Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 9

Date: Mar 30, 2017 10:15 AM

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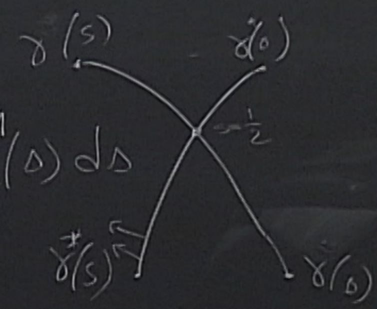
Abstract:

$$E_\gamma = \int h_{\gamma\gamma^*}(s)\, e_a(\gamma^*(s))\, \dot{\gamma}^*(s)\, h^{-1}_{\gamma\gamma^*}(s)\, ds$$



$$e_a = e_a^k T_k .$$

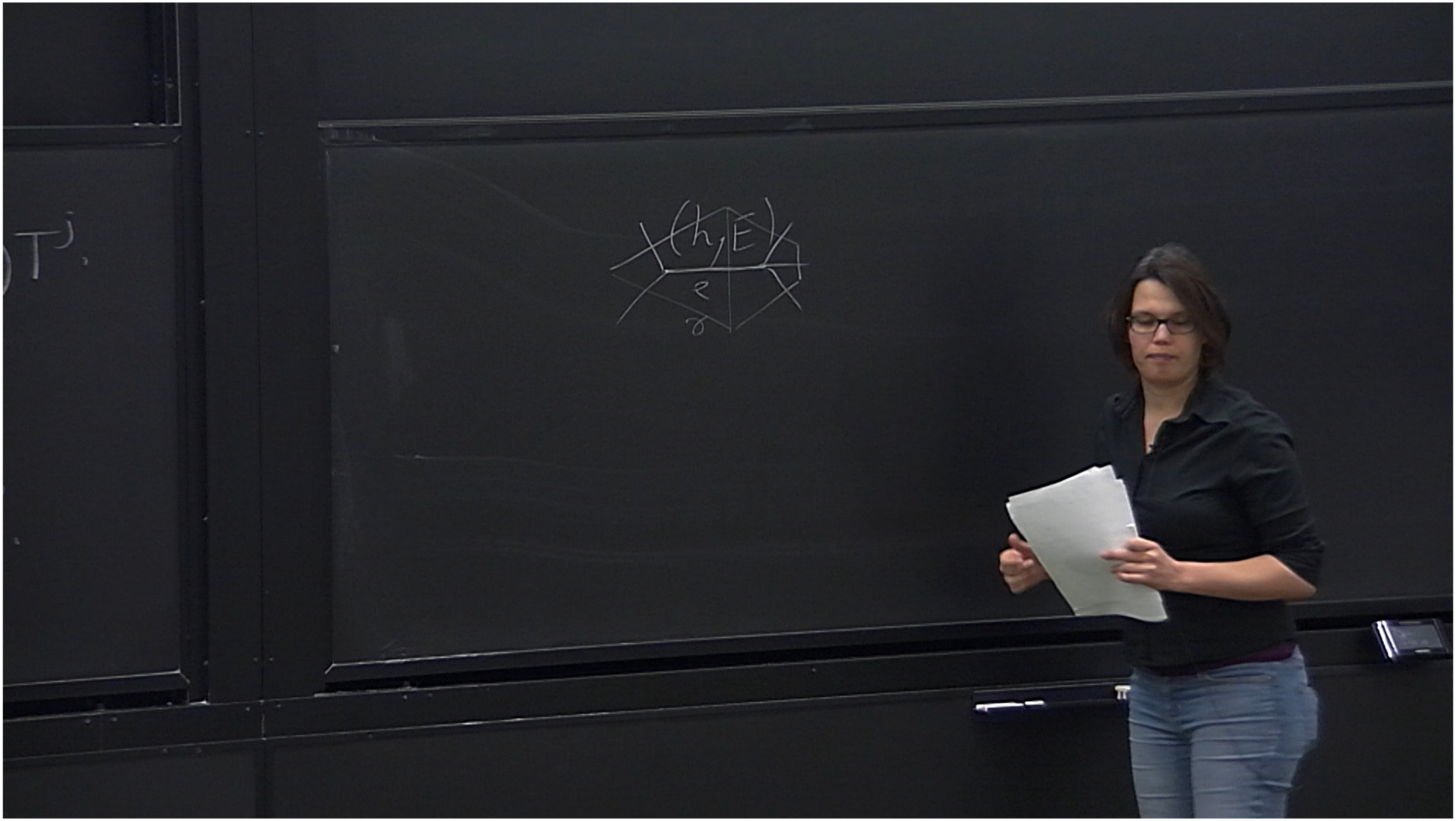
$$e_a = \epsilon_{ab} E^{bj} T_j .$$

$$\{A, E\} \longrightarrow \{(h_\gamma)_{MN}, E^j_\gamma\} = (h_{MN}) T^j .$$

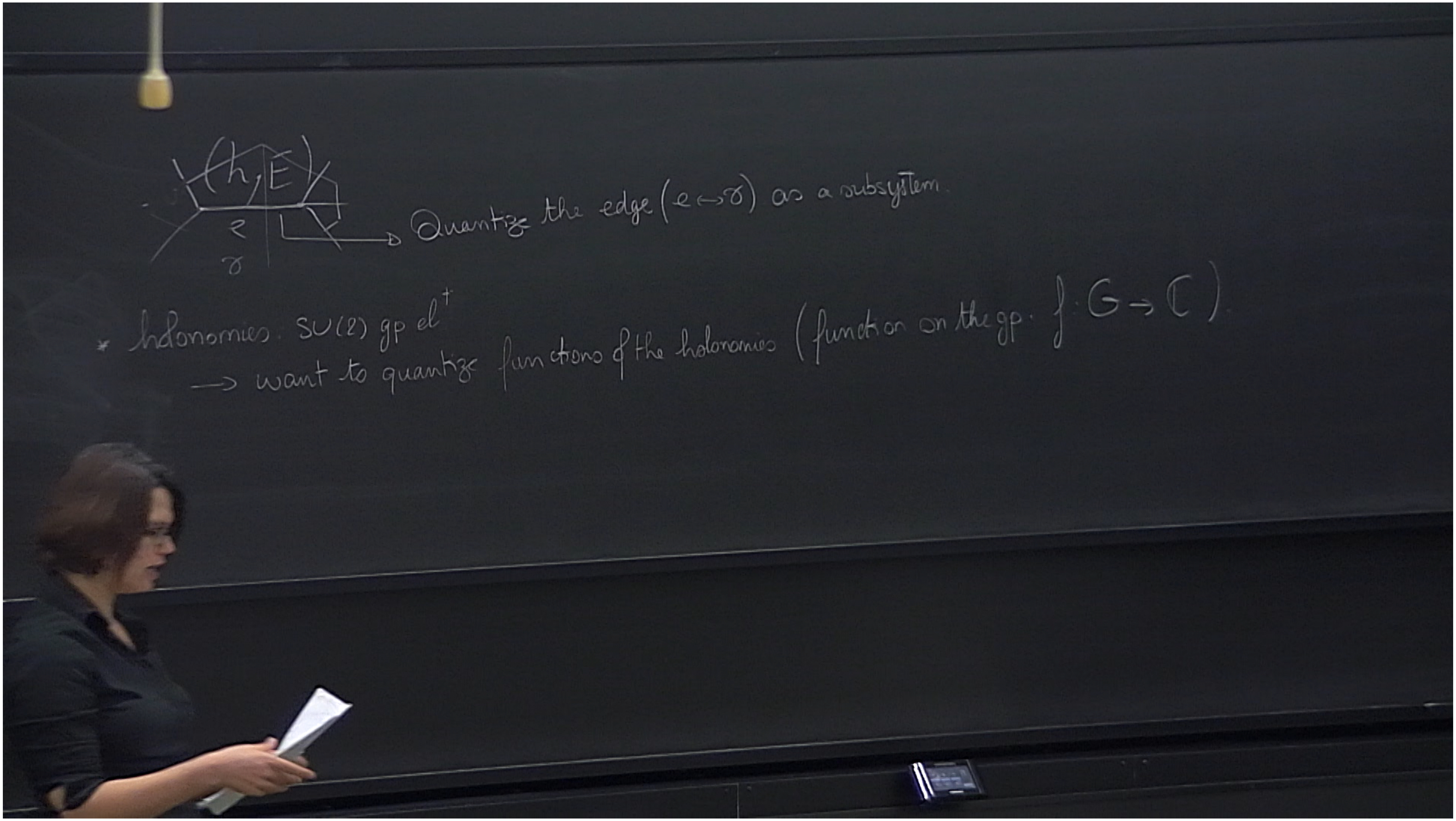
$$\{(h_\gamma)_{MN}, (h_\gamma)_{MN}\} = 0 .$$

$$\{E^j_\gamma, E^k_\gamma\} = \epsilon_{jk}{}^l E^l .$$

$T^* SU(2)$



(h, E)


(h, E)
e
γ
Quantize the edge (e ↔ γ) as a subsystem.
* holonomies. SU(2) gp el^t
→ want to quantize functions of the holonomies (function on the gp. f: G → ℂ).

γ

* holonomies: SU(2) gp el^t

$\longrightarrow$ want to quantize functions of the holonomies (function on the gp. $f: G \to \mathbb{C}$)

$\Rightarrow$ Quantized as multiplication operators

$$\Psi_e : SU(2) \longrightarrow \mathbb{C} \qquad \hat{f}\,\Psi(g) = f(g)\,\Psi(g)$$
$$g \longmapsto \Psi_e(g)$$

For a state, in the holonomy rep. $\langle g | \Psi_e \rangle = \Psi_e(g)$

$|\Psi_e\rangle$

→ want to quantize functions of the

⇒ Quantized as multiplication operators

$$\psi_e : SU(2) \longrightarrow \mathbb{C}$$
$$g \longmapsto \psi_e(g)$$

$$\hat{f}\,\psi(g) = f(g)\,\psi(g)$$

For a state in the holonomy rep: $\langle g | \psi_e \rangle = \psi_e(g)$

$|\psi_e\rangle$

$$\hat{f}\,|\psi_e\rangle = |f_e \psi_e\rangle$$

Multiplication op. commute: OK.

* **Flux**

↳ $\{E_e^j, (h_e)_{mn}\} = (h_e T_j)_{mn} = \frac{d}{dt}\Big|_{t=0} \left(h_e e^{tT_j}\right)_{mn} = L_e^j (h_e)_{mn}$

↳ left invariant derivative.

$$\{E_e^j, f_e(h_e)\} = (L_e^j f_e)(h_e) = \frac{d}{dt}\Big|_{t=0} f_e\left(h_e e^{tT_j}\right)$$

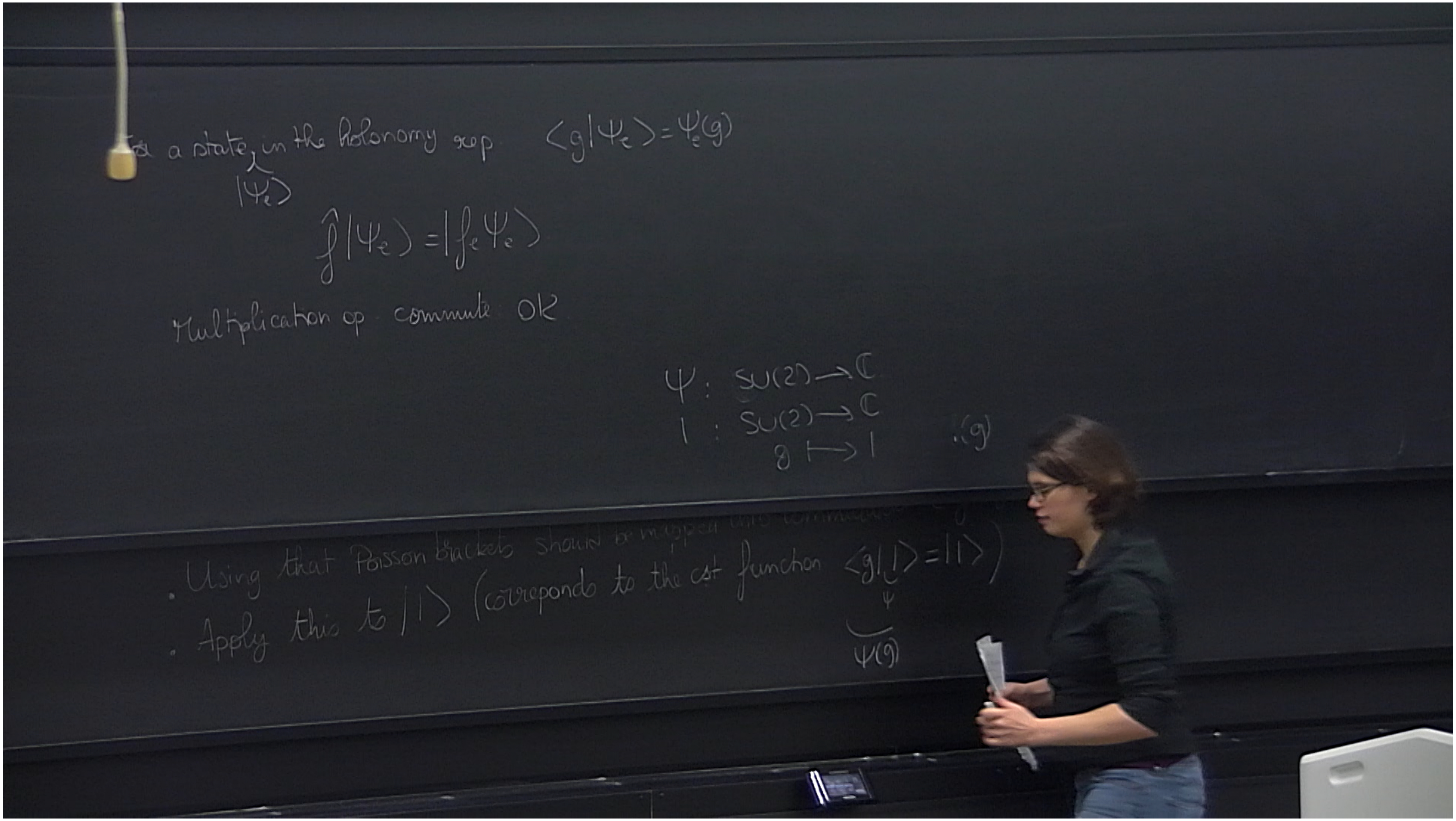
* Flux

↳ $\{E_e^j, (h_e)_{mn}\} = (h_e T_j)_{mn} = \frac{d}{dt}\Big|_{t=0} \big(\underbrace{h_e e^{tT_j}}\big)_{mn} = L_e^j (h_e)_{mn}$

↳ left invariant derivative

$$\{E_e^j, f_e(h_e)\} = (L_e^j f_e)(h_e) = \frac{d}{dt}\Big|_{t=0} f_e(h_e e^{tT_j})$$

- Using that Poisson brackets should be mapped into commutators $[\hat{f}, \hat{g}] = i\hbar\, \widehat{\{f, g\}}$
- Apply this to $|1\rangle$ (correponds to the cst function $\langle g|1\rangle = |1\rangle$)


For a state in the holonomy rep ⟨g|ψe⟩ = ψe(g)
|ψe⟩
f̂|ψe) = |fe ψe⟩
multiplication op commute OK
ψ : SU(2) → ℂ
1 : SU(2) → ℂ
g ↦ 1
. Using that Poisson brackets should be mapped into commutators
. Apply this to |1⟩ (corresponds to the cst function ⟨g|1⟩ = |1⟩)
ψ(g)

$$i\hbar\,\widehat{(L_e^j f_e)}\,|1\rangle = i\hbar\,\widehat{\{E^j, f_e(h_e)\}}\,|1\rangle$$

$$= [\hat{E}^j, \hat{f}_e]\,|1\rangle$$

$$= \hat{E}^j\,|f_e\rangle - \underbrace{\hat{f}_e\,\hat{E}^j\,|1\rangle}_{=0}$$

$\Rightarrow$ $\hat{E}$ acts as a derivative op.

$\Rightarrow$ action of the fluxes E_e^j on a state $|\psi_e\rangle = \hat{\psi}_e\,|1\rangle$

$$\boxed{\hat{E}^j\,|\psi_e\rangle = i\hbar\,|L_e^j\,\psi_e\rangle}$$

$$E^j|\Psi_e\rangle = i\hbar\,|L_e \Psi_e\rangle$$

Consistency of the final commutation relations between fluxes?

$$[\hat{E}^j_e, \hat{E}^k_e]|\Psi_e\rangle = (\hat{E}^j_e \hat{E}^k_e - \hat{E}^k_e \hat{E}^j_e)|\Psi_e\rangle = (i\hbar)^2\,|(L^j_e \triangleright L^k_e - L^k_e \triangleright L^j_e)\Psi_e\rangle$$

$$\overset{?}{=} i\hbar\,\{E^j, E^k\}|\Psi_e\rangle$$

$$(L^j_e \triangleright L^k_e - L^k_e \triangleright L^j_e)(h_e)_{mn} = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0}\left[(h_e\, e^{sT^j} e^{tT^k})_{mn} - (h_e\, e^{tT^k} e^{sT^j})_{mn}\right]$$

$$=$$

$$= 0$$

$\Rightarrow$ action of the fluxes

$$\boxed{\hat{E}^j |\Psi_e\rangle = i\hbar\, |L_e^j \Psi_e\rangle}$$

$$[\hat{E}^j_e, \hat{E}^k_e] |\Psi_e\rangle = (\hat{E}^j_e \hat{E}^k_e - \hat{E}^k_e \hat{E}^j_e) |\Psi_e\rangle = (i\hbar)^2 \,|(L_e^j \triangleright L_e^k - L_e^k \triangleright L_e^j)\Psi_e\rangle$$

$$\overset{?}{=} i\hbar \{E^j, E^k\} |\Psi_e\rangle$$

$$(L_e^j \triangleright L_e^k - L_e^k \triangleright L_e^j)(h_e)_{mn} = \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \Big[(h_e\, e^{sT^j} e^{tT^k})_{mn} - (h_e\, e^{tT^k} e^{sT^j})_{mn} \Big]$$

$$= (h_e [T^j, T^k])_{mn} = (h_e\, \varepsilon^{jk}{}_l T^l)_{mn} = \varepsilon^{jk}{}_l \frac{d}{dt}\Big|_{t=0} (h_e\, e^{tT^l}) = \varepsilon^{jk}{}_l L_e^l (h_e)_{mn}$$

$$\Rightarrow [\hat{E}^j_e, \hat{E}^k_e] = i\hbar\, \varepsilon^{jk}{}_l \hat{E}^l_e$$

③ Inner product

↳ measure inv under right translation ; for SU(2) can do better than that

Haar measure inv under left/right transl

Left and right actions on a group for $h \in G$

$$(L_h f)(g) = f(h^{-1} g) \qquad L_{h_1 h_2} = L_{h_1} \circ L_{h_2}$$

$$(R_h f)(g) = f(g h) \qquad R_{h_1 h_2} = R_{h_1} \circ R_{h_2}$$

$$L^a = \frac{d}{dt}\Big|_{t=0} R_{\exp(tT^a)} \; ; \; R^a = \frac{d}{dt}\Big|_{t=0} L_{\exp(-tT^a)}$$

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi_1(g)}\, \Psi_2(g)\, \underbrace{d\mu(g)}_{\text{Haar measure}}$$

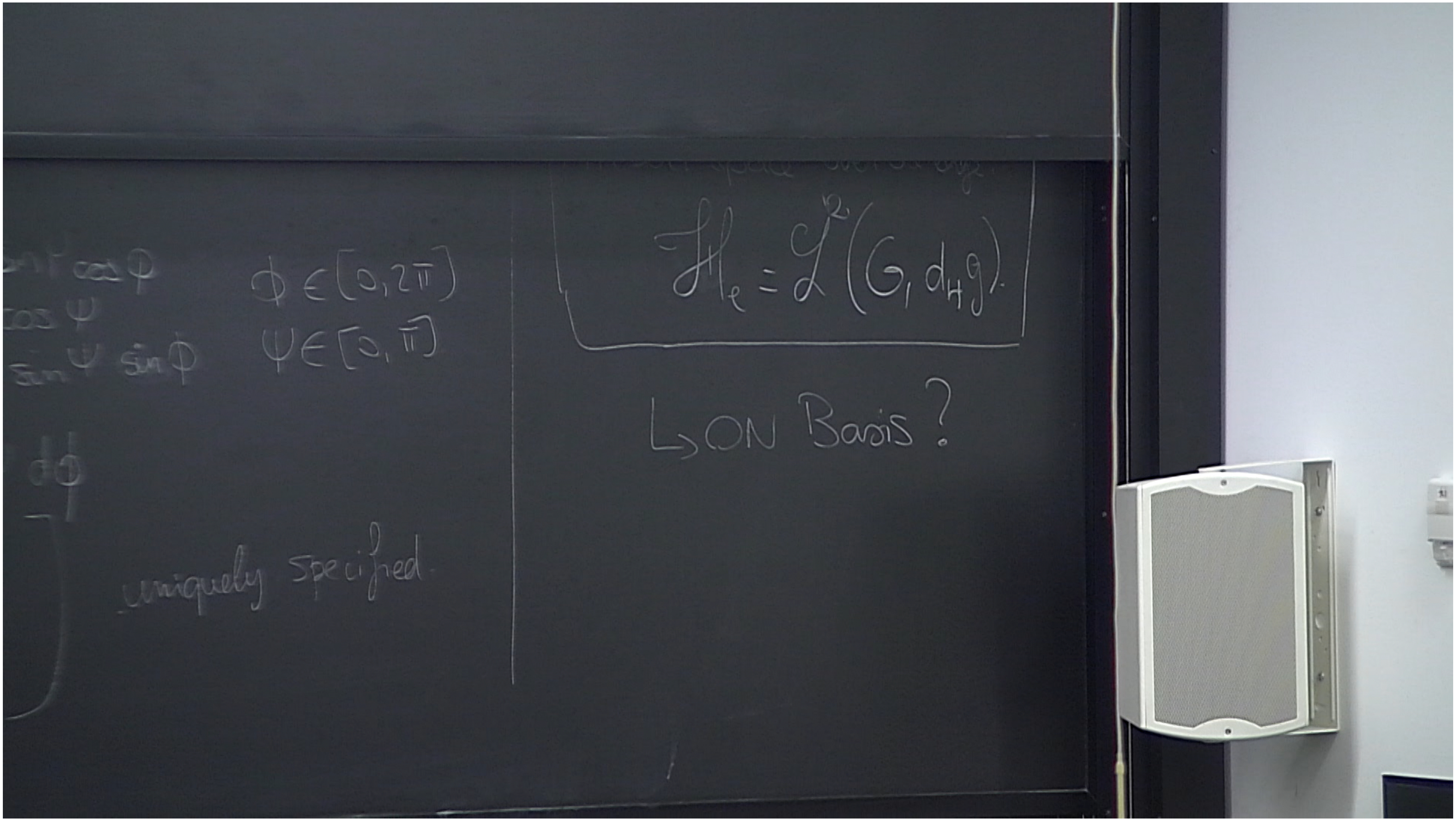
$$g = \exp(\alpha\, T \cdot \hat{n})$$

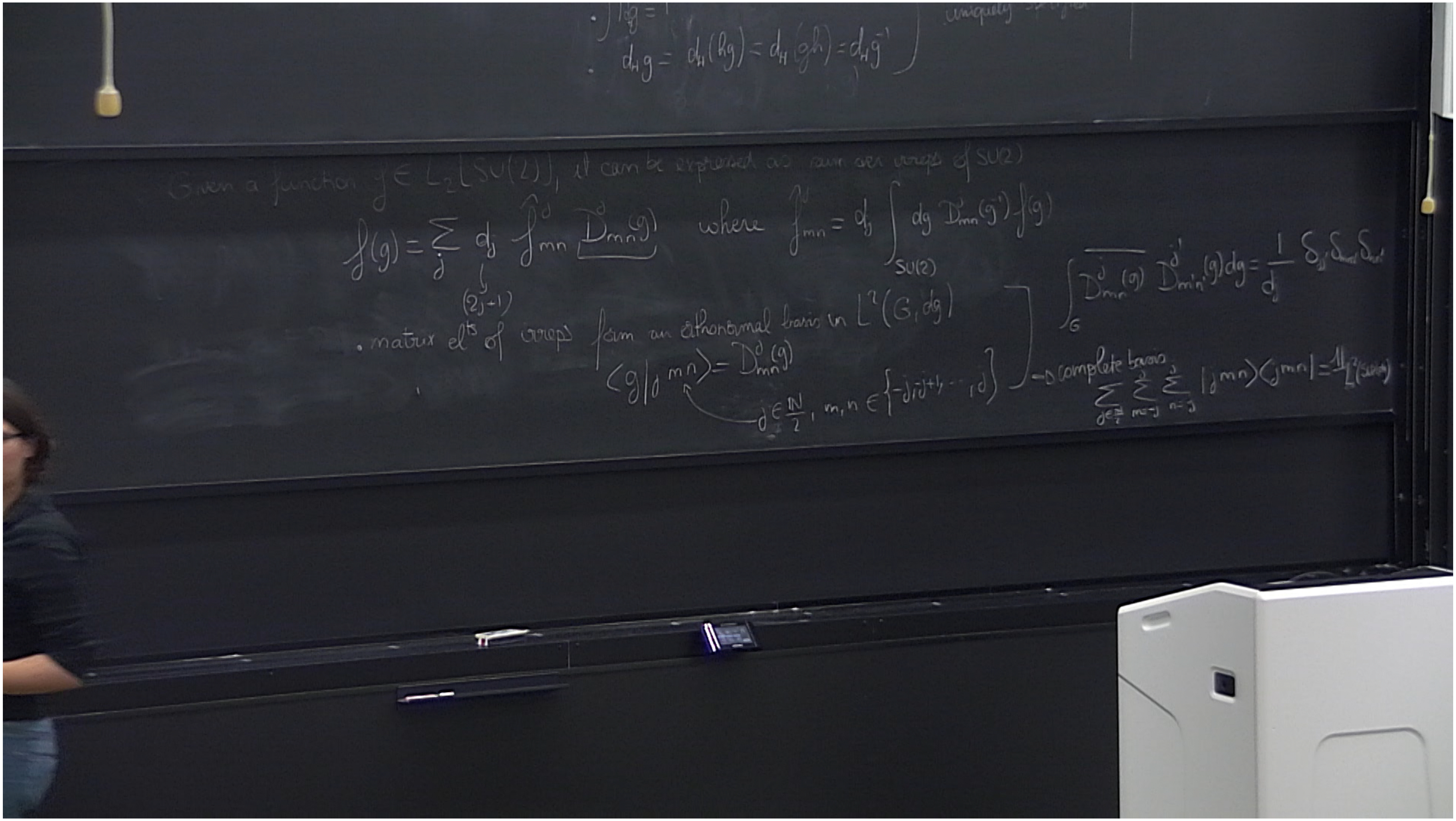
$$g : \underset{[0,2\pi]}{\alpha}\ ,\ \hat{n} \begin{cases} n_1 = \sin\psi \cos\phi \\ n^2 = \cos\psi \\ n^3 = \sin\psi \sin\phi \end{cases} \qquad \begin{array}{l} \phi \in [0, 2\pi) \\ \psi \in [0, \pi] \end{array}$$

$$d_H g = \frac{1}{4\pi^2} \sin^2\left(\frac{\alpha}{2}\right) \sin\psi \, d\alpha\, d\psi\, d\phi$$

$$\left.\begin{array}{l} \bullet \int 1\, d_H g = 1 \\ \bullet\ d_H g = d_H(hg) = d_H(gh) = d_H g^{-1} \end{array}\right\} \text{uniquely specified}$$

Hilbert sp


ϕ ∈ [0, 2π)
ψ ∈ [0, π]
uniquely specified.
$\mathcal{H}_e = \mathcal{L}^2(G, d_H g)$.
↳ ON Basis?



$d_H g = d_H(hg) = d_H(gh) = d_H g^{-1}$
Given a function $f \in L_2[SU(2)]$, it can be expressed as sum over irreps of SU(2)
$f(g) = \sum_j d_j \hat{f}^j_{mn} D^j_{mn}(g)$ where $\hat{f}^j_{mn} = d_j \int_{SU(2)} dg\, D^j_{mn}(g^{-1}) f(g)$
$(2j+1)$
. matrix el^ts of irreps form an orthonormal basis in $L^2(G, dg)$
$\langle g | j\, m n\rangle = D^j_{mn}(g)$
$j \in \frac{\mathbb{N}}{2}$, $m, n \in \{-j, -j+1, \dots, j\}$
$\int_G \overline{D^j_{mn}(g)}\, D^{j'}_{m'n'}(g)\, dg = \frac{1}{d_j} \delta_{jj'} \delta_{mm'} \delta_{nn'}$
→ complete basis
$\sum_{j \in \frac{\mathbb{N}}{2}} \sum_{m=-j}^{j} \sum_{n=-j}^{j} |j\, m n\rangle\langle j\, m n| = \mathbb{1}_{L^2(SU(2))}$

of the final commutation relations between fluxes ?

Hilbert space associated to one edge. ← dual rep

$$\mathcal{L}^2(G, d_H g) = \bigoplus_j V_j \otimes V_j^* = \mathcal{H}_e .$$

g

n —— m ← angular momentum associated to both ends of the edges.

$|j\,m\,n\rangle$

$D^j_{mn}(g)$