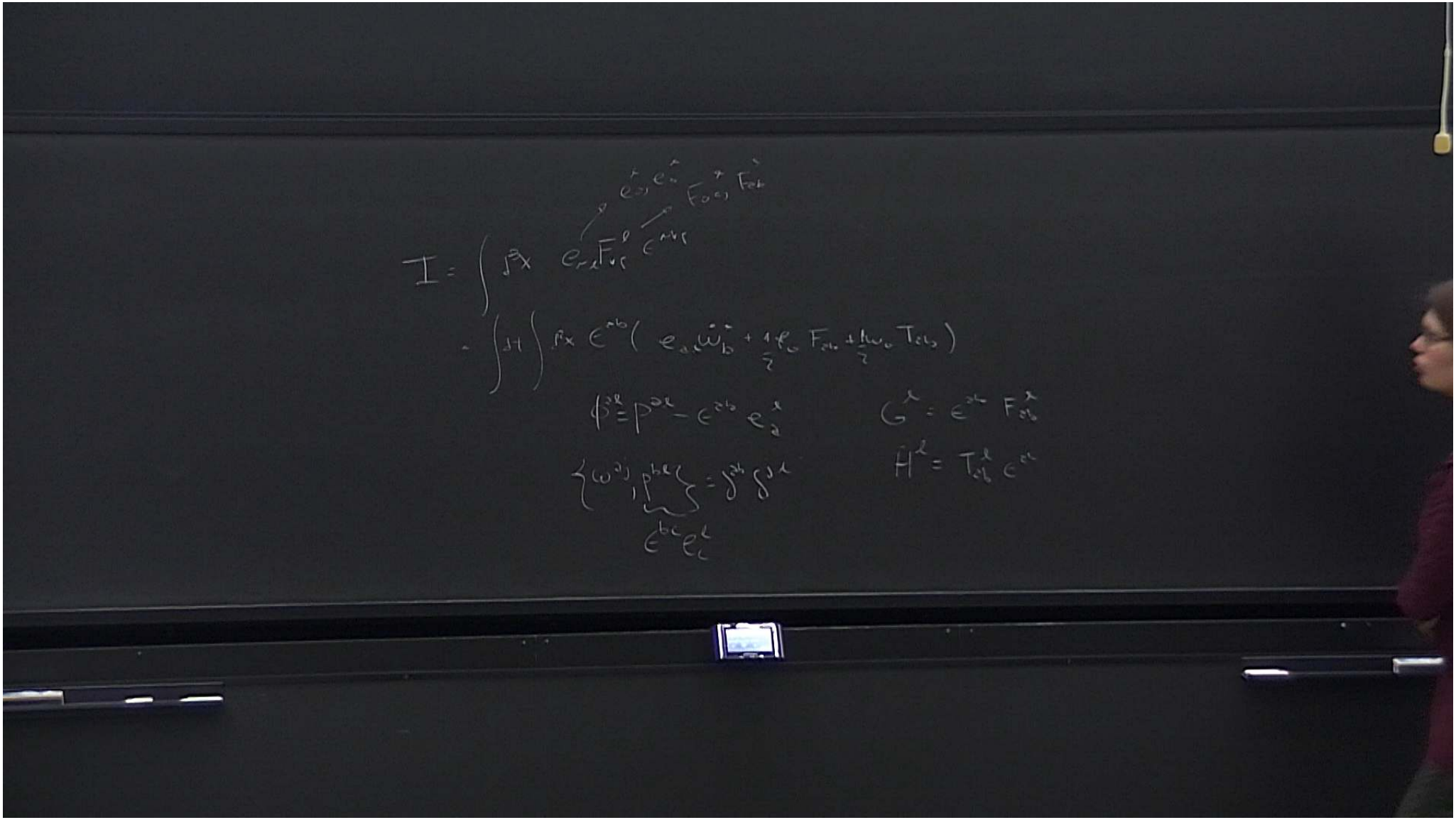
Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 6

Date: Mar 27, 2017 10:15 AM

URL: http://pirsa.org/17030073

Abstract:



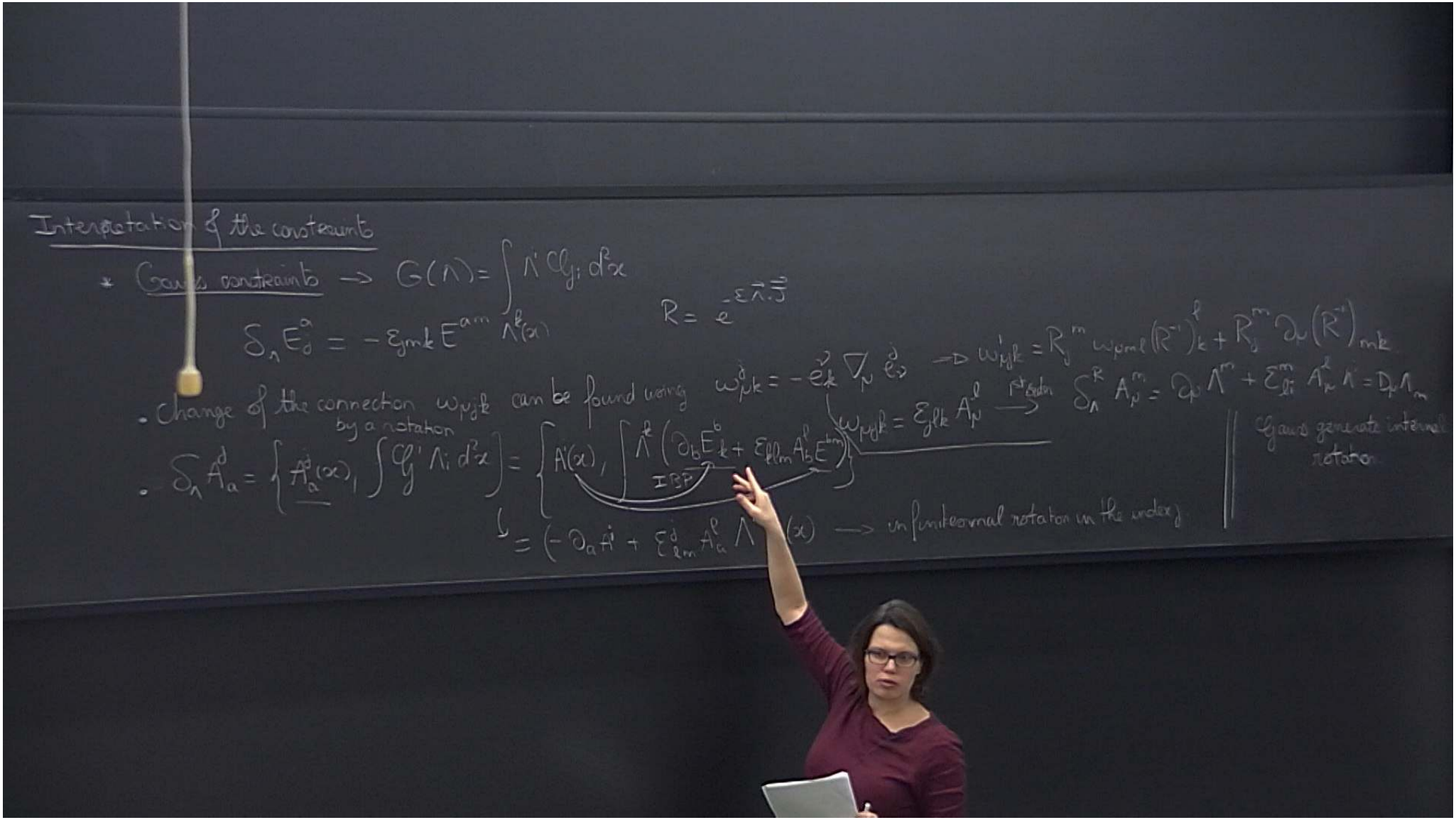
$$e^{\mu}_{a}\, e^{\nu}_{b} \qquad F^{\mu}_{a}\, F^{\nu}_{b}$$

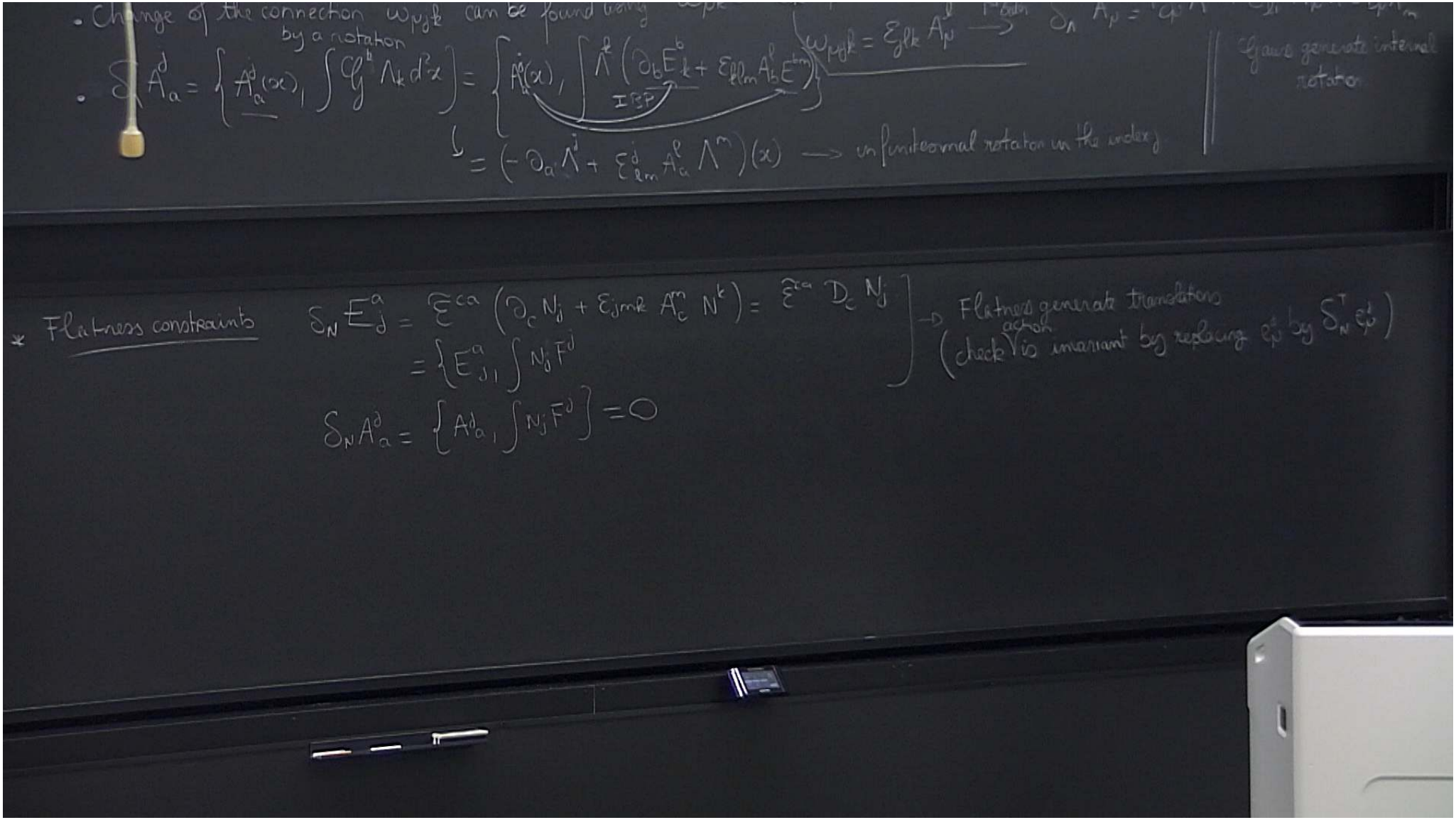
$$I = \int d^3x \; e_{\mu l} F^{l}_{\nu\rho}\, \epsilon^{\mu\nu\rho}$$

$$= \int dt \int d^3x \; \epsilon^{ab} \left( e_{a\alpha} \dot{\omega}^{\alpha}_{b} + \frac{1}{2} \varphi_{a} F_{ab} + \frac{1}{2} \omega_{a} T_{ab} \right)$$

$$\phi^{al} = p^{al} - \epsilon^{ab} e^{l}_{a} \qquad G^{l} = \epsilon^{ab} F^{l}_{ab}$$

$$\{\omega^{aj}, \underbrace{p^{bl}}_{\epsilon^{bc} e^{l}_{c}}\} = \delta^{ab} \delta^{jl} \qquad H^{l} = T^{l}_{ab}\, \epsilon^{ab}$$


Interpretation of the constraints
* Gauss constraints → $G(\Lambda)=\int \Lambda^i \mathcal{G}_i \, d^3x$
$\delta_\Lambda E^a_i = -\varepsilon_{jmk} E^{am} \Lambda^k(x)$
$R = e^{-\varepsilon \vec{\Lambda}\cdot\vec{J}}$
$\omega^i_{\mu jk} = R^m_j \omega_{\mu ml} (R^{-1})^l_k + R^m_j \partial_\mu (R^{-1})_{mk}$
- Change of the connection $\omega_{\mu jk}$ can be found using $\omega^i_{\mu k} = -e^{\nu}_k \nabla_\mu e^i_\nu$ → $\omega_{\mu jk} = \varepsilon_{jlk} A^l_\mu$ → $\delta_\Lambda A^m_\nu = \partial_\nu \Lambda^m + \varepsilon^m_{li} A^l_\nu \Lambda^i = D_\nu \Lambda_m$
by a rotation
- $\delta_\Lambda A^d_a = \{ A^b_a(x), \int \mathcal{G}^i \Lambda_i \, d^3x \} = \{ A^i(x), \int \Lambda^k (\partial_b E^b_k + \varepsilon_{klm} A^l_b E^{bm}) \}$
IBP
↳ $= (-\partial_a \Lambda^i + \varepsilon^d_{lm} A^l_a \Lambda$ (x) → infinitesimal rotation in the index
$\mathcal{G}_i$ are generate internal rotation


Change of the connection $\omega_{\mu jk}$ can be found
by a rotation
$\delta A^j_a = \{A^j_a(x), \int \mathcal{G}^k \Lambda_k d^3x\} = \{A^j_a(x), \int \Lambda^k (\partial_b E^b_k + \varepsilon_{klm} A^l_b E^{bm})\}$
IBP
$= (-\partial_a \Lambda^j + \varepsilon^j_{lm} A^l_a \Lambda^m)(x)$ → infinitesimal rotation in the index j
$\omega_{\mu jk} = \varepsilon_{jlk} A^l_\mu$
Gauss generate internal rotation
* Flatness constraints
$\delta_N E^a_j = \tilde{\varepsilon}^{ca} (\partial_c N_j + \varepsilon_{jmk} A^m_c N^k) = \tilde{\varepsilon}^{ca} D_c N_j$
$= \{E^a_j, \int N_j F^j$
$\delta_N A^j_a = \{A^j_a, \int N_j F^j\} = 0$
→ Flatness generate translations
(check V is invariant by replacing $\xi^a_\mu$ by $\delta^\tau_N \xi^a_\mu$)

$\delta_N A^a_a = \{ A^a_a , \int N_j F^j \} = 0$

* Where are the diffeo ?

↳ $\delta^D_v e^a_\mu = \mathcal{L}_v e^a_\mu = ?\ (v^\nu \partial_\nu e^a_\mu + e^a_\nu \partial_\mu v^\nu)$

$\delta^D_v A^a_\mu = \mathcal{L}_v A^a_\mu = v^\nu \partial_\nu A^a_\mu + A^a_\nu \partial_\mu v^\nu$

| symmetry of the action

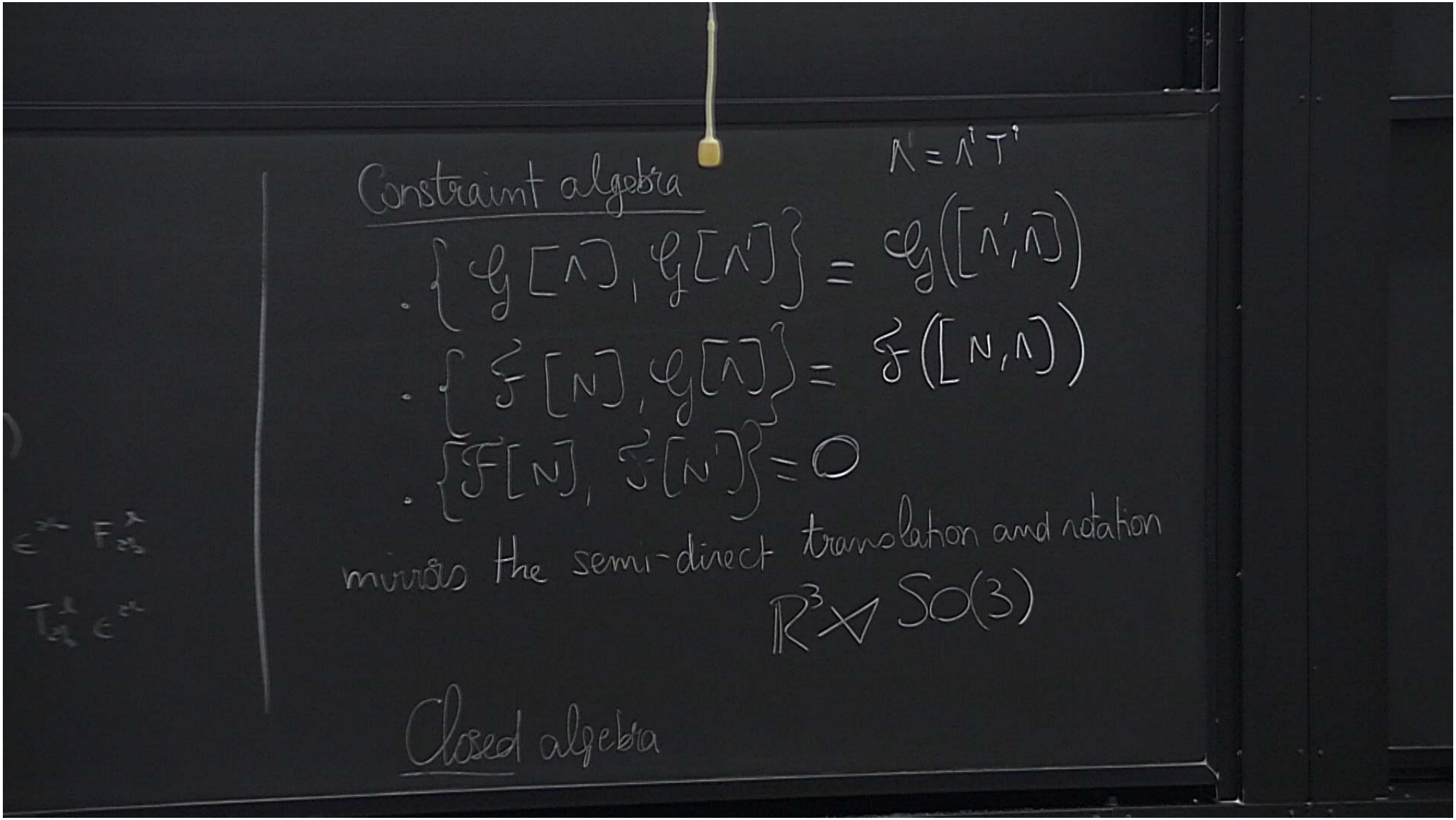
→ Symmetries are not indpt → relation among symmetry generators.

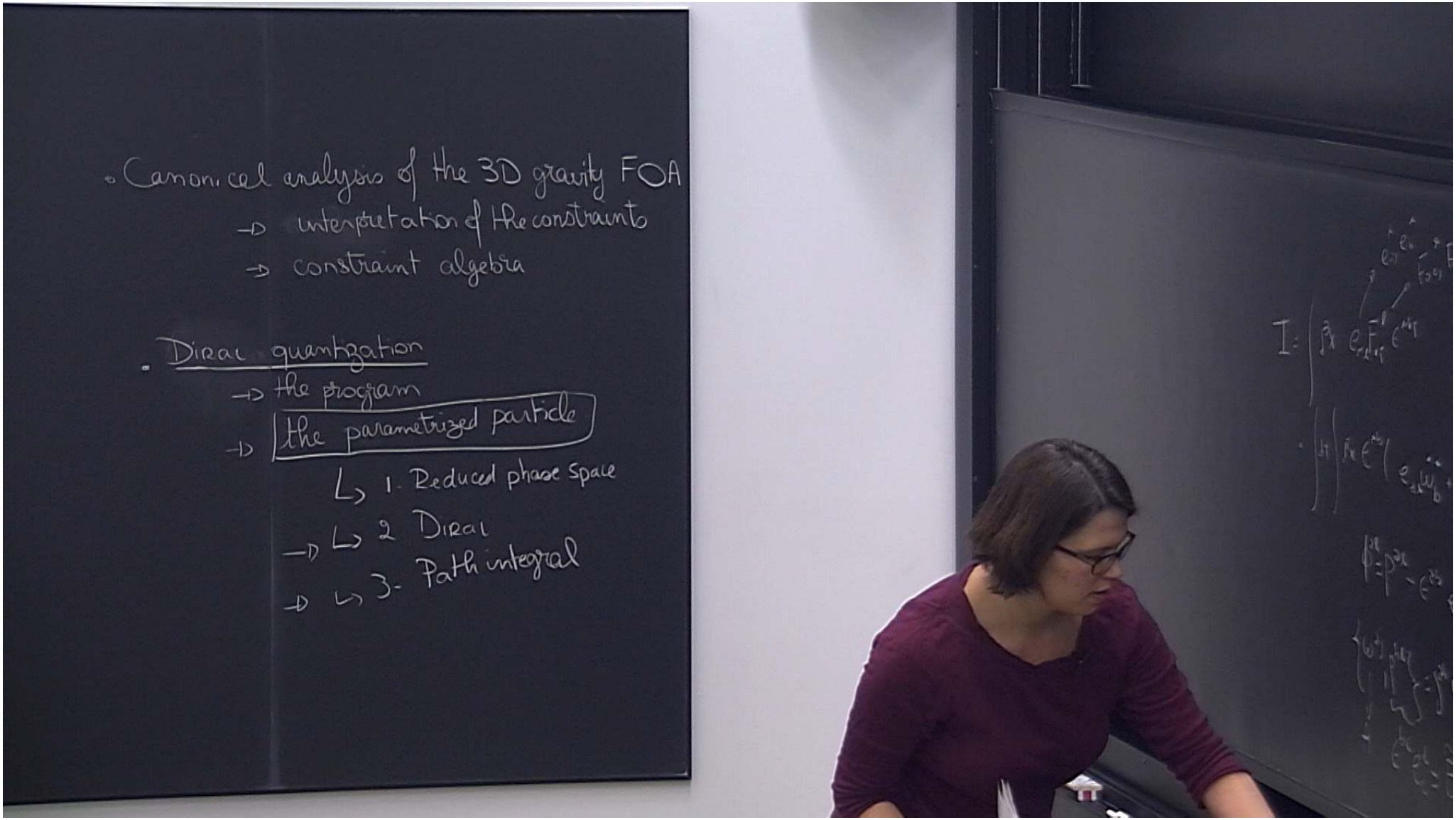
- $\delta^R_{\omega(v)} e^a_\mu + \delta^T_{e(v)} e^a_\mu = \delta^D_v e^a_\mu + T^i_{\mu\nu} v^\nu$
- $\delta^R_{\omega(v)} \omega + \delta^T_{e(v)} \omega = \delta^D_v \omega^a_\mu + F^i_{\mu\nu} v^\nu$

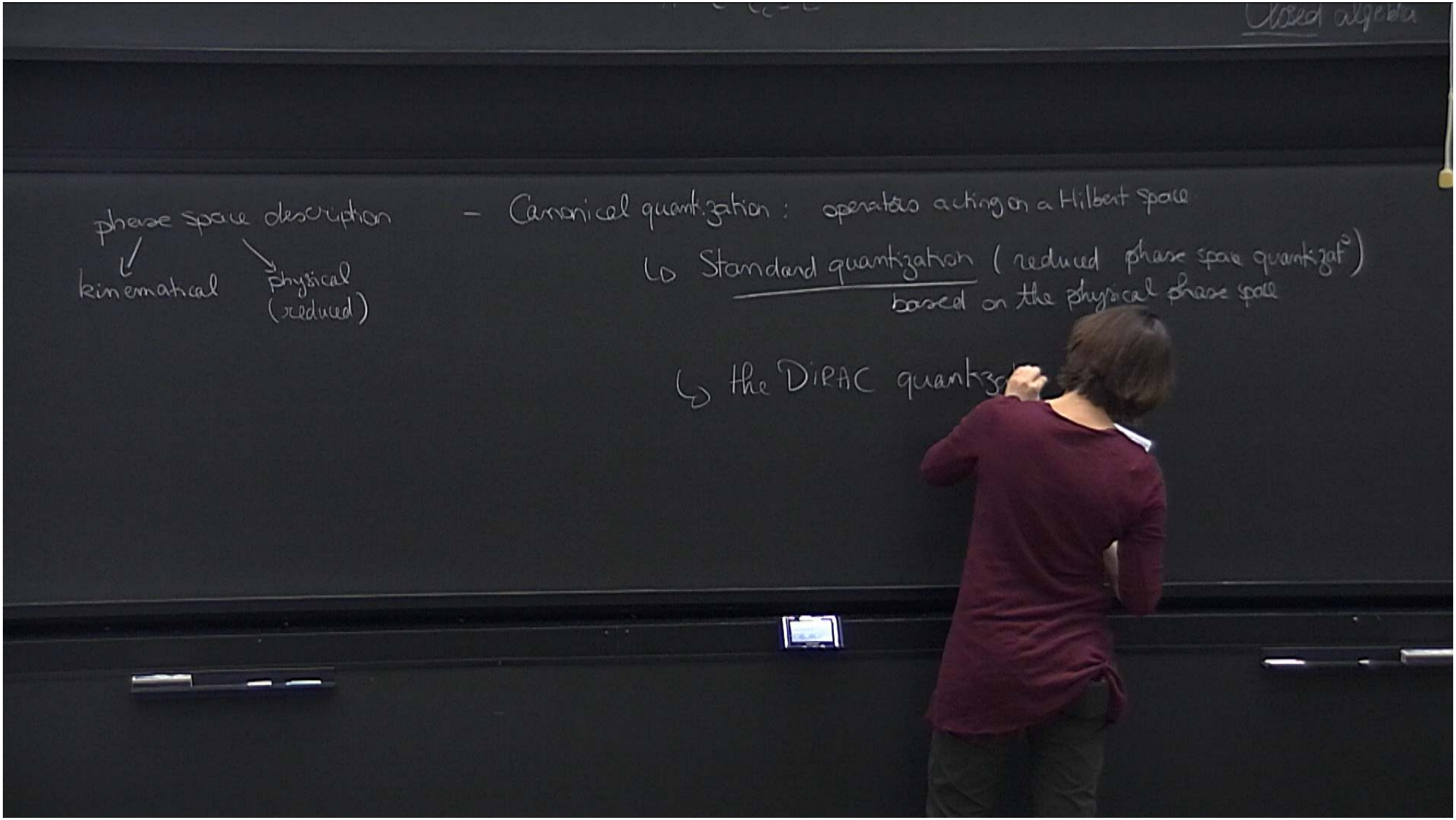
$\omega(v) \Rightarrow \omega^i_\nu v^\nu = \Lambda^i$

$e(v) \Rightarrow e^i_\nu v^\nu = N^i$

⇒ 3D gravity very simple. Diffeo can be replaced by a translation symmetry.


Constraint algebra
$\Lambda' = \Lambda^i T^i$
$\{\mathcal{G}[\Lambda], \mathcal{G}[\Lambda']\} = \mathcal{G}([\Lambda', \Lambda])$
$\{\mathcal{F}[N], \mathcal{G}[\Lambda]\} = \mathcal{F}([N, \Lambda])$
$\{\mathcal{F}[N], \mathcal{F}[N']\} = 0$
mirrors the semi-direct translation and rotation
$\mathbb{R}^3 \rtimes SO(3)$
Closed algebra


• Canonical analysis of the 3D gravity FOA
→ interpretation of the constraints
→ constraint algebra
• Dirac quantization
→ the program
→ the parametrized particle
↳ 1. Reduced phase space
→ ↳ 2 Dirac
→ ↳ 3. Path integral


Closed algebra
phase space description
kinematical
physical (reduced)
– Canonical quantization: operators acting on a Hilbert space
↳ Standard quantization (reduced phase space quantizat°)
based on the physical phase space
↳ the DIRAC quantiz

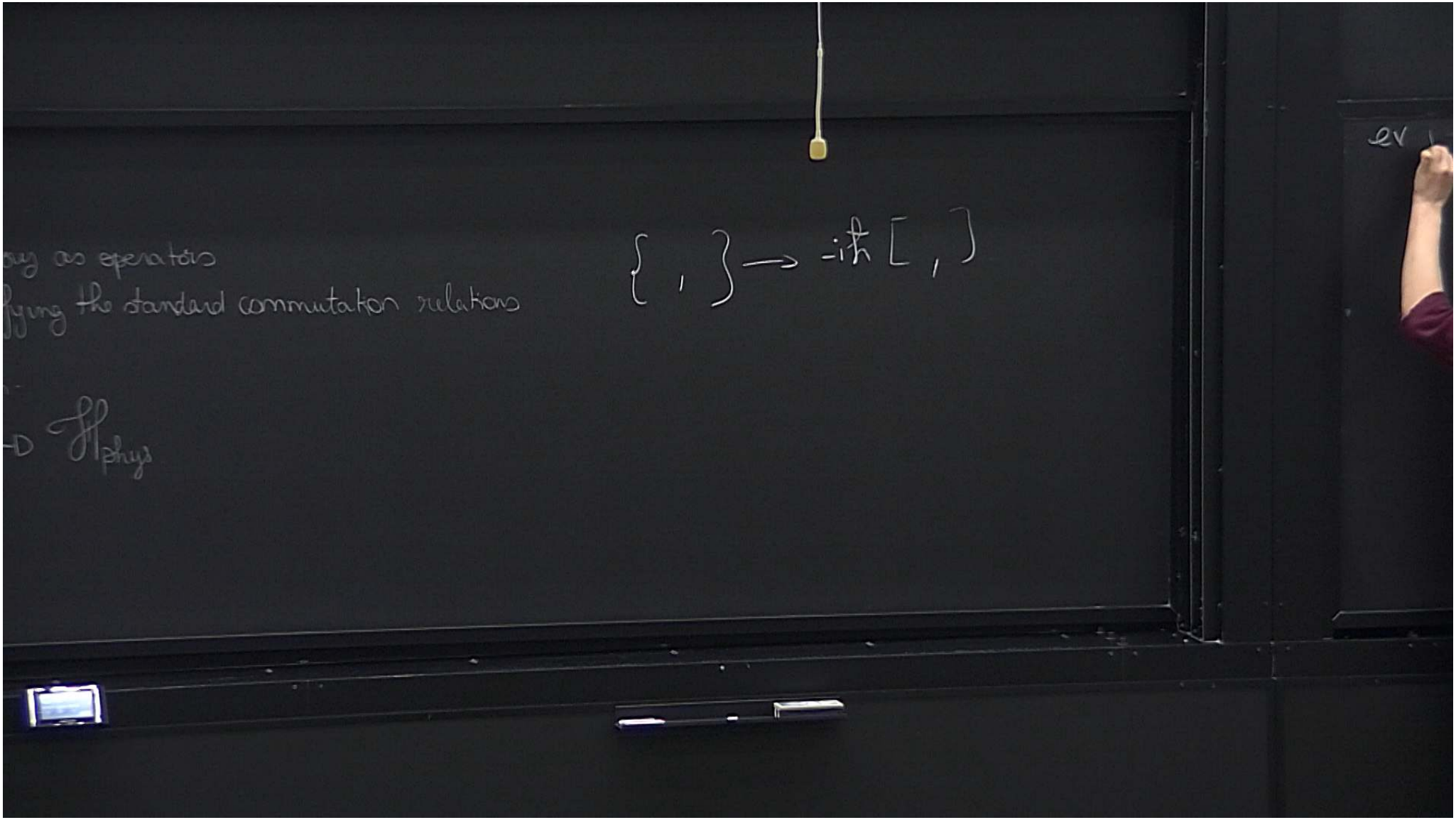
## The Dirac program

(i) Find a representation of the phase space variables of the theory as operators in a kinematical Hilbert space $\mathcal{H}_{kin}$ satisfying the standard commutation relations

(ii) Promote the constraints to (self-adjoint) operators in $\mathcal{H}_{kin}$.

(iii) Space of solutions of the constraints + inner product $\longrightarrow$ $\mathcal{H}_{phys}$

(iv) Find a (complete) set of gauge invariant observables


ay as operators
fying the standard commutation relations
{ , } → -ih [ , ]
ev

ex Parametrized particle

(i) kinematical phase space $(q, t, p_q, p_t)$ 1d

$$\mathcal{H}_{kin} = \mathcal{L}^2(\mathbb{R}^2)$$

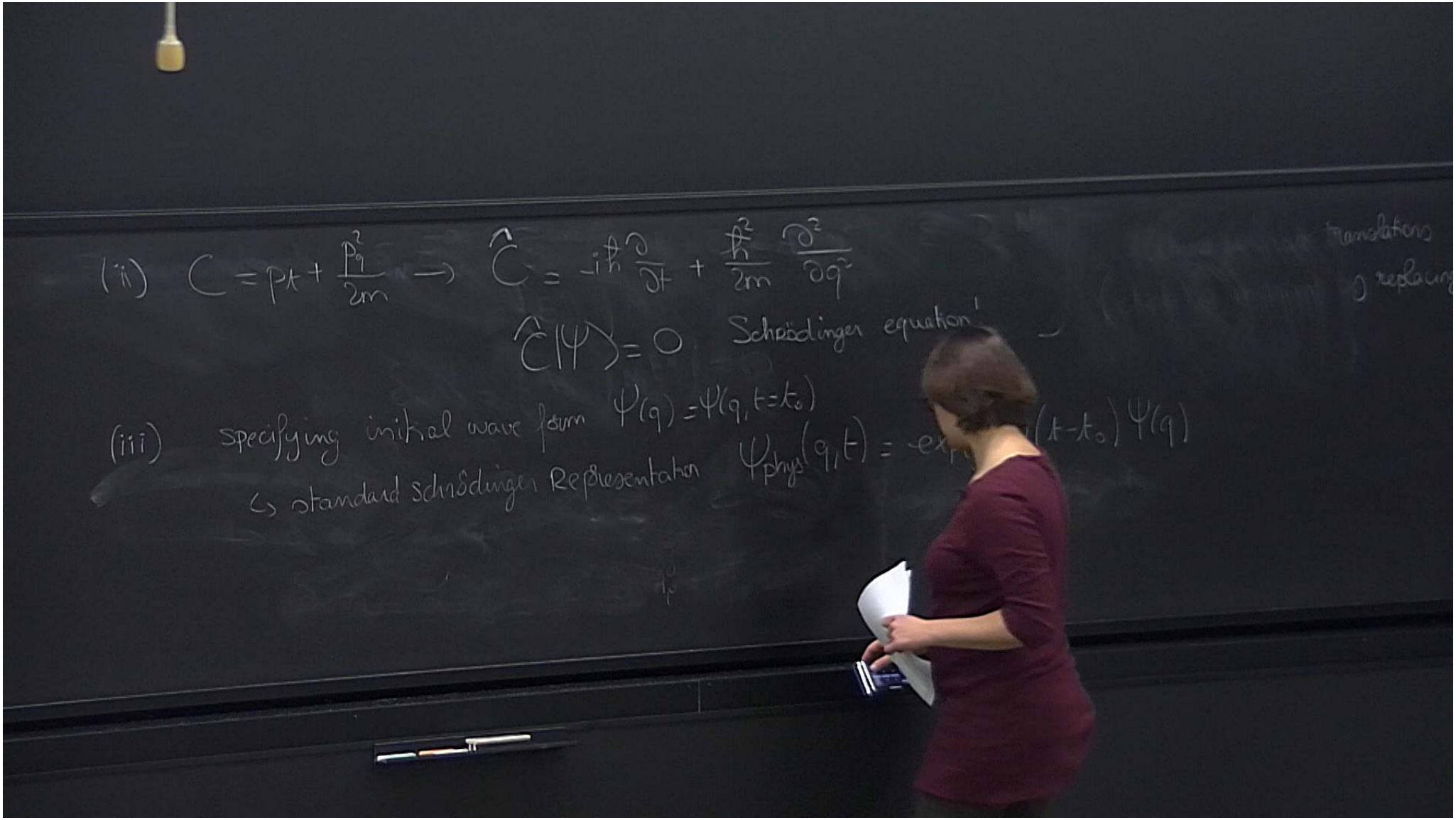
wave functions $\psi(q,t)$

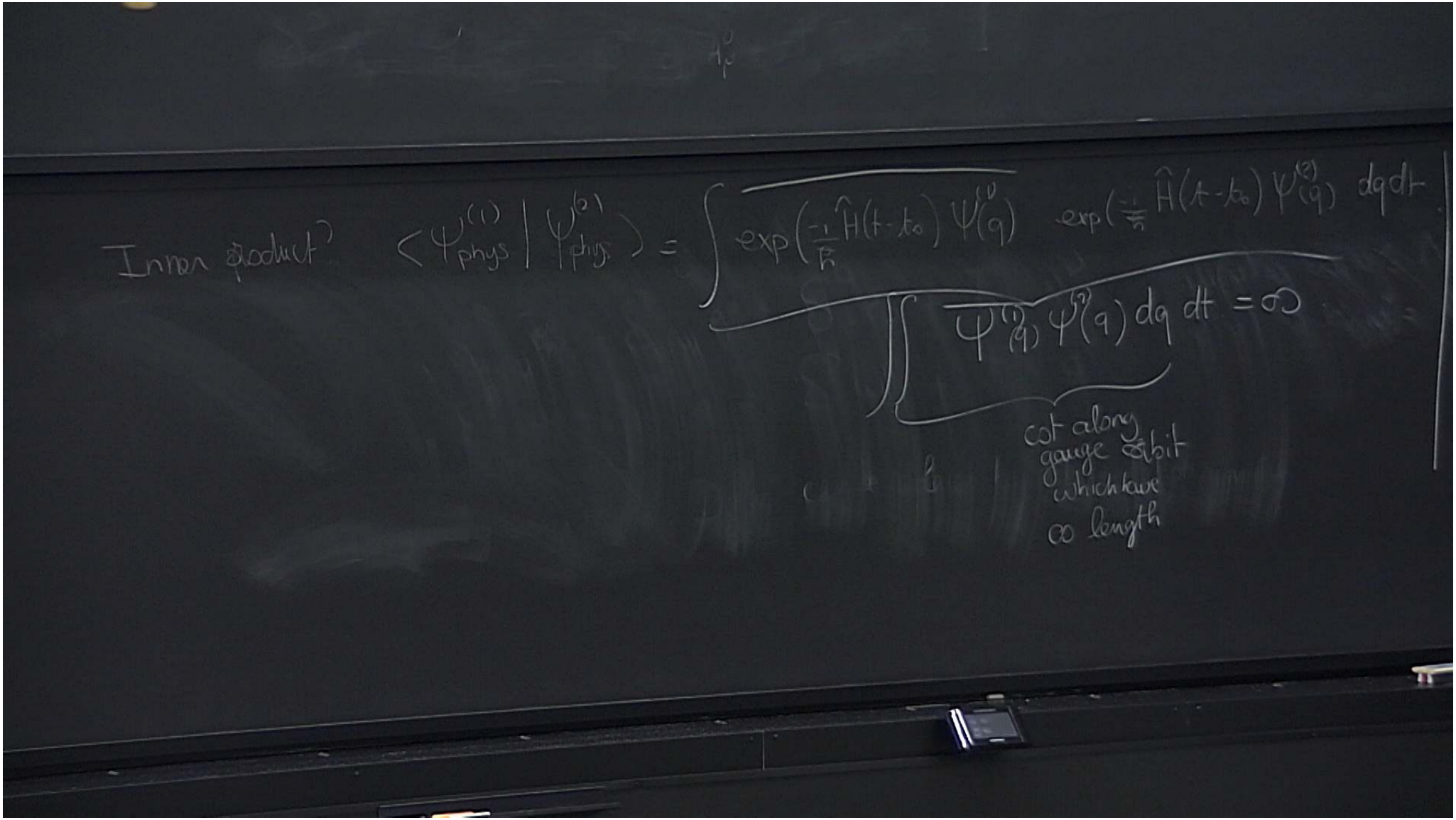
+ inner product $\langle \phi | \psi \rangle = \int dq\, dt\, \overline{\phi(q,t)}\, \psi(q,t)$

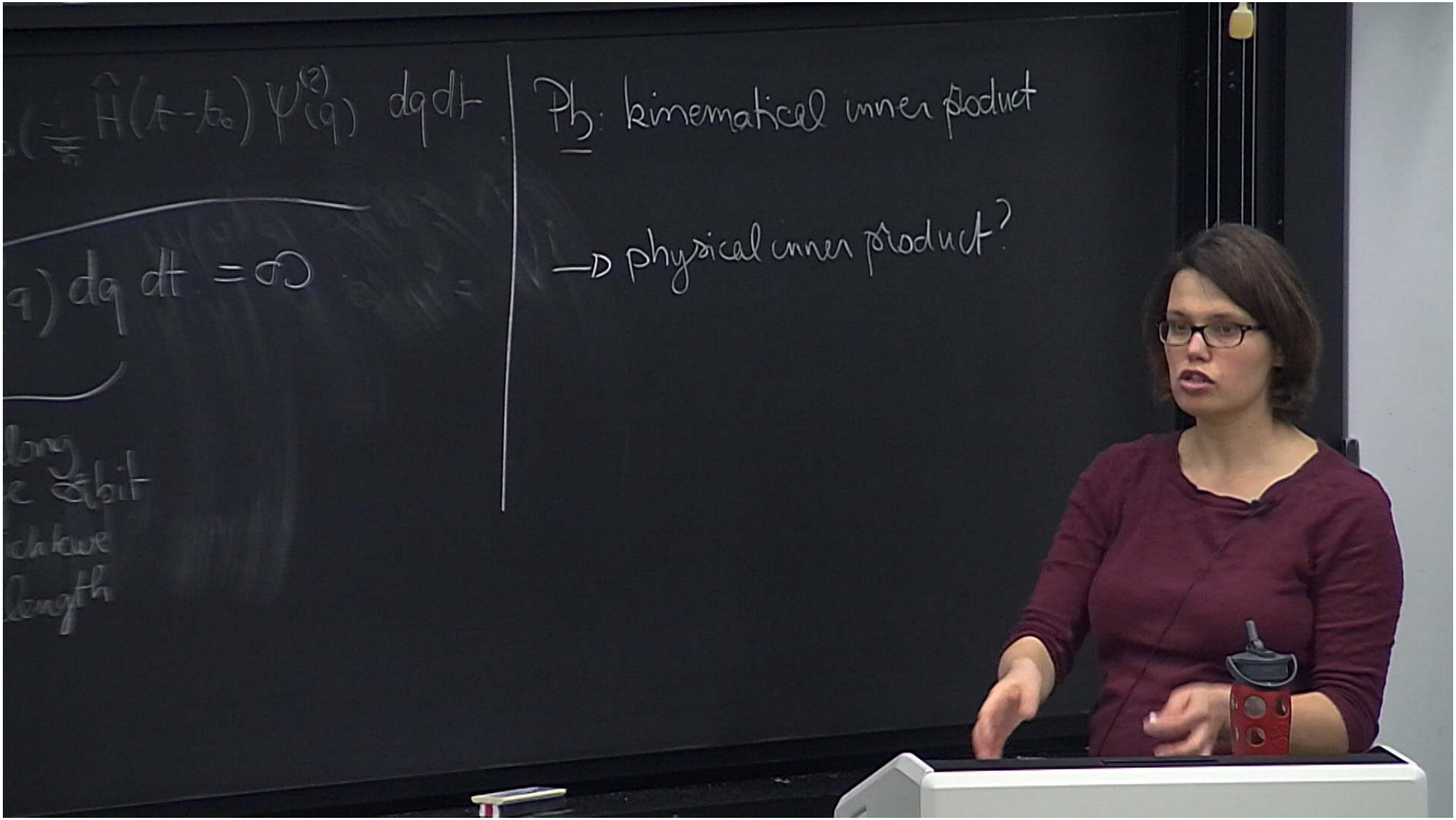
↳ $q \longrightarrow \hat{q}\psi = q\psi$ | $p_q \longrightarrow \hat{p}_q = -i\hbar \frac{\partial}{\partial q}$

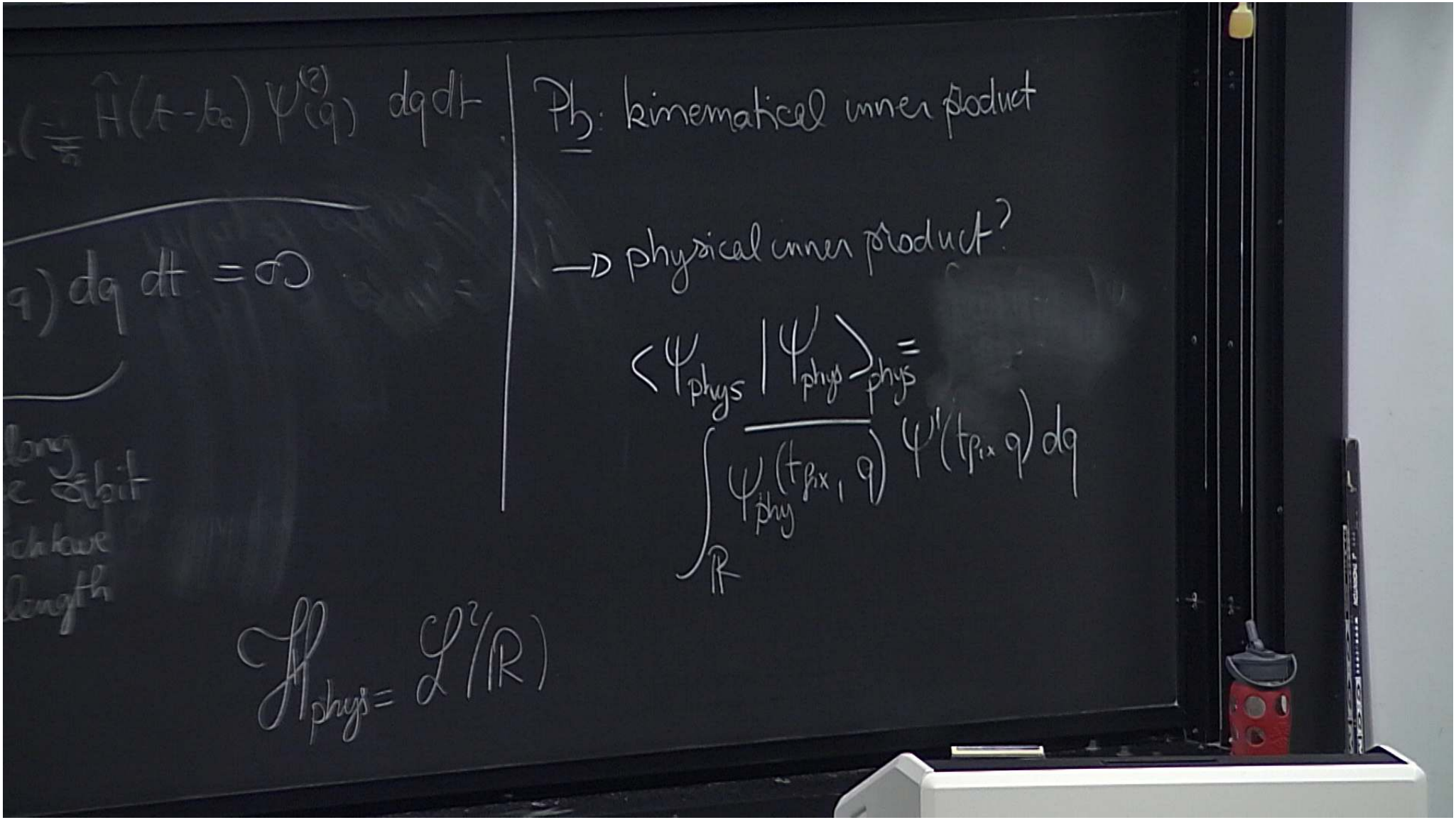
$t \longrightarrow \hat{t}\psi = t\psi$ | $p_t \longrightarrow \hat{p}_t = -i\hbar \frac{\partial}{\partial t}$

check that commutators reproduce poisson brackets (modulo $i\hbar$)


(ii) $C = pt + \frac{p_q^2}{2m} \rightarrow \hat{C} = -i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}$
$\hat{C}|\psi\rangle = 0$ Schrödinger equation
(iii) specifying initial wave form $\psi(q) = \psi(q, t=t_0)$
↳ standard schrödinger Representation $\psi_{phys}(q,t) = \exp$ $(t-t_0)\,\psi(q)$
translations
replacing


Inner product? $\langle \psi^{(1)}_{phys} | \psi^{(2)}_{phys} \rangle = \int \overline{\exp\left(\frac{-1}{\hbar}\hat{H}(t-t_0)\psi^{(1)}(q)\right)} \exp\left(\frac{-1}{\hbar}\hat{H}(t-t_0)\psi^{(2)}(q)\right) dq\,dt$
$\int \overline{\psi^{(1)}(q)}\,\psi^{(2)}(q)\,dq\,dt = \infty$
cst along gauge orbit which have $\infty$ length


Pb: kinematical inner product
→ physical inner product?


Pb: kinematical inner product
→ physical inner product?
$\langle \psi_{phys} | \psi_{phys} \rangle_{phys} =$
$\int_{\mathbb{R}} \overline{\psi_{phy}(t_{fix}, q)}\, \psi'(t_{fix}, q)\, dq$
$\mathcal{H}_{phys} = \mathcal{L}^2(\mathbb{R})$

Dirac program

(i) Find a representation of the phase space variables of the theory as operators in a kinematical Hilbert space $\mathcal{H}_{kin}$ satisfying the standard commutation relations

(ii) Promote the constraints to (self-adjoint) operators in $\mathcal{H}_{kin}$.

(iii) Space of solutions of the constraints + inner product $\longrightarrow$ $\mathcal{H}_{phys}$

(iv) Find a (complete) set of gauge invariant observables $\longrightarrow$ $F_q \to \hat{F}_q$, $F_{p_q} \to \hat{F}_{p_q}$