with specified Young diagram asymptotics



- level $\leftrightarrow$ number of boxes
- this is exactly the topological vertex of topological strings
- so the topological vertex $\leftrightarrow$ characters of $\mathcal{W}_{1+\infty}$ reps
- truncation to integer $\lambda \leftrightarrow$ restriction the height