Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 7

Date: Mar 28, 2017 10:15 AM

URL: http://pirsa.org/17030074

Abstract:

$$S = \int d^3x \sqrt{g} R - - \int e_{0a} F^{l}_{\mu\nu} \tilde{\epsilon}^{0\mu\nu} d^3x$$

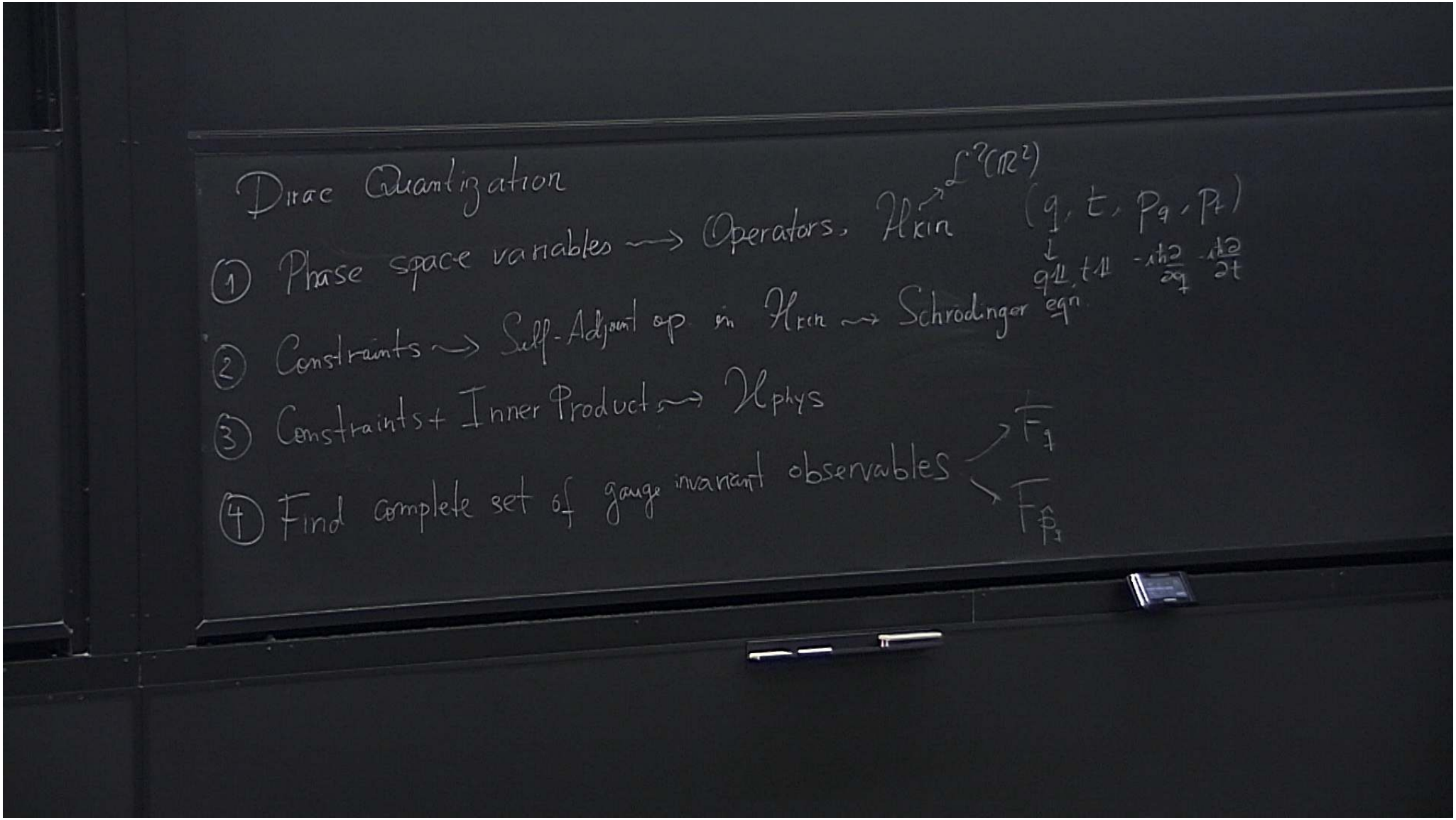
Canonical variables: A^{j}_{a} ; E^{k}_{b}

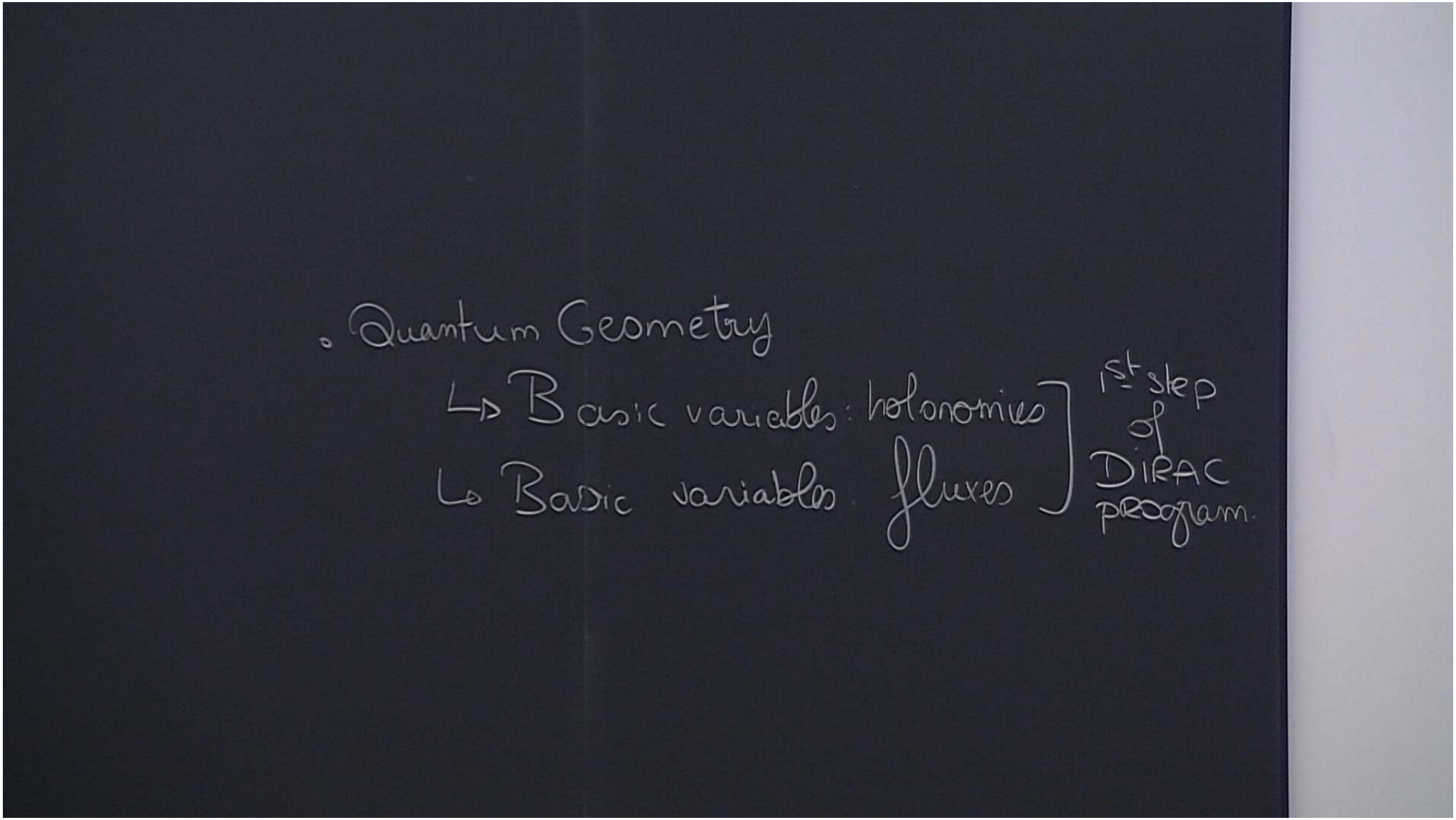
From e.o.m → Get constraints → Gauss

→ Flatness

$\rightsquigarrow$ Smear out → $\delta_\Lambda E^{a}_{j} = -\epsilon_{jmk} E^{am} \Lambda^{k}$

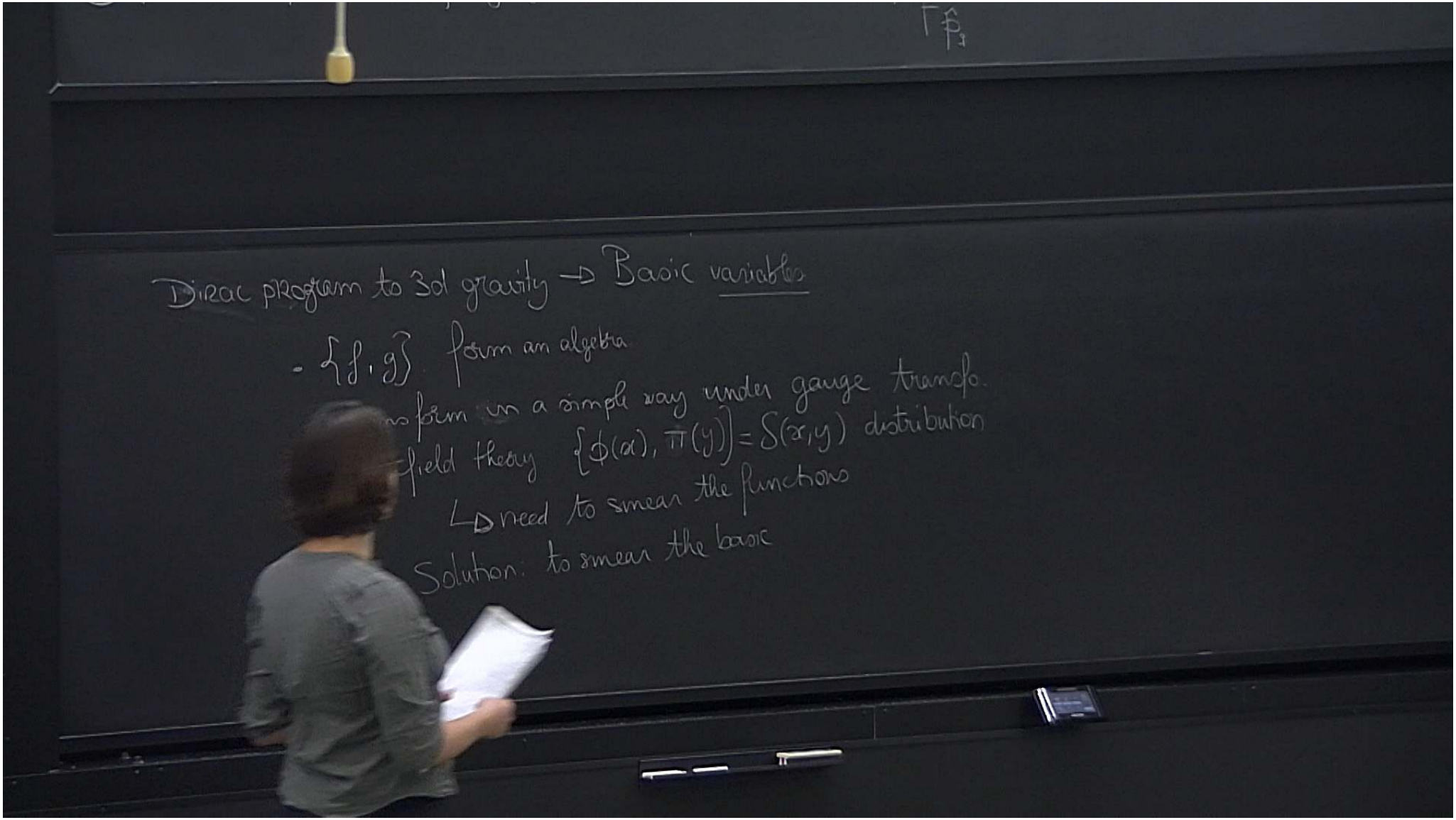
$\delta_\Lambda A^{j}_{a} = -(\partial_A \Lambda^{j} + \epsilon^{j}_{lm} A^{l}_{a} \Lambda^{m})$


Dirac Quantization
(1) Phase space variables ⟶ Operators, $\mathcal{H}_{kin}$ → $L^2(\mathbb{R}^2)$
(q, t, p_q, p_t)
$\downarrow$
$\hat{q}, \hat{t}$ $-i\hbar\frac{\partial}{\partial q}$ $-i\hbar\frac{\partial}{\partial t}$
(2) Constraints ⟶ Self-Adjoint op in $\mathcal{H}_{kin}$ ⟶ Schrodinger eqn
(3) Constraints + Inner Product ⟶ $\mathcal{H}_{phys}$
(4) Find complete set of gauge invariant observables
F_1
$F_{\hat{p}_t}$


• Quantum Geometry
↳ Basic variables: holonomies
↳ Basic variables: fluxes
1st step of DIRAC program.

Dirac program to 3d gravity → Basic variables

- $\{f, g\}$ form an algebra
- transform in a simple way under gauge transfo.
- ⚠ field theory $\{\phi(x), \pi(y)\} = \delta(x,y)$ distribution


Dirac program to 3d gravity → Basic variables
- {f, g} form an algebra
- form in a simple way under gauge transfo.
field theory [φ(x), π(y)] = δ(x,y) distribution
need to smear the functions
Solution: to smear the basic

Find complete set of gauge invariant observables → F_1, $F_{\hat{P}_3}$

Dirac program to 3d gravity → Basic variables

- $\{f, g\}$ form an algebra
- transform in a simple way under gauge transfo.
- ⚠ field theory $\{\phi(x), \pi(y)\} = \delta(x,y)$ distribution
  - ↳ need to smear the functions

Solution: to smear the basic variables over k-dimensional hypersurfaces and the conjugated variables over $(d-k)$ d ones

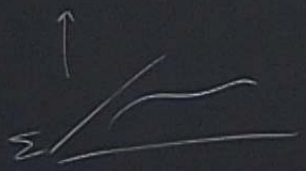
↑ configuration

↳ need to smear

Solution: to smear the basic variables (configuration) over 1-dimensional ... ⇒ LQG holonomies-fluxes

- Configuration: connection = 1-form
  ↳ integrate over one dimensional object: curve γ → holonomy (finite // transport along γ)

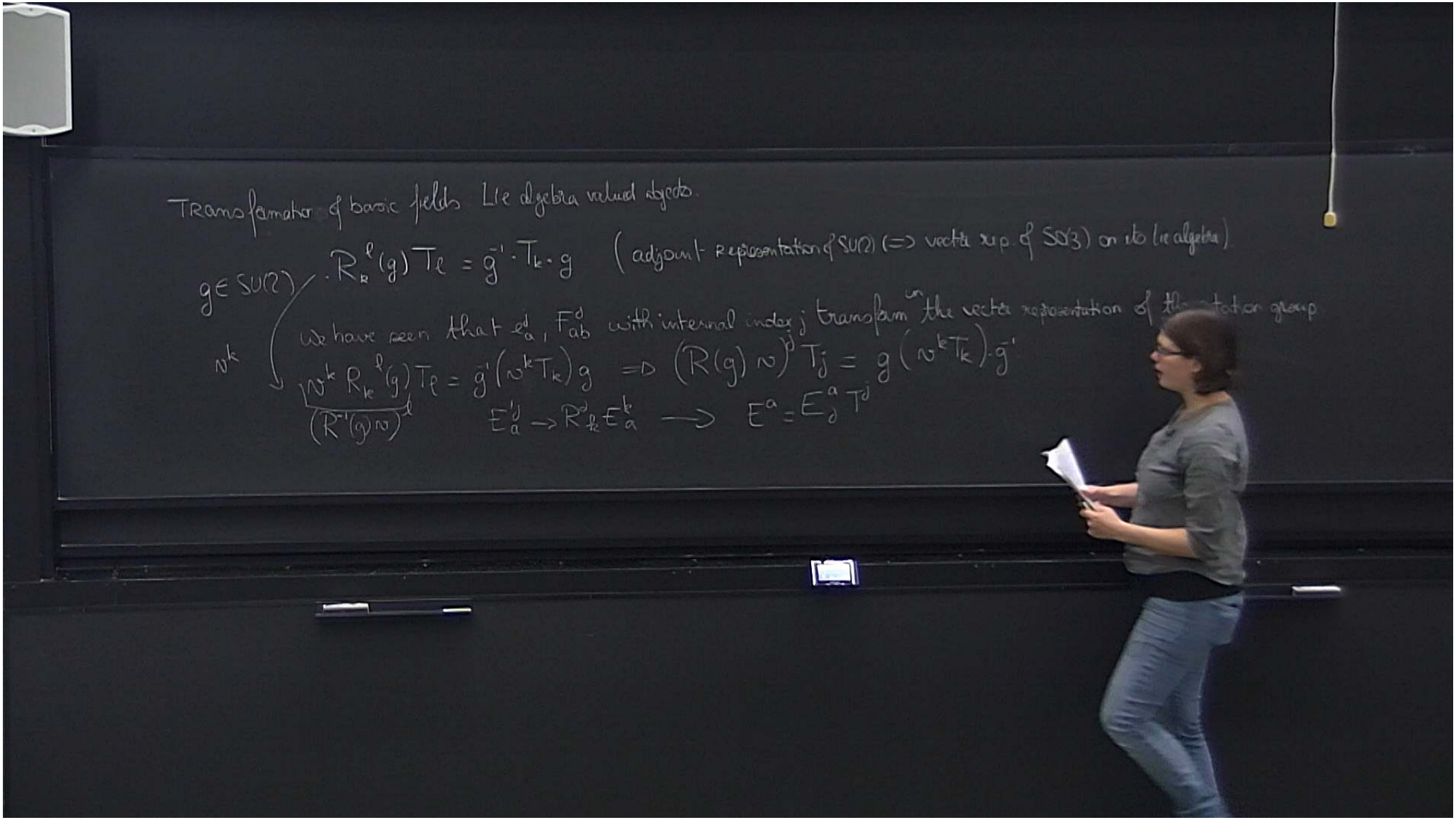
$$h_\gamma[A] \in SU(2)$$
$$h_\gamma[A] = \mathrm{Pexp}-\int_\gamma A$$



- Conjugated variables: triads → co-triad (one-form) $e_a = \underbrace{e_a^j T_j}_{} = \varepsilon_{ab} E^{bj} T^j$ smeared over a curve γ^*

$$T_j = -\frac{i}{2}\sigma_i \qquad [T_i, T_j] = \varepsilon_{ijk} T^k \quad (\text{su(2)} \cong \text{so(3)} \text{ algebra})$$

↳ Lie value one form


Transformation of basic fields Lie algebra valued objects.
$g \in SU(2)$
$\cdot R_k^{\ell}(g) T_\ell = g^{-1} \cdot T_k \cdot g$ (adjoint representation of $SU(2)$ (=> vector rep. of $SO(3)$) on its Lie algebra)
v^k
We have seen that e_a^i, F_{ab}^i with internal index j transform in the vector representation of the rotation group
$v^k R_k^{\ell}(g) T_\ell = g^{-1}(v^k T_k) g \Rightarrow (R(g)v)^j T_j = g(v^k T_k) \cdot g^{-1}$
$(R^{-1}(g)v)^{\ell}$
$E_a^j \to R^j_k E_a^k \longrightarrow E^a = E^a_j T^j$

Transformation of basic fields. Lie algebra valued objects.

$g \in SU(2)$ $\quad R^l_k(g)\, T_l = g^{-1} \cdot T_k \cdot g$ (adjoint representation of $SU(2)$ ($\Rightarrow$ vector rep. of $SO(3)$ on its Lie algebra)

v^k $\quad$ We have seen that e^j_a, F^j_{ab} with internal index j transform in the vector representation of the rotation group

$v^k R^l_k(g)\, T_l = g^{-1}(v^k T_k)\, g \Rightarrow \boxed{(R(g)\,v)^j T_j = g\,(v^k T_k)\,g^{-1}}$

$(R^{-1}(g)\,v)^j$

$E^j_a \to R^j_k E^k_a \longrightarrow E^a = E^a_j T^j \longrightarrow \boxed{g\,E^a\,g^{-1}}$

$F_{ab} = F^j_{ab} T_j \longrightarrow \boxed{g\,F_{ab}\,g^{-1}}$

A_a

We have seen that e^i_a, F^i_{ab} with internal index j transform the vector representation of the Rotation group

v^k

$v^k R_k{}^l(g) T_l = g^{-1}(v^k T_k) g \Rightarrow \boxed{(R(g)v)^j T_j = g(v^k T_k) g^{-1}}$

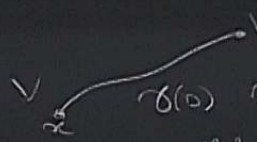
$(R^{-1}(g)v)^l$

$E^j_a \to R^j_k E^k_a \longrightarrow E^a = E^a_j T^j \longrightarrow \boxed{g E^a g^{-1}}$

$F_{ab} = F^j_{ab} T_j \longrightarrow \boxed{g F_{ab} g^{-1}}$

$\delta_D \; g A_a g^{-1} + g \partial_a g^{-1}$

Holonomies Given an internal vector $V = V^i T_i$ at a point x



$\sigma \in [0,1]$ (tangent vector $\dot\gamma(\sigma) = \frac{d}{d\sigma}\gamma(\sigma)$

$\Rightarrow$ parallel transport along the curve γ_σ

V transforms in the adjoint rep

So we define the // transport of V by $\boxed{V(\sigma) = h(\sigma)\, V(0)\, h(\sigma)^{-1}}$ $h(\sigma)$ holonomy from $\gamma(0) = x$ to $\gamma(\sigma)$

→ the transformation under local gauge transfo. $g(x) \in SU(2)$ of the holonomy

$$h(s) \longrightarrow g(\gamma(s))\, h(s)\, g(\gamma(s))^{-1}$$

$$\partial_a v^j = \partial_a v^j + \varepsilon_{ijk} A_i^a v^j$$

Covariant derivative with the motivation to // transport objects

$$D_a V = \partial_a V + [A_a, V]$$

$$\left([A_a^i T_i, V^j T_j] = A_a^i V^j [T_i, T_j] \right)$$

Lie algebra valued scalar

Solution to smear the basic variables configuration

LQG holonomies-fluxes ⇒

• configuration connection = 1-form

... one dimensional object: curve γ → holonomy (finite // transport along γ

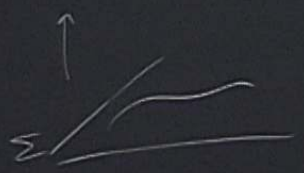
Solution to smear the basic variables configuration

LQG holonomies-fluxes

- configuration connection = 1-form
  - ↳ integrate over one dimensional object: curve γ → holonomy (finite // transport along γ)

$$h_\gamma[A] \in SU(2)$$

$$h_\gamma[A] = P\exp-\int_\gamma A$$



- conjugated variables triads → co-triad (one-form) $e_a = \underbrace{e_a^j T_j}_{} = \varepsilon_{ab} E^{bj} T^j$ smeared over a curve γ^*

$$T_j = -\frac{i}{2}\sigma_i \qquad [T_i, T_j] = \varepsilon_{ijk} T^k \quad \left(\begin{matrix} su(2) \\ so(3) \end{matrix} \text{ algebra}\right)$$

↳ Lie valued one form

v^k We have seen that e^j_a, F^j_{ab} with internal index j transform the vector representation of the rotation group.

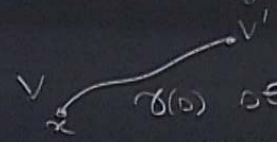
$$\underbrace{v^k R_k{}^l(g)}_{(R^{-1}(g)v)^l} T_l = g^{-1}(v^k T_k)\, g \Rightarrow \boxed{(R(g)\,v)^j T_j = g\,(v^k T_k)\cdot g^{-1}}$$

$$E^j_a \to R^j{}_k E^k_a \longrightarrow E^a = E^a_j T^j \longrightarrow \boxed{g\, E^a\, g^{-1}}$$

$$F_{ab} = F^j_{ab} T_j \longrightarrow \boxed{g\, F_{ab}\, g^{-1}}$$

$$\hookrightarrow \boxed{g A_a g^{-1} + g\,\partial_a g^{-1}}$$

Holonomies Given an internal vector $V = V^j T_j$ at a point x



$\gamma(s)$ $s \in [0,1]$ (tangent vector $\dot\gamma(s) = \frac{d}{ds}\gamma(s)$

$\Rightarrow$ parallel transport along the curve γ_s (*)

V transforms in the adjoint rep.

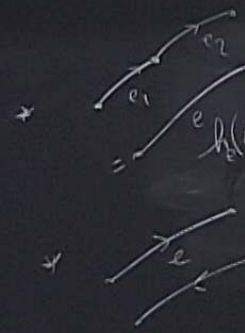
So we define the // transport of V by $\boxed{V(s) = h(s)\, V(0)\, h(s)^{-1}}$ $h(s)$ holonomy from $\gamma(0) = x$ to $\gamma(s)$

using ②

$$= \left[\frac{dh(\sigma)}{d\sigma} h^{-1}, h V h^{-1}\right] + \left[\dot{\sigma}^a A_a, h V h^{-1}\right]$$

$$\Rightarrow \boxed{\frac{dh}{d\sigma} = -\dot{\sigma}^a A_a(\gamma(\sigma))\, h(\sigma)}$$

differential eq with initial condition $h(0) = \mathbb{1}$



$$h_e(A) = h_{e_1}[A]\, h_{e_2}[A]$$

$$h_{e^{-1}}[A] = h_e^{-1}[A]$$

step by step

$$h(\sigma) = \mathbb{1} - \int_0^\sigma \dot{\sigma}\, A_a(\gamma(\sigma))\, h(\sigma')\, d\sigma'$$

↳ iterate this functional solution

$$\Rightarrow h_\gamma(\sigma) = \mathbb{P}\exp\left[-\int_\gamma A\right] = \sum_{n=0}^{\infty} (-1)^n \int_0^\sigma d\sigma_n \int_0^{\sigma_n} d\sigma_{n-1} \cdots \int_0^{\sigma_2} d\sigma_1\, A(\sigma_n)\, A(\sigma_{n-1}) \cdots A(\sigma_2)\, A(\sigma_1)$$

where $A(\sigma) = \dot{\gamma}^a(\sigma)\, A_a(\gamma(\sigma))$