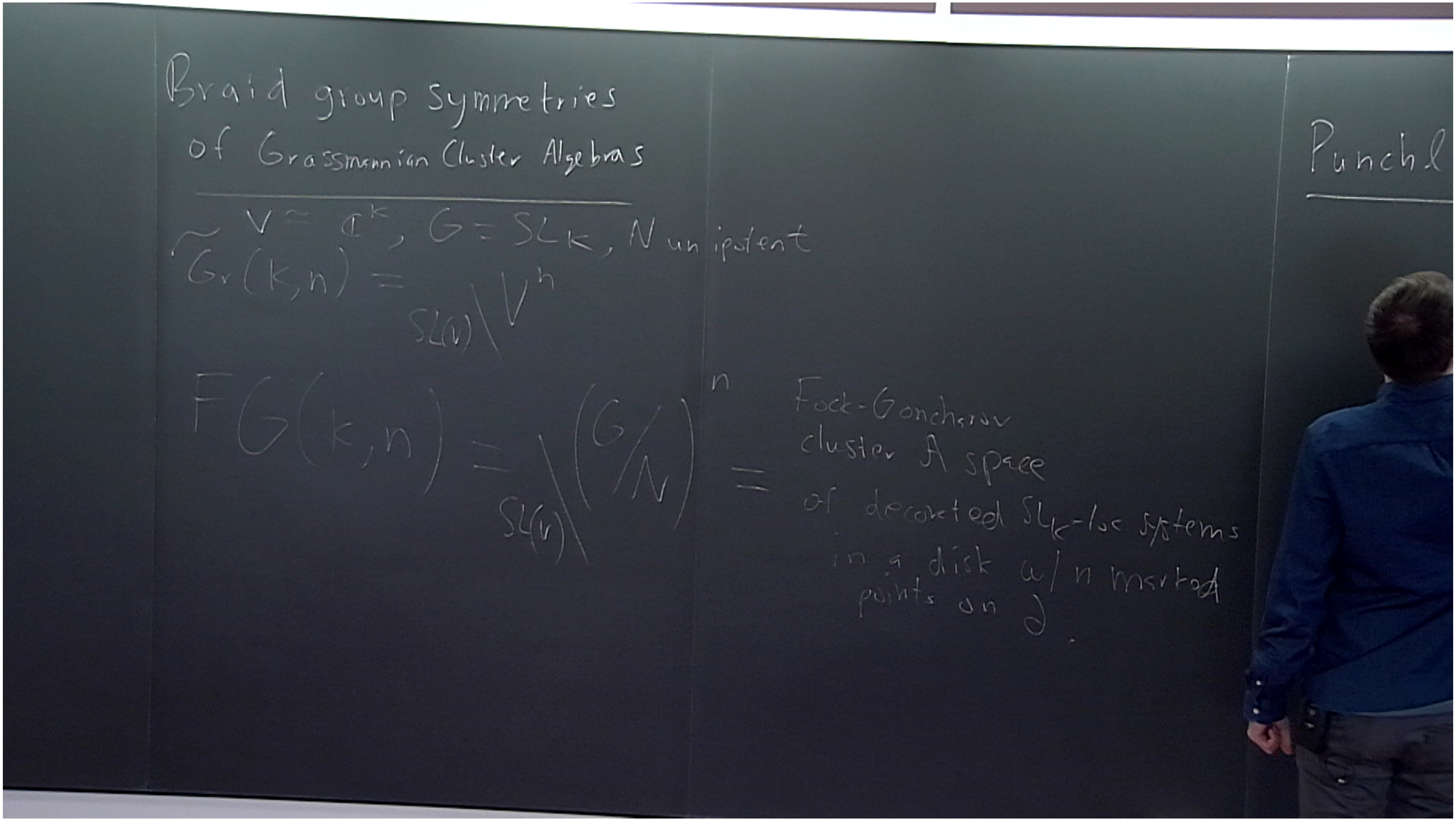
Title: Braid group symmetries of Grassmannian cluster algebras

Date: Mar 13, 2017 02:00 PM

URL: http://pirsa.org/17030081

Abstract: <p>We define an action of the k-strand braid group on the set of cluster variables for the Grassmannian Gr(k,n), provided k divides n. The action sends clusters to clusters, preserving the underlying quivers, defining a homomorphism from the braid group to the cluster modular group for Gr(k,n). Our results can be translated to statements about clusters in Fock-Goncharov configuration spaces of affine flags, provided the number of flags is even. Finally, we apply our results to the two "Grassmannians of finite mutation type," proving in these cases versions of conjectures made by Fomin and Pylyavskyy describing cluster variables as SL_k web invariants. </p>



Braid group symmetries of Grassmannian Cluster Algebras

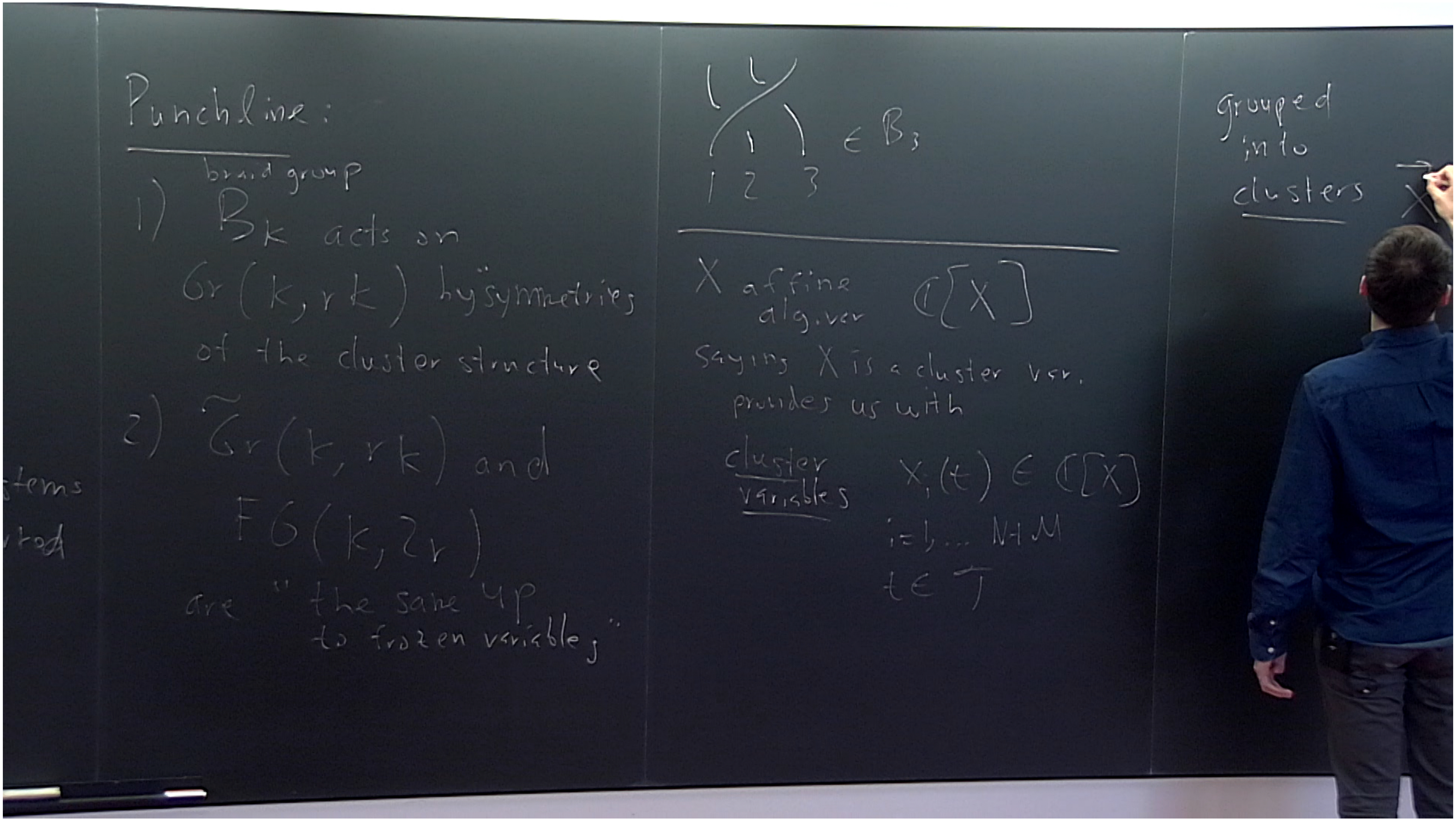
$V \simeq \mathbb{C}^k$, $G = SL_k$, N unipotent

$$\widetilde{Gr}(k,n) = {}_{SL(V)}\backslash V^n$$

$$FG(k,n) = {}_{SL(V)}\backslash \left(G/N \right)^n =$$

Fock-Goncharov cluster $\mathcal{A}$ space of decorated SL_k-loc systems in a disk w/ n marked points on ∂.

Punchl



stems

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Punchline:

1) B_k acts on
braid group
$Gr(k, rk)$ by "symmetries"
of the cluster structure

2) $\widetilde{Gr}(k, rk)$ and
$FG(k, 2r)$
are "the same up
to frozen variables"

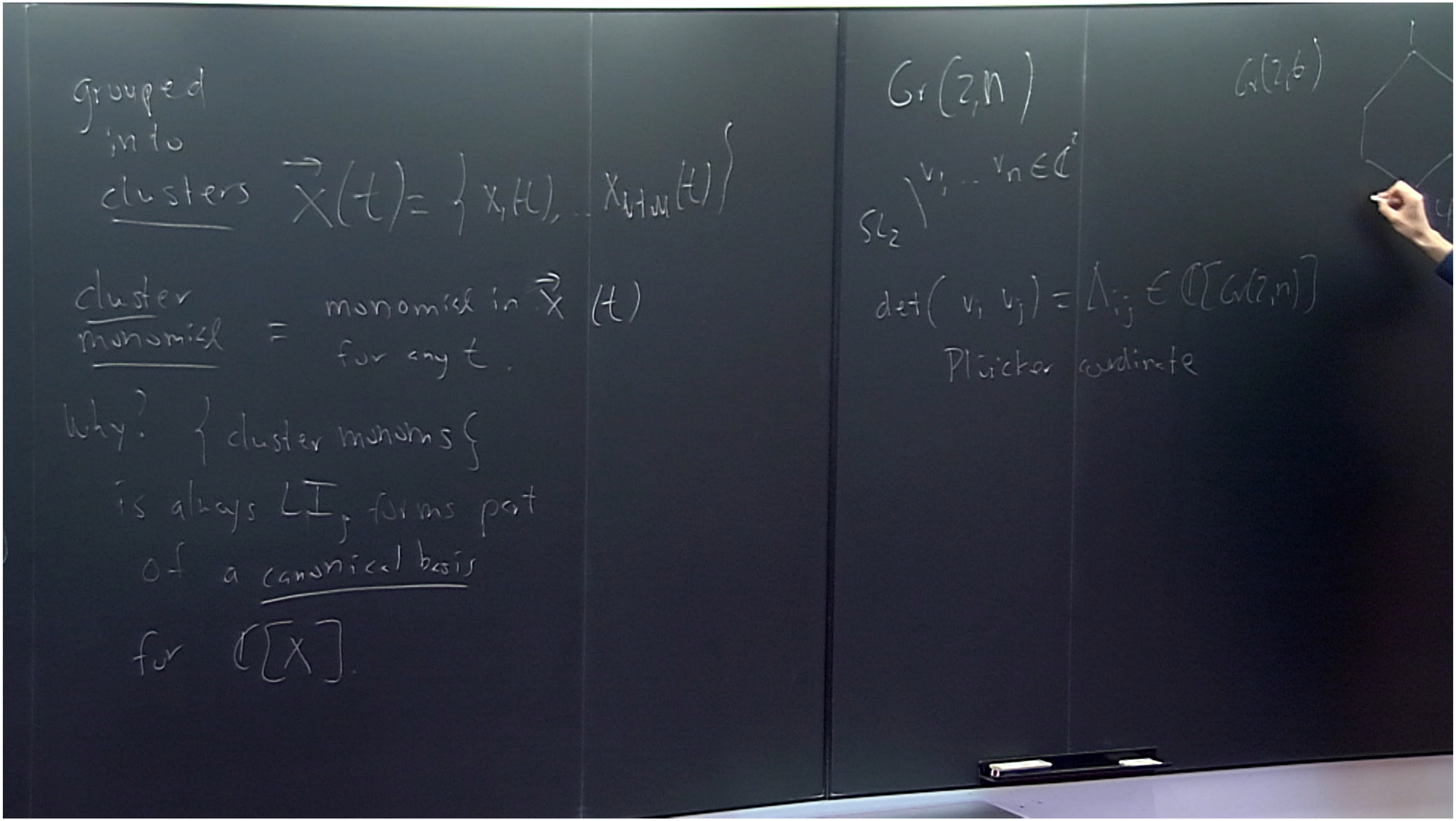
$\in B_3$
1 2 3

X affine
alg. var
$\mathbb{C}[X]$

Saying X is a cluster var.
provides us with

cluster variables
$x_i(t) \in \mathbb{C}[X]$
$i = 1, \dots N+M$
$t \in T$

grouped
into
clusters
$\vec{X}$



grouped into clusters $\vec{X}(t) = \{ x_1(t), \ldots x_{N+M}(t) \}$

cluster monomial $=$ monomial in $\vec{X}(t)$ for any t.

Why? $\{$cluster monoms$\}$ is always LI, forms part of a canonical basis for $\mathbb{C}[X]$.

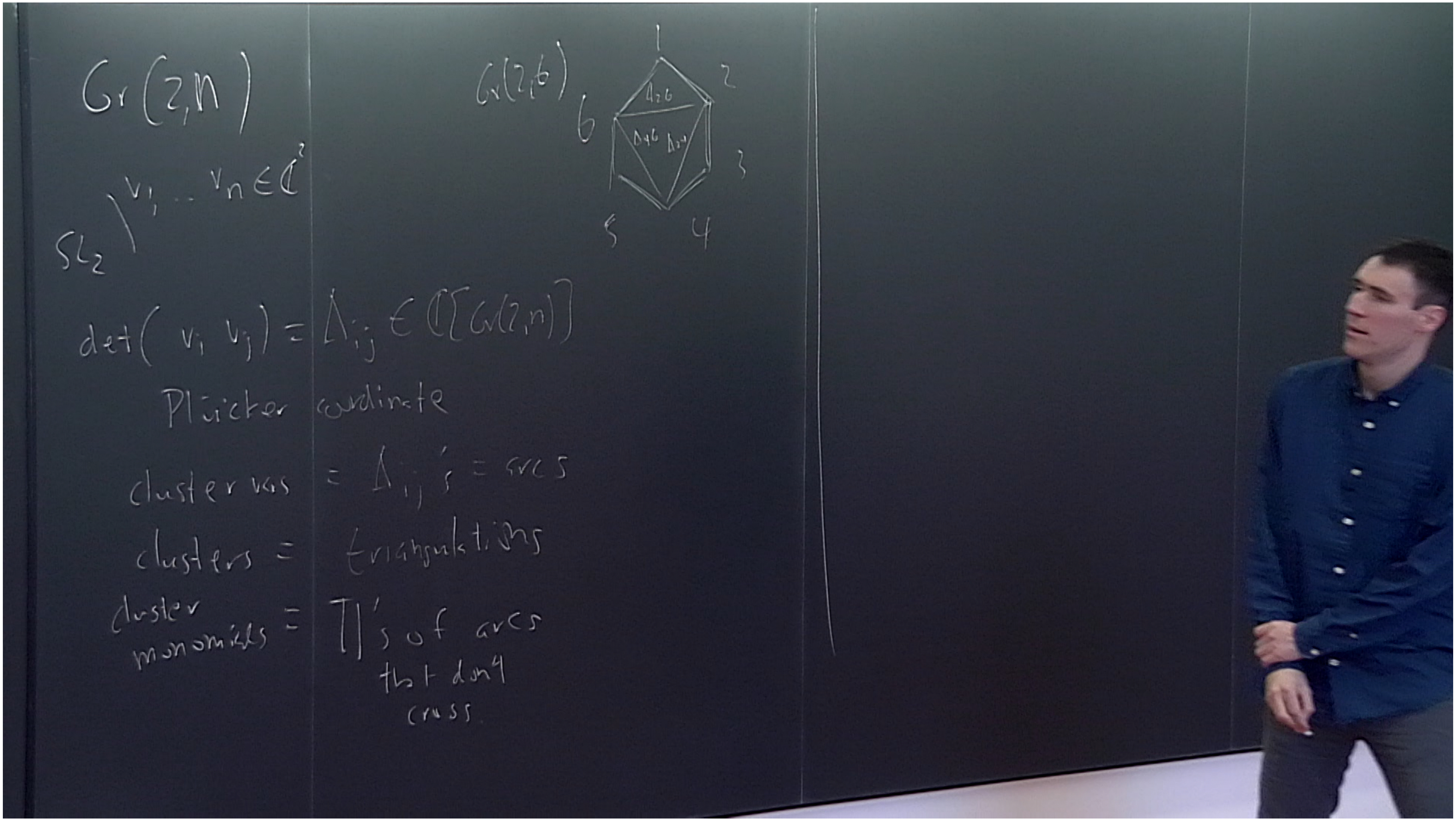
$Gr(2,n)$

$SL_2 \backslash v_1, \ldots v_n \in \mathbb{C}^2$

$\det(v_i\ v_j) = \Delta_{ij} \in \mathbb{C}[Gr(2,n)]$

Plücker coordinate

$Gr(2,6)$



$Gr(2,n)$

$SL_2 \backslash v_1 \ldots v_n \in \mathbb{C}^2$

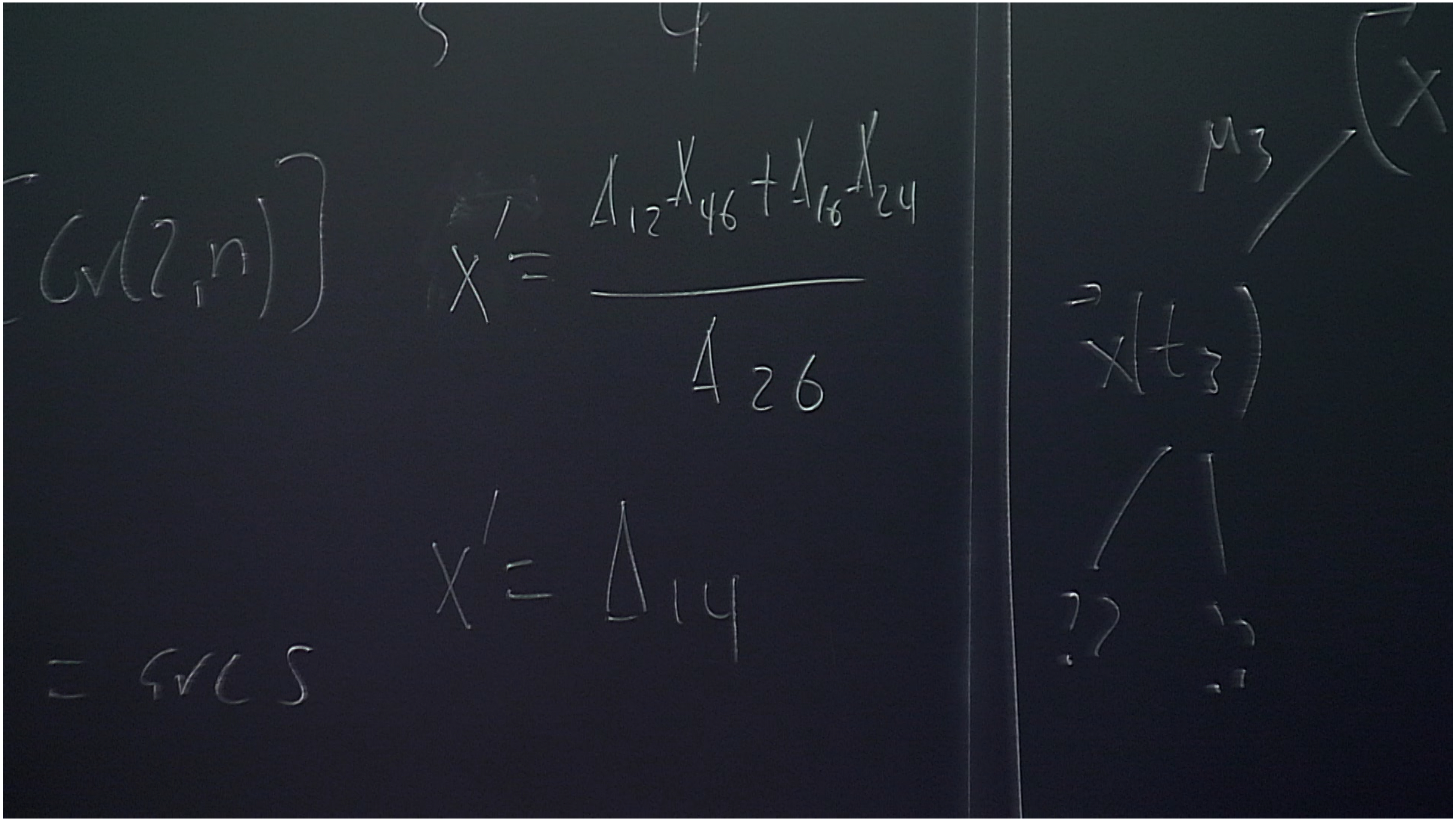
$\det(v_i\ v_j) = \Delta_{ij} \in \mathbb{C}[Gr(2,n)]$

Plücker coordinate

cluster vars $= \Delta_{ij}$'s $=$ arcs

clusters $=$ triangulations

cluster monomials $= \prod$'s of arcs that don't cross.



$[Gr(2,n)]$

$$x' = \frac{\Delta_{12}\Delta_{46} + \Delta_{16}\Delta_{24}}{\Delta_{26}}$$

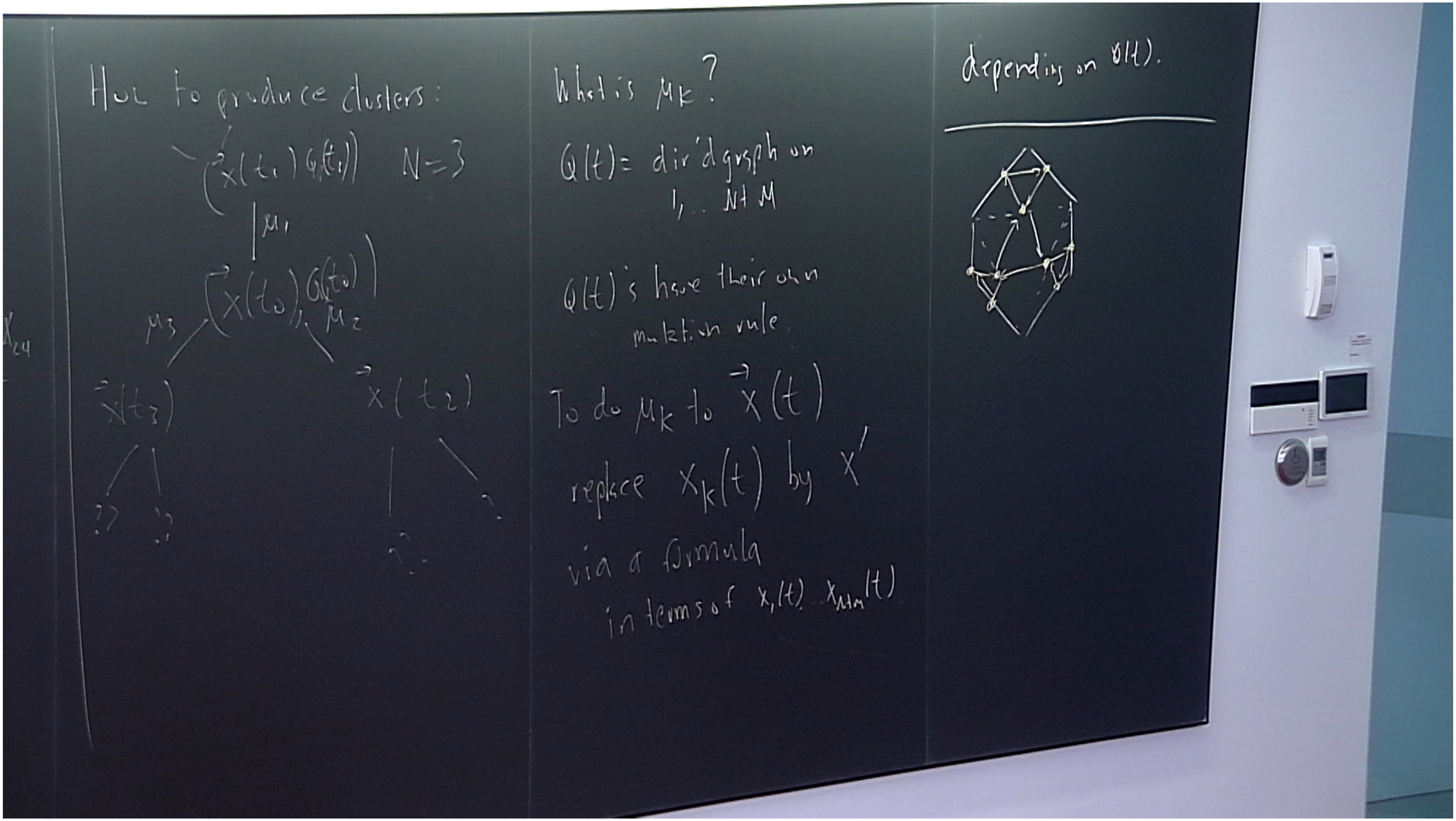
$$x' = \Delta_{14}$$

$= GrCS$

μ_3

$\Rightarrow x(t_3)$

??



X_{24}

How to produce clusters:

$(\vec{x}(t_1), Q(t_1))$ $N=3$

μ_1

$(\vec{x}(t_0), Q(t_0))$

μ_3 μ_2

$\vec{x}(t_3)$ $\vec{x}(t_2)$

?? ?? ?? ??

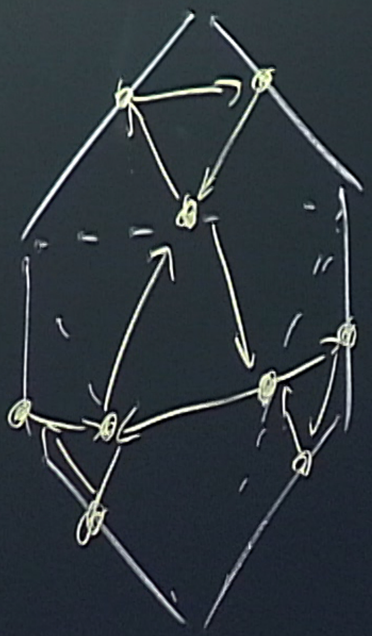
What is μ_k?

$Q(t)$ = dir'd graph on $1, .. N+M$

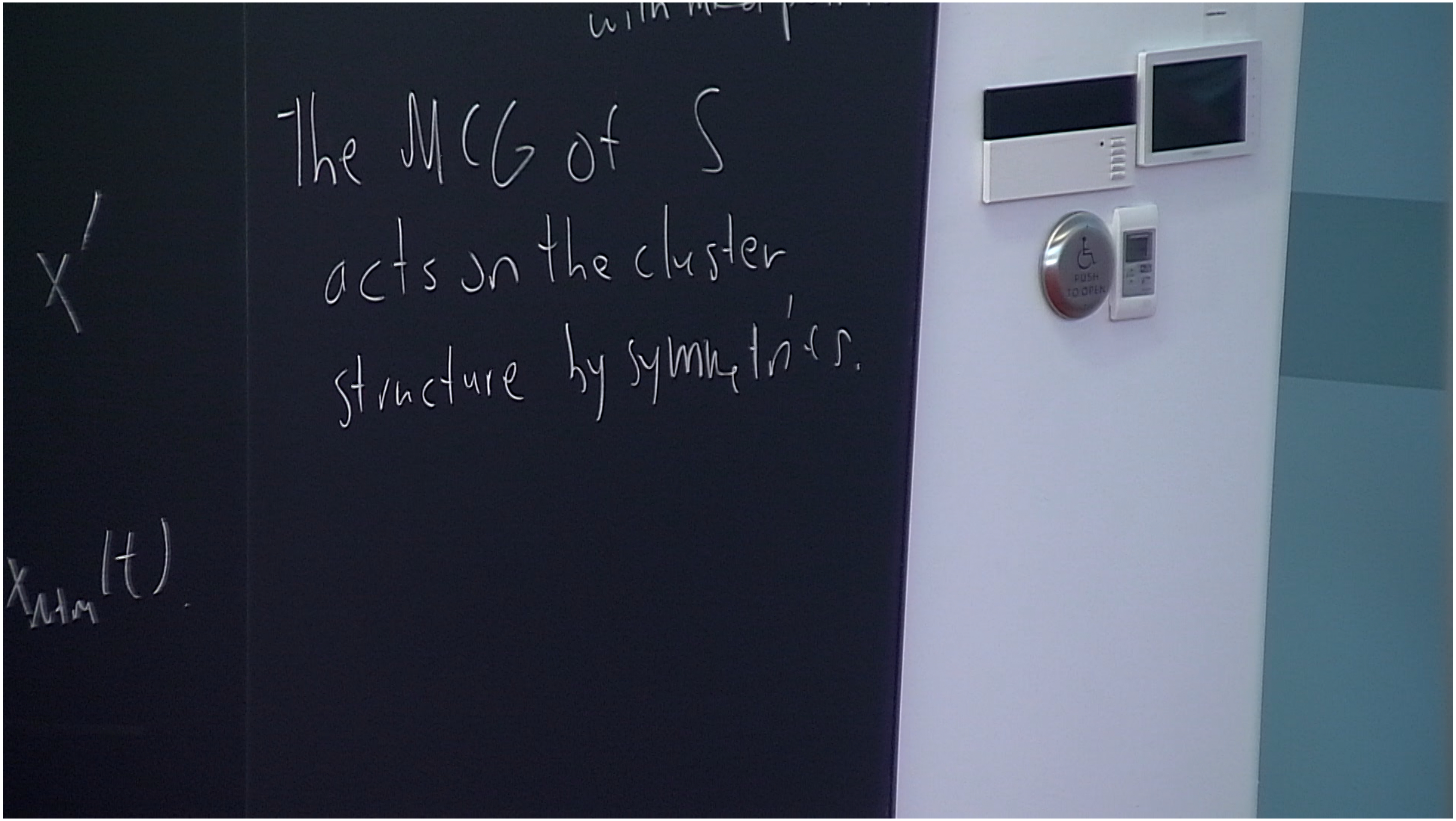
$Q(t)$'s have their own mutation rule.

To do μ_k to $\vec{x}(t)$ replace $x_k(t)$ by x' via a formula in terms of $x_1(t) .. x_{N+M}(t)$

depending on $Q(t)$.



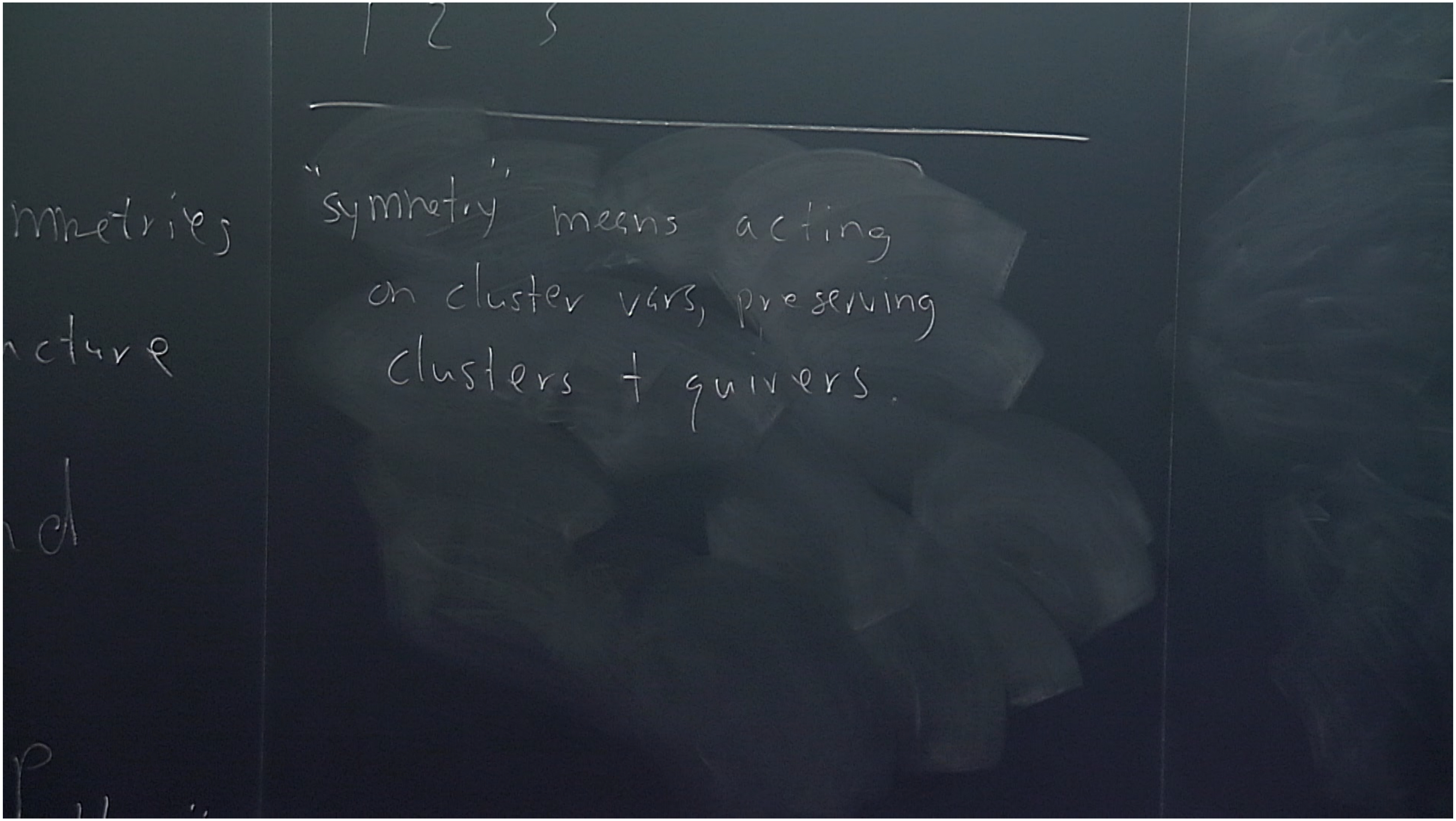
This can be generalized to a surface S w/ ∂ and with mkd points.



X'

$X_{n+m}(t)$.

The MCG of S acts on the cluster structure by symmetries.



1 2 3

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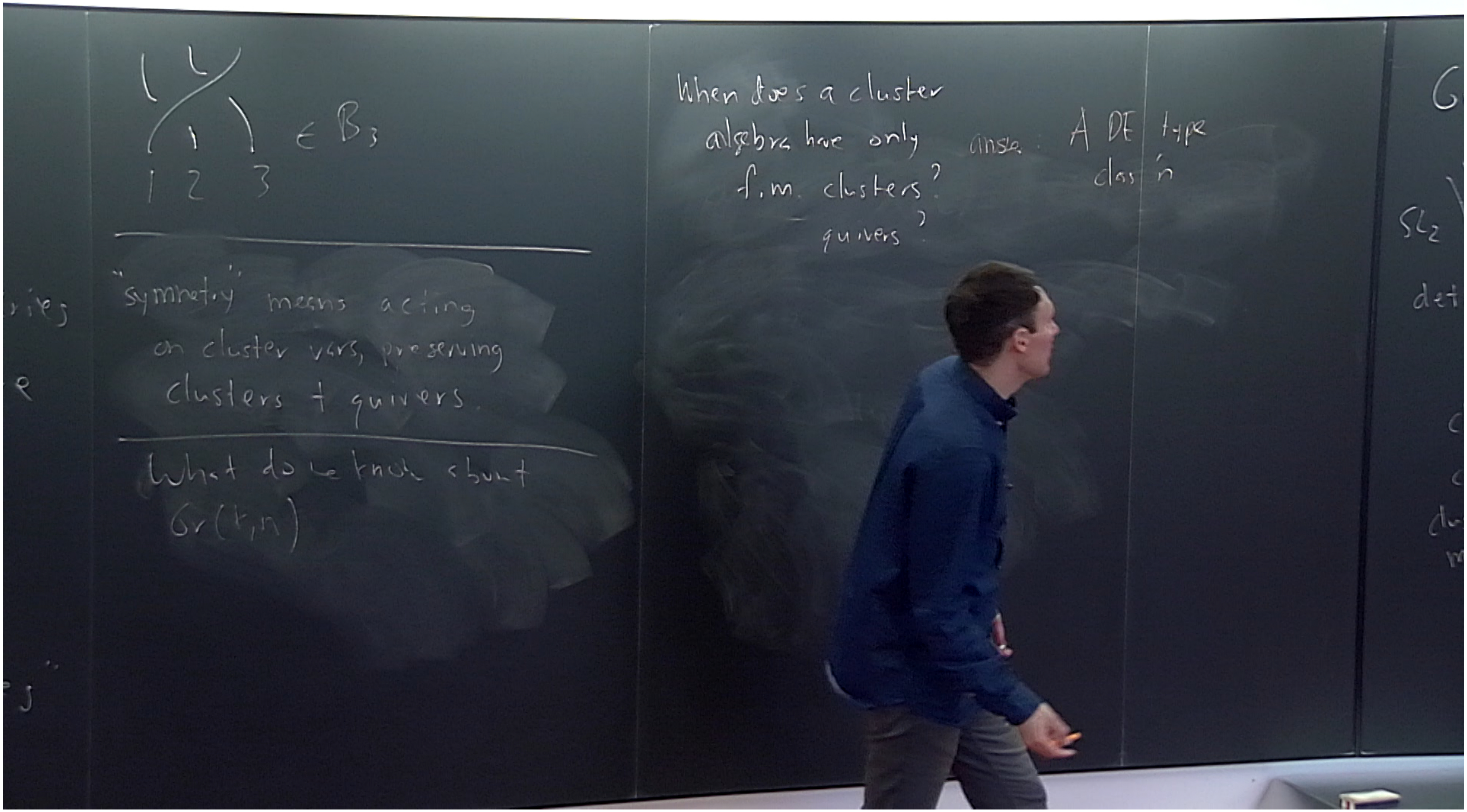
"symmetry" means acting on cluster vars, preserving clusters + quivers.



"symmetry" means acting on cluster vars, preserving clusters + quivers.

What do we know about Gr(k,n)

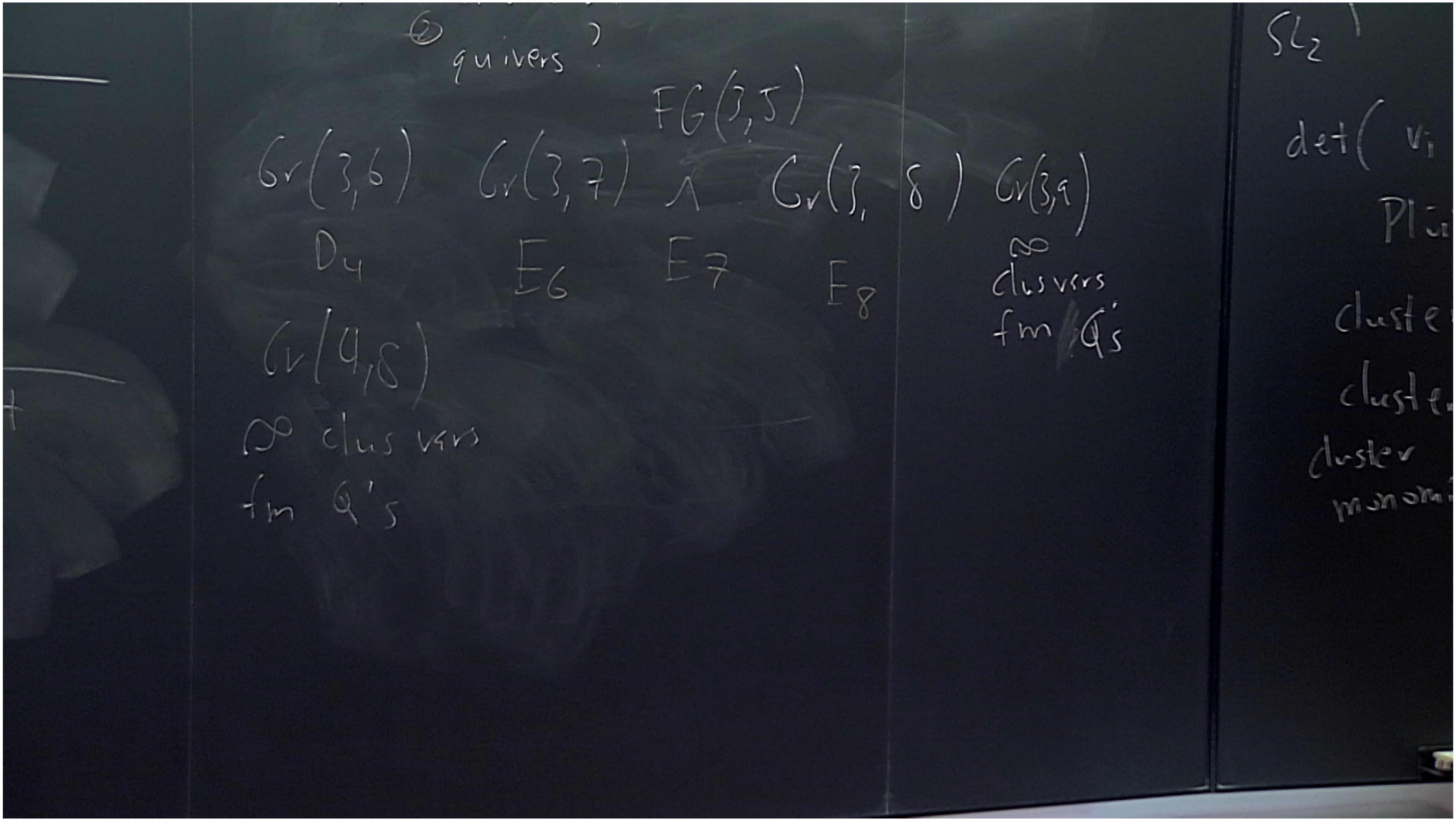
When does a cluster algebra have only f.m. clusters? quivers?

answer: ADE type class'n

$n \in \mathbb{C}^2$

$Gr(2,n)$

A_{n-3}



② quivers ?

$FG(3,5)$

$Gr(3,6)$ $Gr(3,7)$ ∧ $Gr(3,8)$ $Gr(3,9)$

D_4 E_6 E_7 E_8 ∞ clus vars fm Q's

$Gr(4,8)$

∞ clus vars fm Q's

SL_2

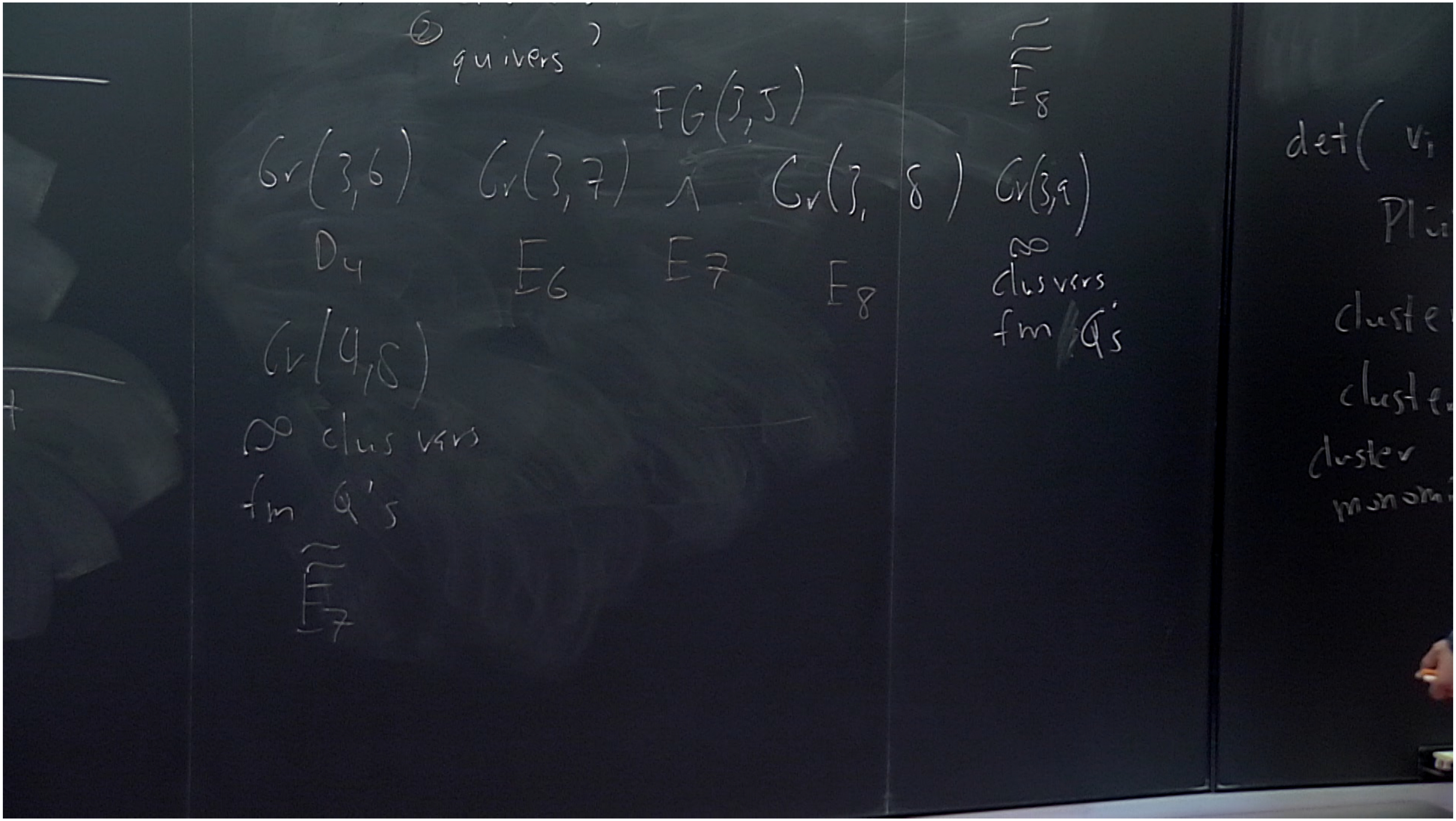
$\det(v_i$

Pl

cluste

cluste

cluster monomi



② quivers ?

FG(3,5)

$\tilde{\tilde{E}}_8$

Gr(3,6) Gr(3,7) $\wedge$ Gr(3, 8) Gr(3,9)

D_4 E_6 E_7 E_8

∞ clus vars fm Q's

Gr(4,8)

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$\tilde{\tilde{E}}_7$

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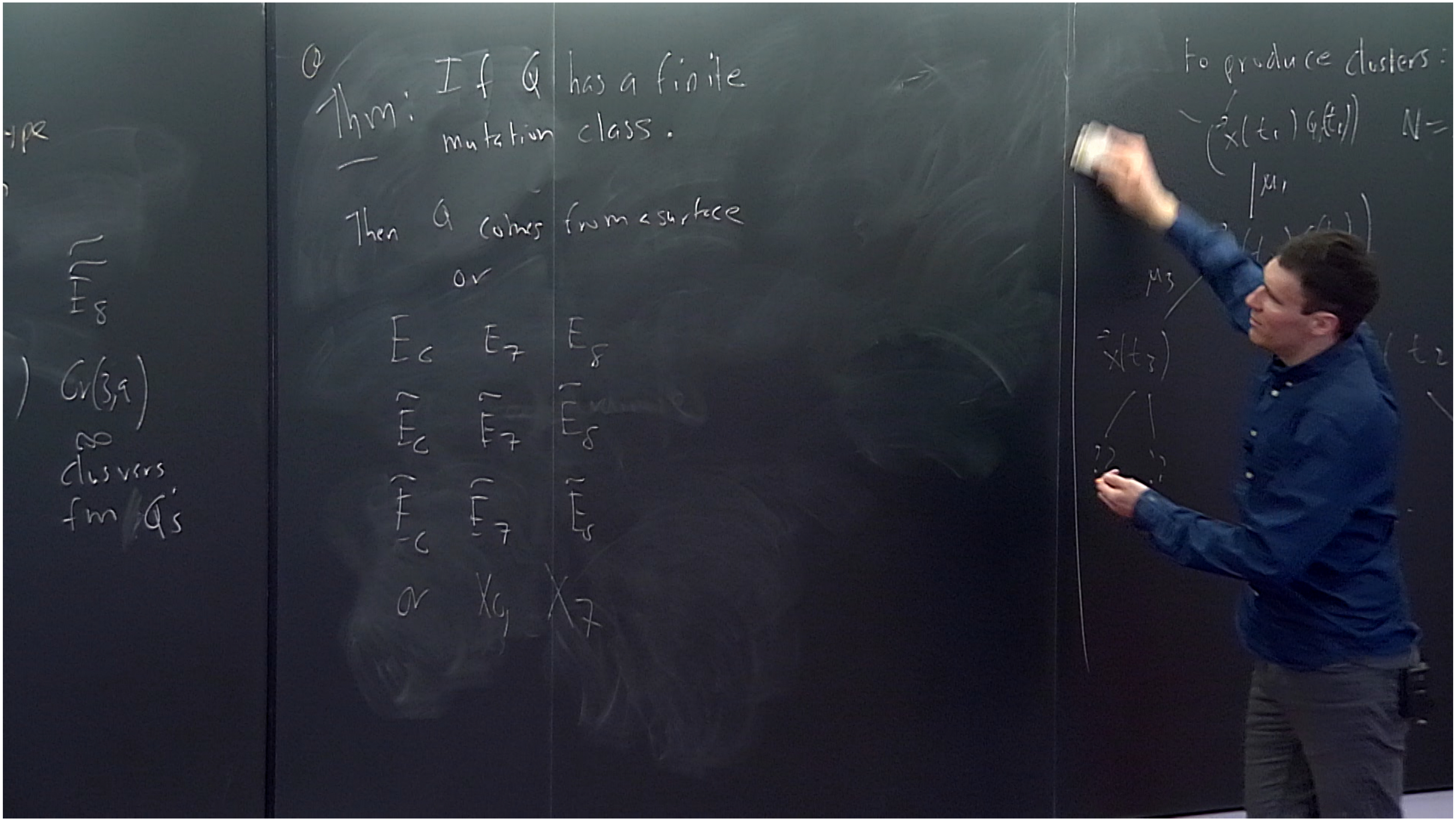


Thm: If Q has a finite mutation class.

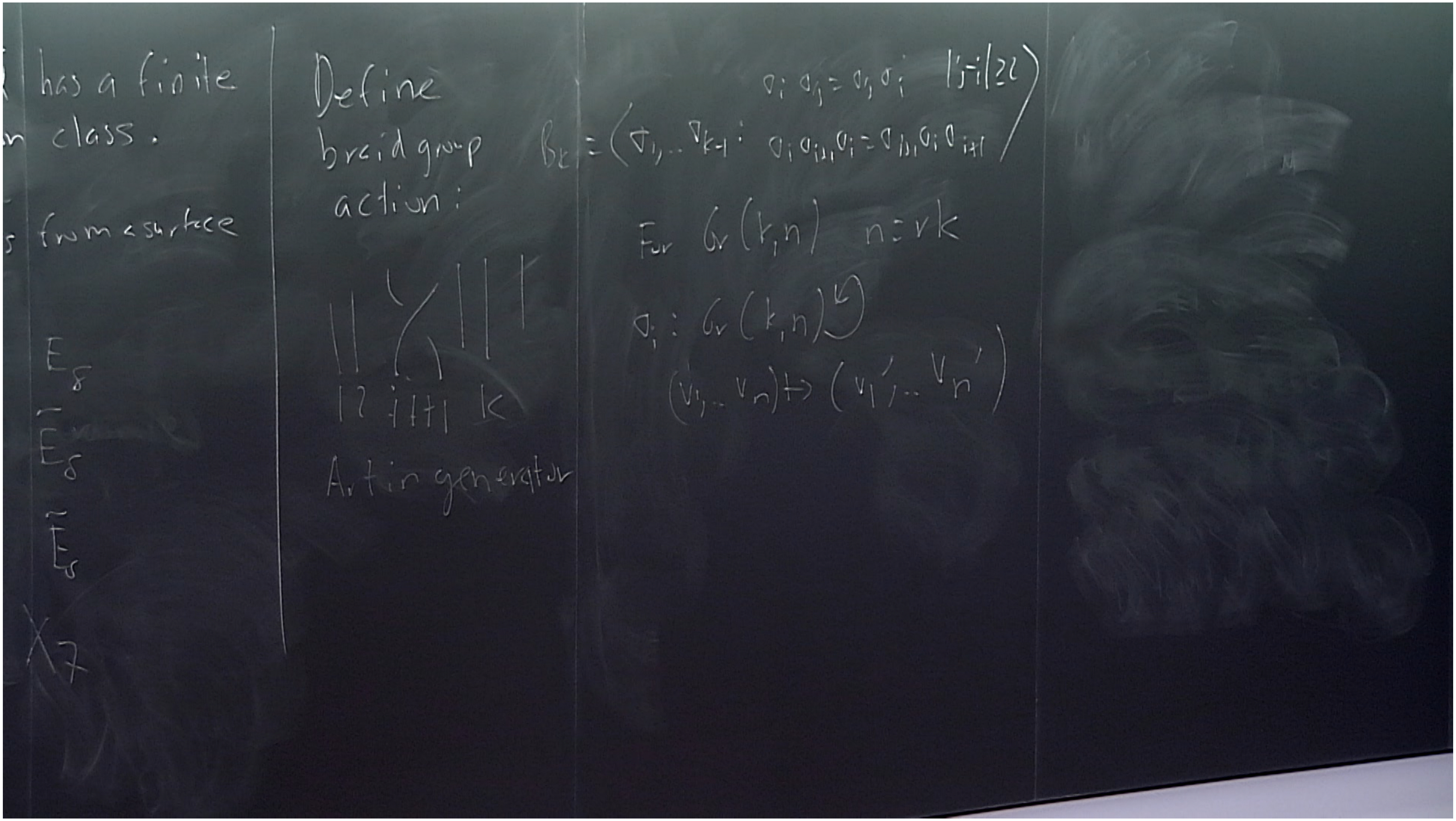
Then Q comes from a surface or

$E_6 \quad E_7 \quad E_8$

$\tilde{E}_6 \quad \tilde{E}_7 \quad \tilde{E}_8$

$\tilde{\tilde{E}}_6 \quad \tilde{\tilde{E}}_7 \quad \tilde{\tilde{E}}_8$

or $X_6 \quad X_7$



has a finite
class.

from a surface

E_8

$\tilde{E}_8$

$\tilde{\tilde{E}}_8$

X_7

Define
braid group action:

1 2 i i+1 k

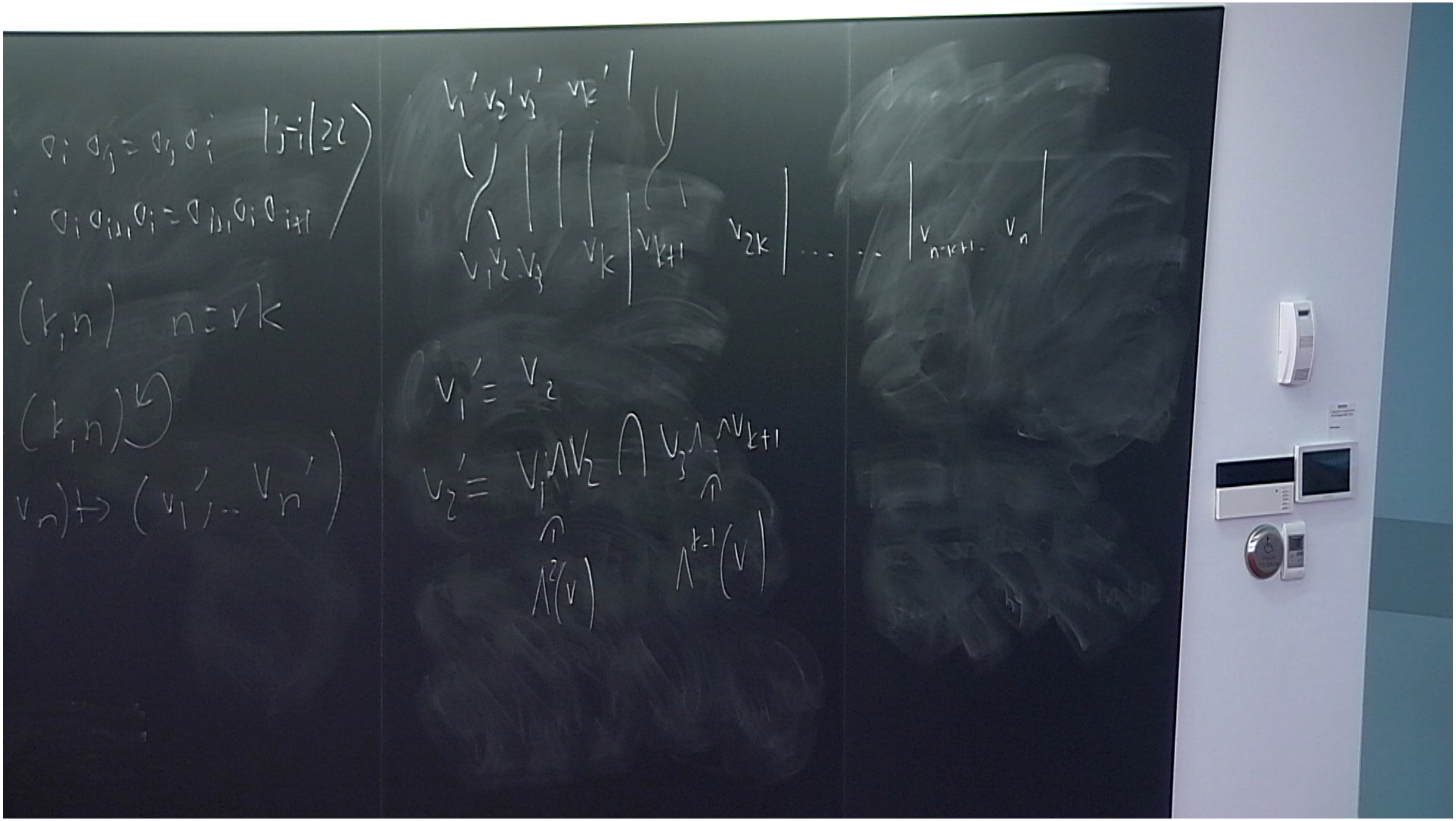
Artin generator

$$B_k = \left\langle \sigma_1, \dots \sigma_{k-1} : \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad |j-i| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$$

For $Gr(k,n)$ $n = rk$

$\sigma_i : Gr(k,n) \circlearrowleft$

$(v_1, .. v_n) \mapsto (v_1', .. v_n')$



$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |j-i| \geq 2$)

: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ /

$(k,n) \quad n = rk$

$(k,n) \circlearrowleft$

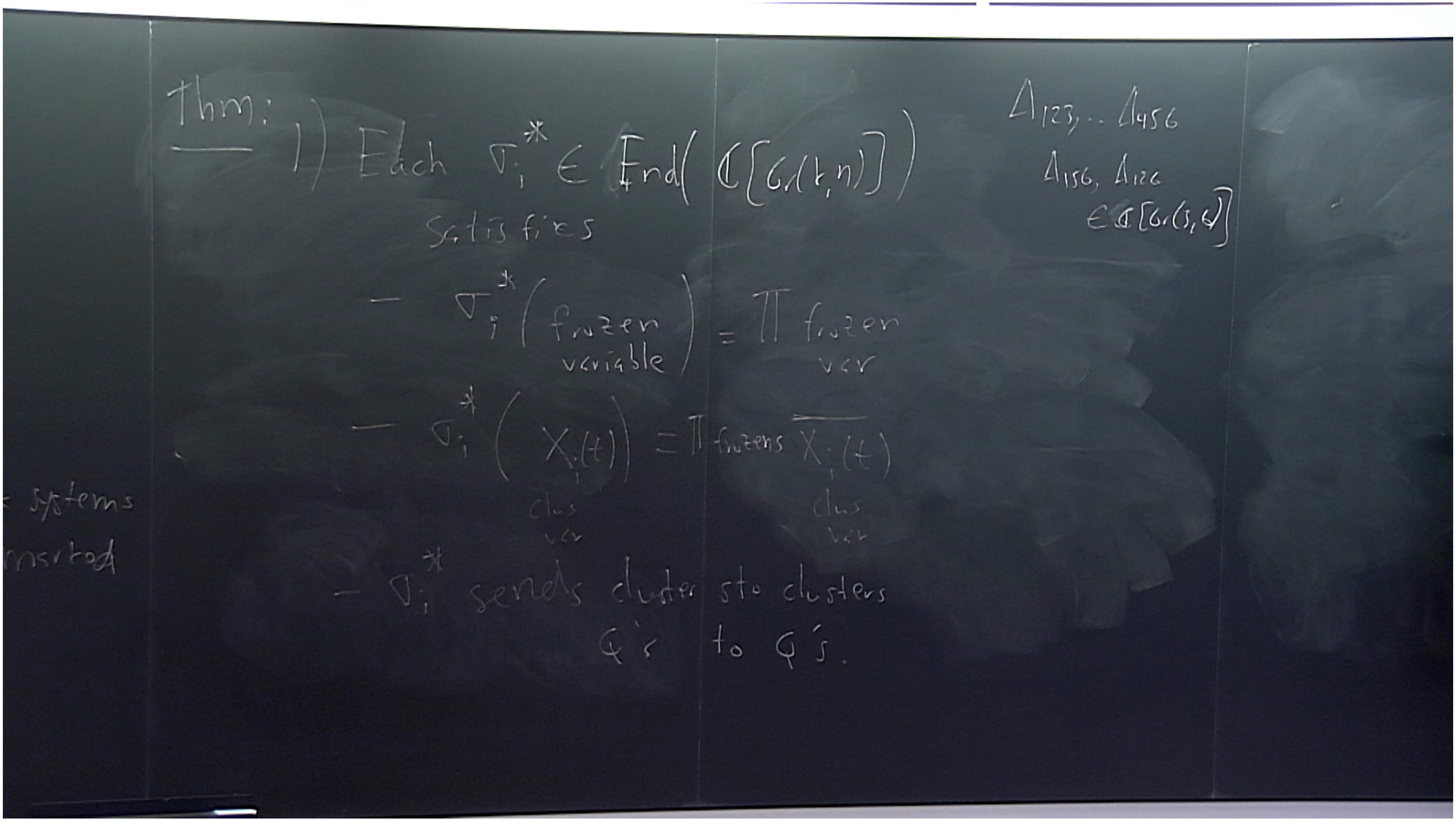
$v_n) \mapsto (v_1', \dots v_n')$

$v_1' \, v_2' \, v_3' \quad v_k'$

$v_1 \, v_2 \, v_3 \quad v_k \mid v_{k+1} \quad v_{2k} \mid \dots \dots \mid v_{n-k+1} \dots v_n \mid$

$v_1' = v_2$

$v_2' = \underset{\underset{\Lambda^2(V)}{\cap}}{v_1 \wedge v_2} \cap \underset{\underset{\Lambda^{k-1}(V)}{\cap}}{v_3 \wedge \dots \wedge v_{k+1}}$



systems

nstructed

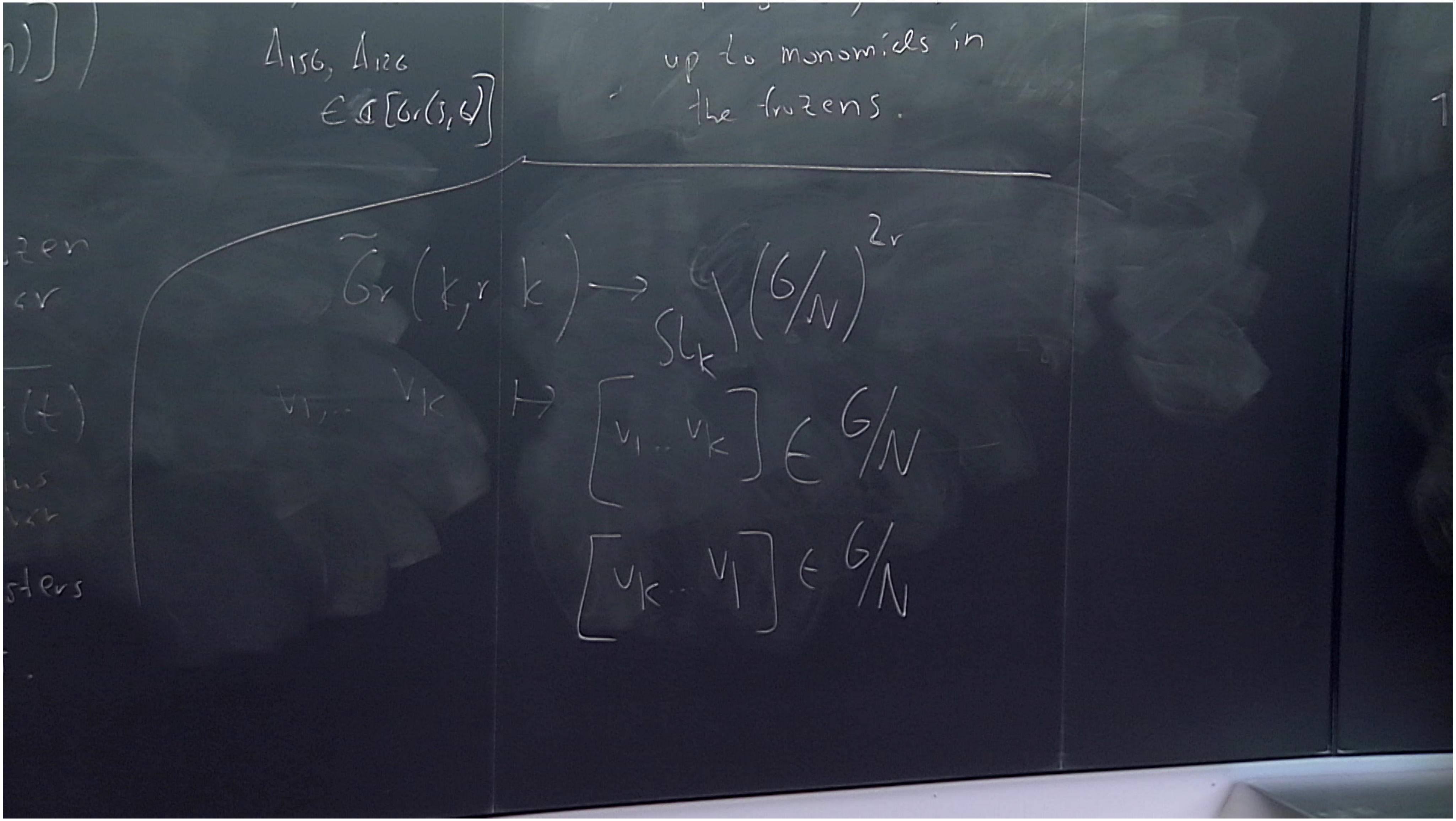
Thm: 1) Each $\sigma_i^* \in \mathrm{End}(\mathbb{C}[Gr(k,n)])$ satisfies

- $\sigma_i^*\left(\text{frozen variable}\right) = \prod \text{frozen var}$
- $\sigma_i^*\left(\underset{\text{clus var}}{X_i(t)}\right) = \prod \text{frozens} \underset{\text{clus var}}{\overline{X_i(t)}}$
- σ_i^* sends cluster s to clusters
  Q's to Q's.

$\Delta_{123}, .. \Delta_{456}$

$\Delta_{156}, \Delta_{126}$

$\in \mathbb{C}[Gr(3,6)]$



n)])

$\Delta_{156}, \Delta_{126}$
$\in \mathbb{C}[Gr(3,6)]$

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the frozens.

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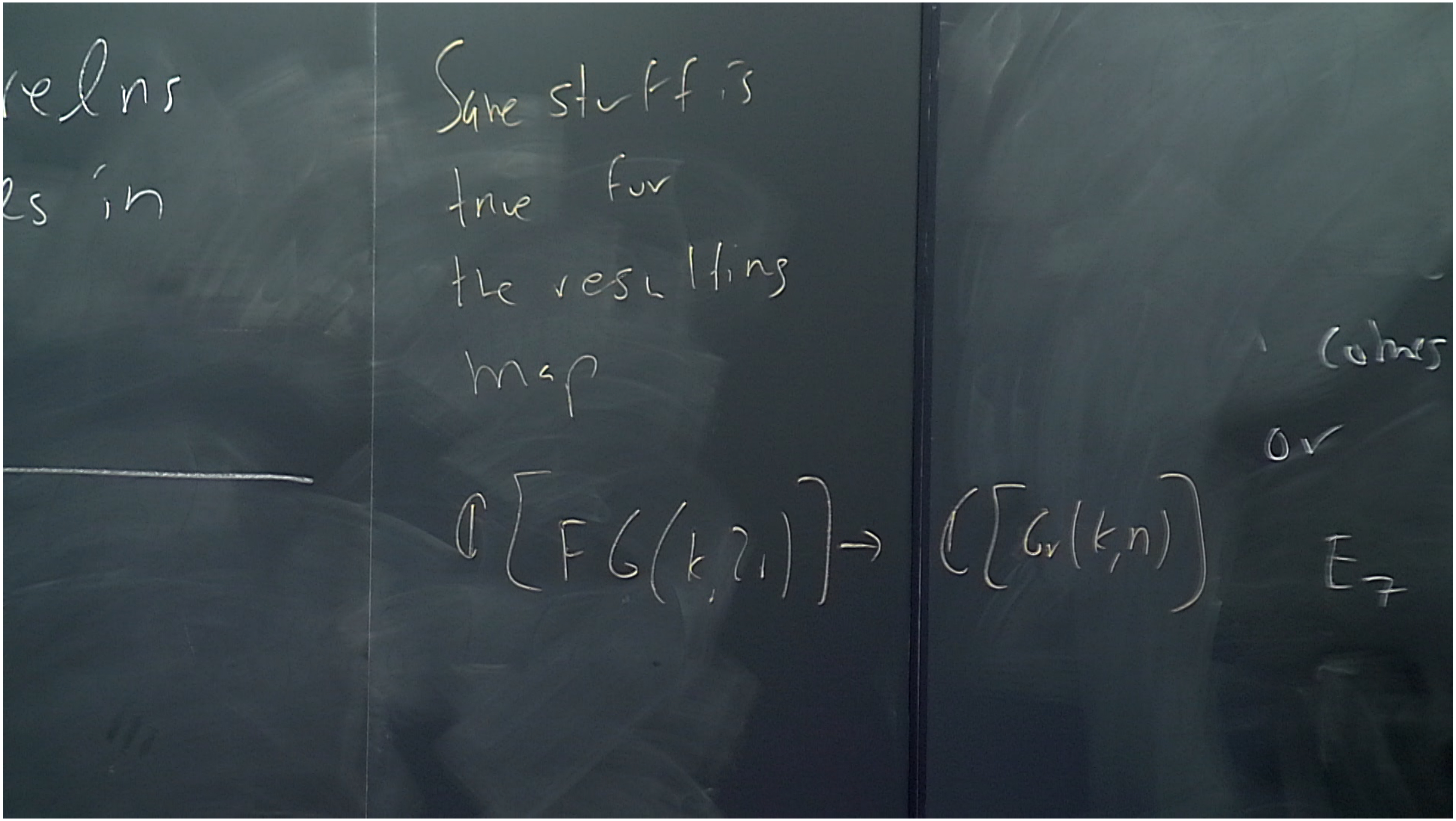
$$\widetilde{Gr}(k, r\,k) \longrightarrow SL_k \backslash (G/N)^{2r}$$

$$v_1, \ldots v_k \mapsto [v_1 \ldots v_k] \in G/N$$

$$[v_k \ldots v_1] \in G/N$$



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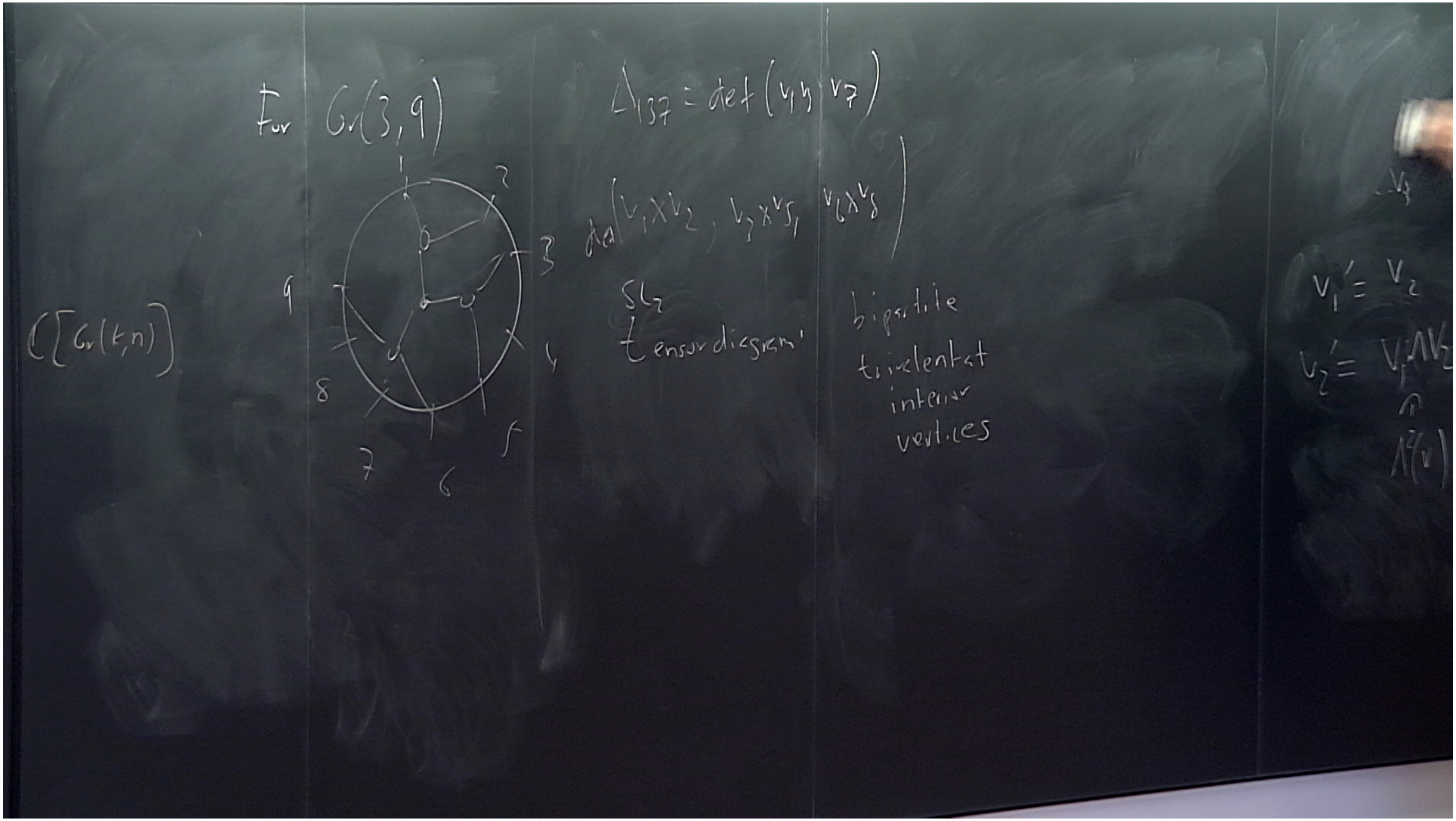
Same stuff is true for the resulting map

$\mathbb{C}[FG(k,n)] \to \mathbb{C}[Gr(k,n)]$

comes

or

E_7



For $Gr(3,9)$

$\mathbb{C}[Gr(k,n)]$

$\Delta_{137} = \det(v_1 v_3 v_7)$

$\det(v_1 \times v_2, v_3 \times v_{5,1} \; v_6 \times v_8)$

SL_3

tensor diagram

bipartite

trivalent

interior

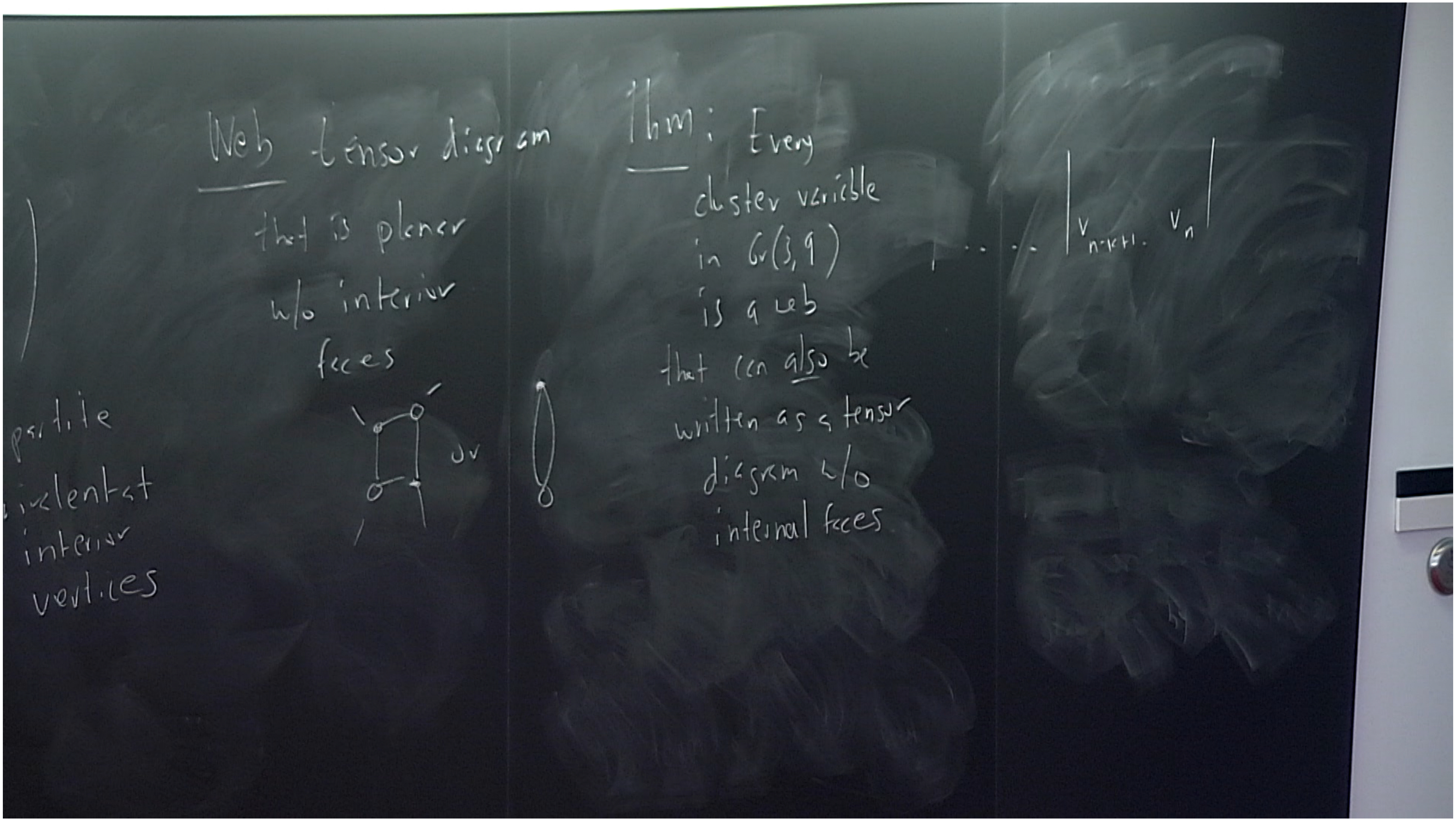
vertices

$\ldots v_3$

$V_1' = V_2$

$V_2' = V_1 \wedge V_2$

$\Lambda^2(V)$



partite

ivalent at

interior

vertices

Web tensor diagram

that is planar

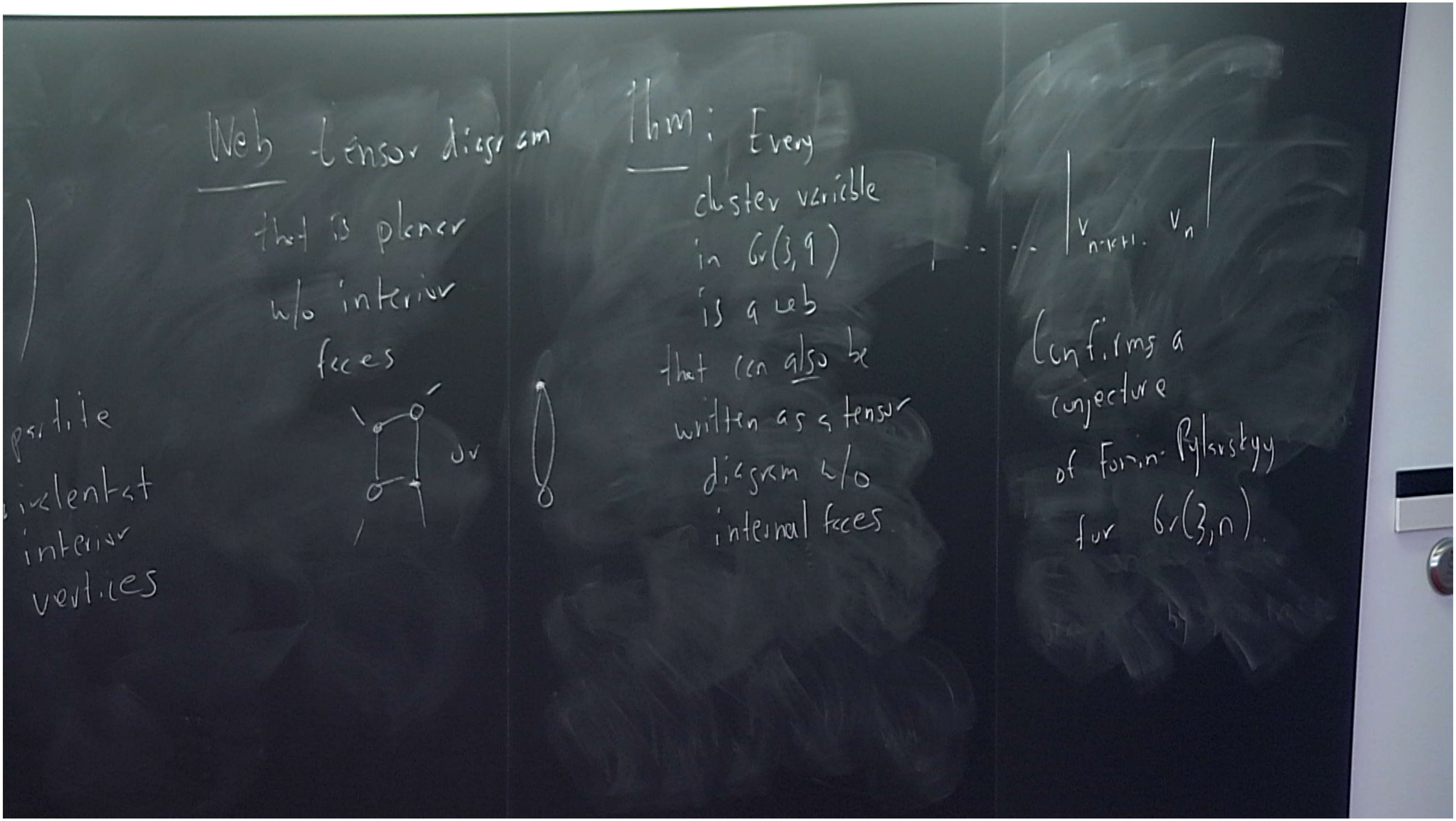
w/o interior

faces

or

Thm: Every cluster variable in $Gr(3,9)$ is a web that can *also* be written as a tensor diagram w/o internal faces.

$$\cdots \; |v_{n-k+1} \ldots v_n|$$



partite

ivalent at

interior

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<u>Web</u> tensor diagram

that is planar

w/o interior

faces

or

<u>Thm</u>: Every

cluster variable

in $Gr(3,9)$

is a web

that can <u>also</u> be

written as a tensor

diagram w/o

internal faces.

$$\cdots \left| v_{n-k+1} \cdots v_n \right|$$

Confirms a

conjecture

of Fomin-Pylyavskyy

for $Gr(3,n)$.