Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 4

Date: Mar 23, 2017 10:15 AM

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Abstract:

<u>Aim</u> : canonical analysis of the 1st-order action for 3D gravity.

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- GR. $\longrightarrow$ constrained system

  Constraint $\phi(p_i, q_i) = 0$ hold at every time

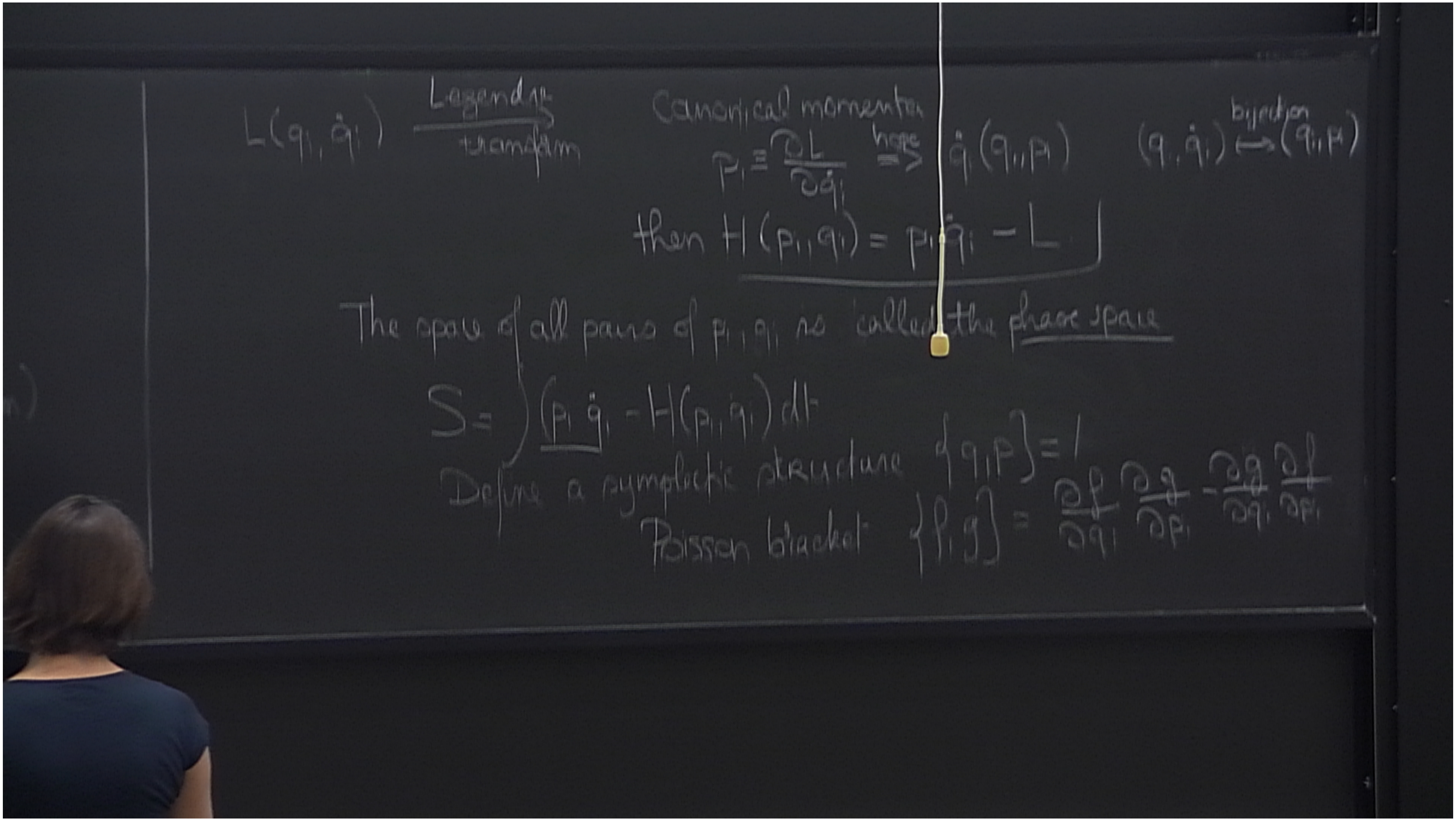
  $\hookrightarrow$ conserved quantity

- Aim : canonical analysis of the 1^{st} order action for 3D gravity.

- GR. ⟶ constrained system

Constraint $\phi(p_i, q_i) = 0$ hold at every time

↳ conserved quantity ⟶ symmetries
(Noether's theorem)
see Tutorial 2.


$L(q_i, \dot{q}_i)$ $\xrightarrow[\text{transform}]{\text{Legendre}}$
Canonical momenta
$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \overset{\text{hope}}{\Longrightarrow} \dot{q}_i(q_i, p_i)$
$(q_i, \dot{q}_i) \overset{\text{bijection}}{\longleftrightarrow} (q_i, p_i)$
then $H(p_i, q_i) = p_i \dot{q}_i - L$
The space of all pairs of p_i, q_i is called the phase space
$S = \int (p_i \dot{q}_i - H(p_i, q_i))\, dt$
Define a symplectic structure $\{q, p\} = 1$
Poisson bracket $\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$

The time evolutions is defined as the flow of the Hamiltonian.

$q_i(t)$
$p_i(t)$.

$$\begin{cases} \dot{q}_i = \{q_i, H\} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \{p_i, H\} = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

If constraints $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ is not unvertible. $\Rightarrow$ $\psi_\alpha(p_i, q_i) = 0$

$$H = p_i \dot{q}_i - L + \lambda_\alpha \psi_\alpha(p_i, q_i)$$

$$L = \frac{1}{2} m \left(v_1^2 + v_2^2 \right) - \frac{1}{2} q_3 \left(q_1^2 + q_2^2 - r^2 \right)$$

constraint $\psi_1(p_i, q_i) = p_3$

ex 1 $\quad L = \frac{1}{2} m \left(v_1^2 + v_2^2 \right) - \frac{1}{2} q_3 \left(q_1^2 + q_2^2 - r^2 \right) \longrightarrow$ Euler-Lagrange

- Primary constraint $\quad \psi_1(p_i, q_i) = p_3$
- Secondary constraint $\quad \psi_2 = \frac{1}{2}\left(q_1^2 + q_2^2 - r^2\right)$
- Tertiary constraint $\quad \dot{\psi}_2 = \{\psi_2, H\} = m^{-1}\left(p_1 q_1 + p_2 q_2\right)$
- $\dot{\psi}_3 = \{\psi_3, H\} = -q_3\left(q_1^2 + q_2^2\right) + m^{-1}\left(p_1^2 + p_2^2\right) \Rightarrow \psi_4 = q_3 - m^{-1} r^{-2}\left(p_1^2 + p_2^2\right)$
- $\dot{\psi}_4 = \{\psi_4, H\} = u + 2 m^{-1} r^{-2} q_3 \left(p_1 q_1 + p_2 q_2\right)$

ex1 $L = \frac{1}{2} m \left(v_1^2 + v_2^2\right) - \frac{1}{2} q_3 \left(q_1^2 + q_2^2 - r^2\right)$ $\rightarrow$ Euler-Lagrange

- $p_i = \frac{\partial L}{\partial v_i}$ $\quad p_3 = 0$

- Primary constraint $\boxed{\psi_1(p_i, q_i) = p_3}$ $\longrightarrow H = (p_i \dot{q}_i - L) + u\,\psi$

  $\{\psi_1, H\} \neq 0$

- Secondary constraint $\psi_2 = \frac{1}{2}\left(q_1^2 + q_2^2 - r^2\right)$
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ex1 $L = \frac{1}{2} m (v_1^2 + v_2^2) - \frac{1}{2} q_3 (q_1^2 + q_2^2 - r^2)$ → Euler-Lagrange

$\cdot\ p_i = \frac{\partial L}{\partial v_i}$ $\quad p_3 = 0$

$\cdot$ Primary constraint $\boxed{\psi_1(p_i, q_i) = p_3}$ ⟶ $H = (p_i \dot{q}_i - L) + u\psi$

$\{\psi_1, H\} \neq 0$

$\cdot$ Secondary constraint $\psi_2 = \frac{1}{2}(q_1^2 + q_2^2 - r^2)$ $\quad \{\psi_2, H\} \neq 0$

$\cdot$ Tertiary constraint $\dot{\psi}_2 = \{\psi_2, H\} = m^{-1}(p_1 q_1 + p_2 q_2)$

$\cdot\ \dot{\psi}_3 = \{\psi_3, H\} = -q_3 \underbrace{(q_1^2 + q_2^2)}_{\psi_2} + m^{-1}(p_1^2 + p_2^2) \Rightarrow \psi_4 = q_3 - m^{-1} r^{-2} (p_1^2 + p_2^2)$

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$\dot{\psi}_3 = \{\psi_3, H\}$

$\approx$

$$L = \frac{1}{2} m \left(v_1^2 + v_2^2 \right) - \frac{1}{2} q_3 \left(q_1^2 + q_2^2 - r^2 \right) \longrightarrow \text{Euler-Lagrange}$$

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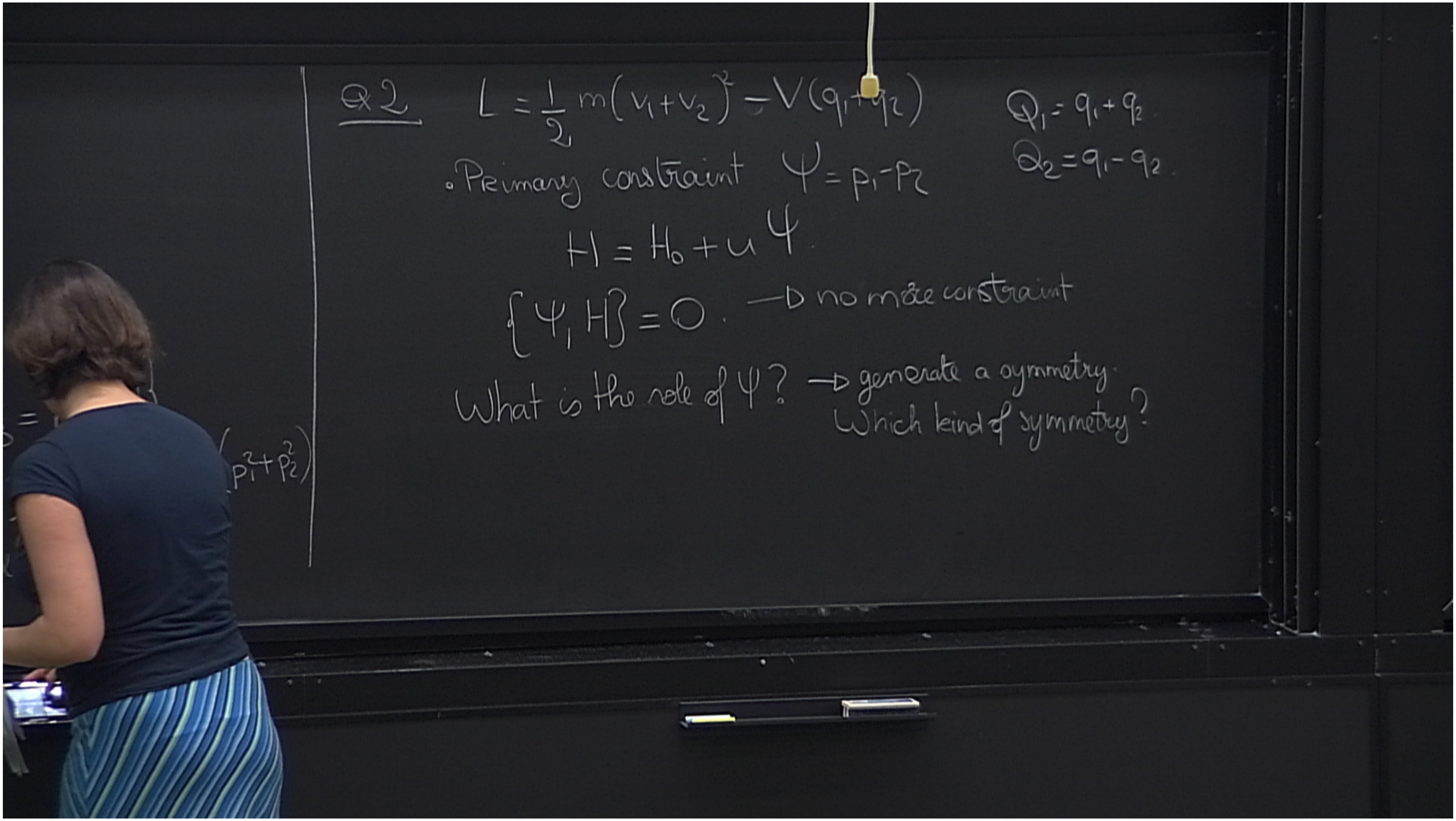
$$\dot{\psi}_4 = \{\psi_4, H\} = u + 2\, m^{-1} r^{-2} q_3 \underbrace{\left(p_1 q_1 + p_2 q_2\right)}_{\psi_3} \approx u$$

$$\dot{\psi}_3 = \{\psi_3, H\} \approx -q_3 + m^{-1} r^{-2} \left(p_1^2 + p_2^2\right)$$

Weakly


Q2 $L = \frac{1}{2} m(v_1+v_2)^2 - V(q_1+q_2)$
$Q_1 = q_1 + q_2$
$Q_2 = q_1 - q_2$
• Primary constraint $\Psi = p_1 - p_2$
$H = H_0 + u\Psi$
$\{\Psi, H\} = 0$ → no more constraint
What is the role of Ψ? → generate a symmetry.
Which kind of symmetry?
$(p_1^2 + p_2^2)$

flow generated by Ψ in the phase space

$$\delta q_1 = u\{q_1, \Psi\} = u \qquad \delta q_2 = u\{q_2, \Psi\} = -u$$

$$\delta p_1 = u\{p_1, \Psi\} = 0 \qquad \delta p_2 = u\{p_2, \Psi\} = 0$$

→ leav… chan…

$$\{f, u\Psi\} = \{f, u\}\Psi + u\{f, \Psi\} \simeq u\{f, \Psi\}$$

$$\delta q_2 = u\{q_2, \Psi\} = -u$$
$$\delta p_2 = u\{p_2, \Psi\} = 0.$$

→ leave the "physical" quantities Q_1 invariant
changes the "internal" d.o.f Q_2.

$$\ldots\{f, \Psi\} \approx u\{f, \Psi\}$$

stinction between constraints

system with constraints ψ_α (primary + secondary . . .)

If there is a subset of constraints ϕ_a, whose brackets with all constraint weakly vanish

$\{\phi_a, \psi_\alpha\} \approx 0$ ϕ_a are the first class constra between constraints

m with constraints ψ_α (primary + secondary ...)

. If there is a subset of constraints ϕ_a, whose brackets with all constraint weakly vanishes

$\{\phi_a, \psi_\alpha\} \approx 0$ $\quad \phi_a$ are the first class constraints.

If we denote the remaining constraints by χ_m, then sub-matrix $\Delta_{mn} = \{\chi_m, \chi_n\}$ is in

(primary + secondary ...)

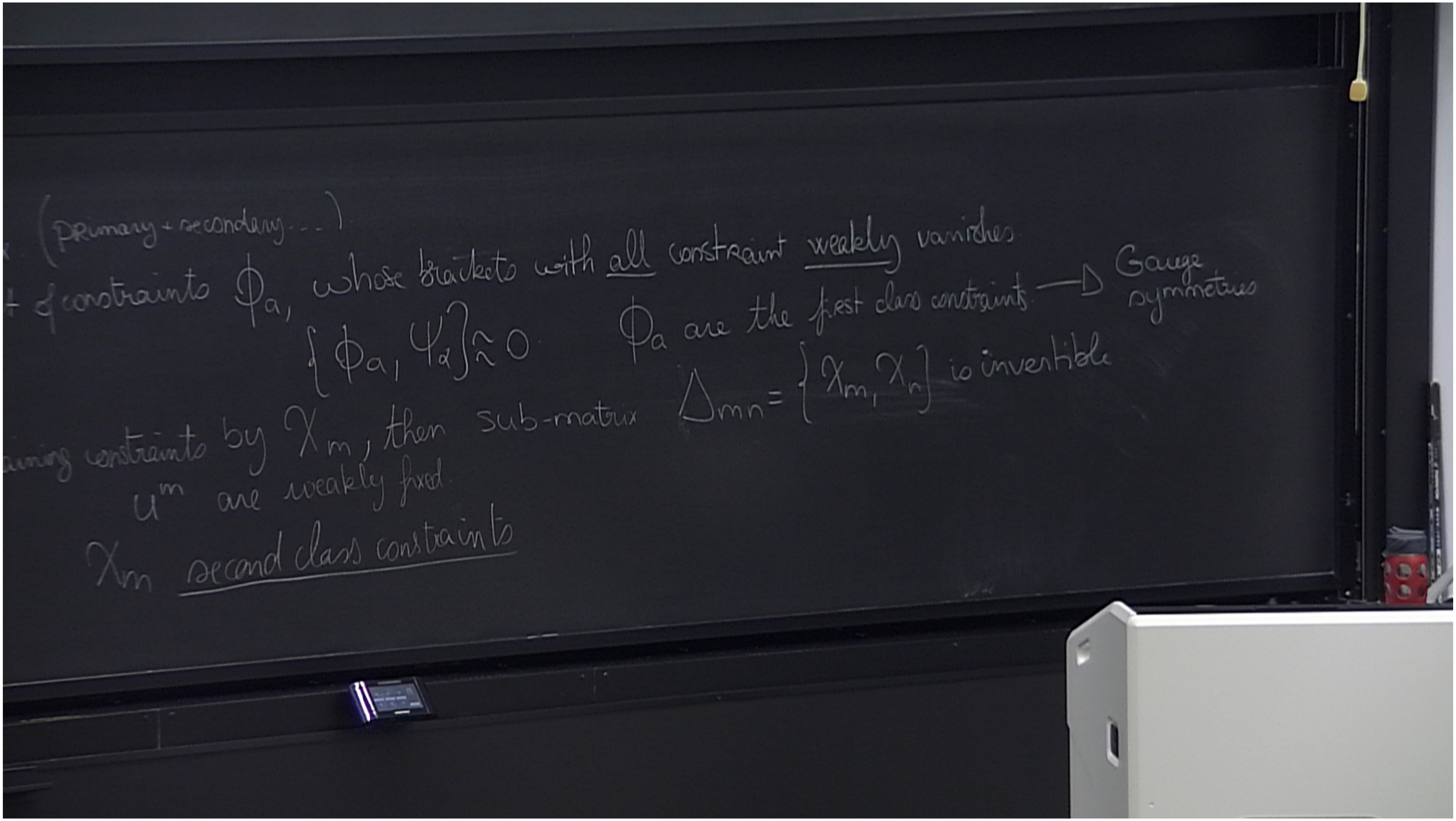
t of constraints ϕ_a, whose brackets with <u>all</u> constraint <u>weakly</u> vanishes

$\{\phi_a, \psi_\alpha\} \approx 0$ $\quad$ ϕ_a are the first class constraints $\longrightarrow$ Gauge symmetries

aining constraints by χ_m, then sub-matrix $\Delta_{mn} = \{\chi_m, \chi_n\}$ is invertible

u^m are weakly fixed

χ_m <u>second class constraints</u>


(primary + secondary ...)
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$\{\phi_a, \psi_\alpha\} \approx 0$
ϕ_a are the first class constraints → Gauge symmetries
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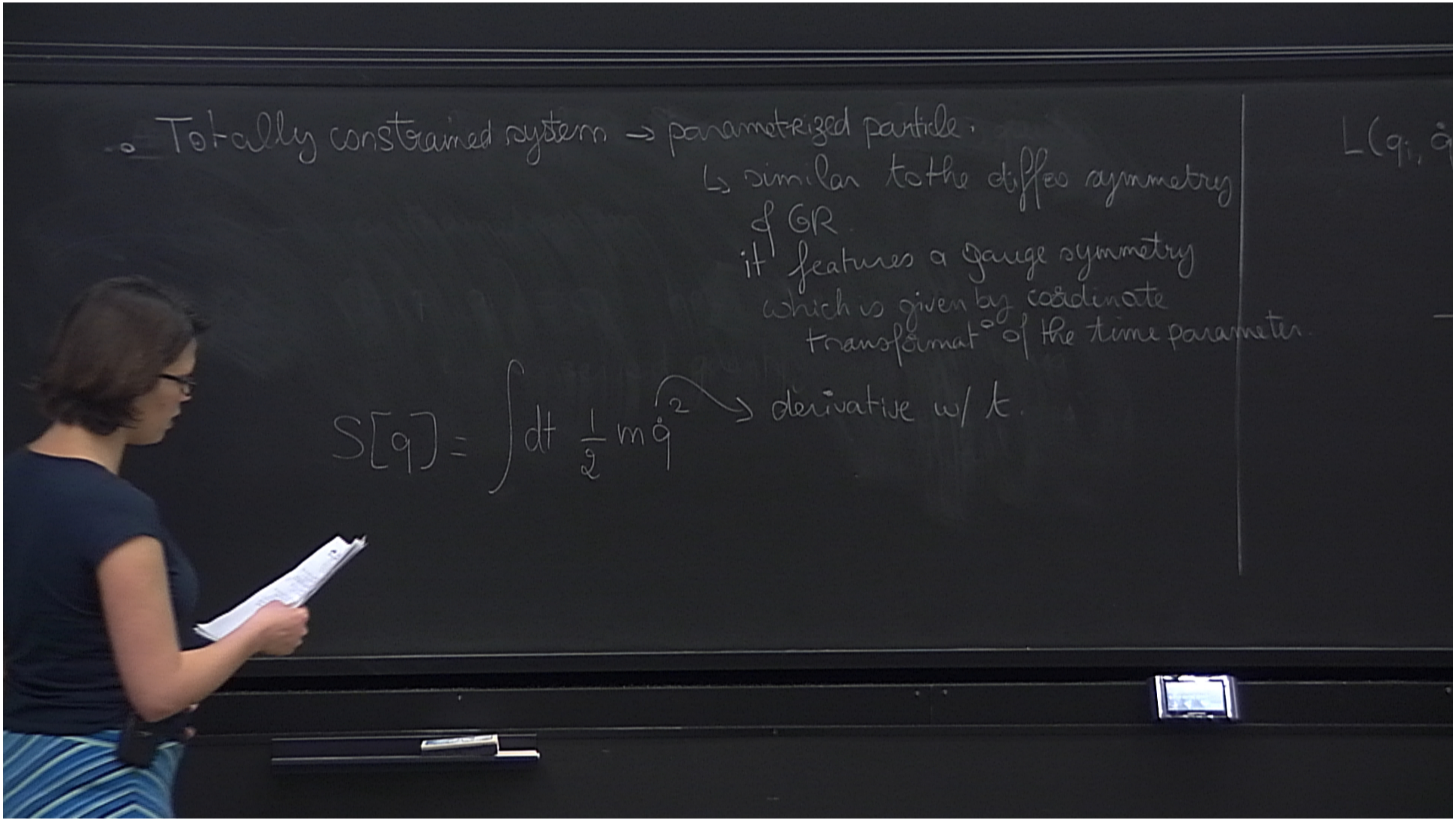
u^m are weakly fixed
χ_m second class constraints

- Totally constrained system → parametrized particle.
  - ↳ similar to the diffeo symmetry of GR

    it features a gauge symmetry which is given by coordinate transformat° of the time parameter.

$L(q_i, \dot{q}$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S[q] = ∫dt ½ m q̇² → derivative w/ t.
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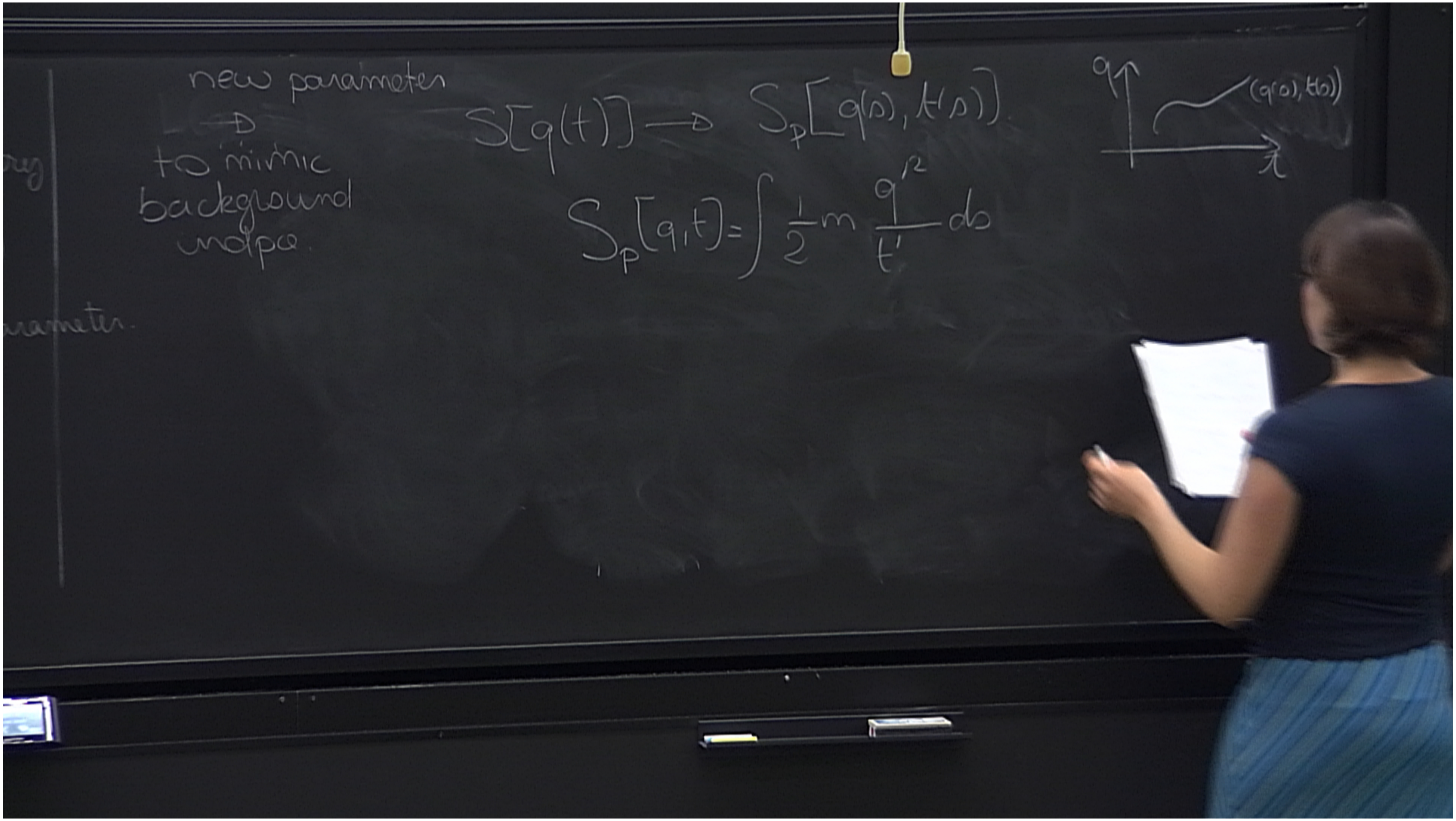
$$S[q] = \int dt\, \frac{1}{2} m \dot{q}^2$$

$\dot{q}$ → derivative w/ t.

$$\hookrightarrow q(\tau) = q + \frac{p}{m}(\tau - t)$$

$L(q_i, \dot{q}$


new parameter
to mimic background
S[q(t)] → S_p[q(s), t(s)]
S_p[q,t] = ∫ 1/2 m q'^2 / t' ds
(q(s), t(s))

Canonical analysis – Hamiltonian picture.

$$p_t = \frac{\delta S}{\delta t'} = -m\frac{(q')^2}{2(t')^2} \qquad p_q = \frac{\delta S}{\delta q'} = m\frac{q'}{t'}$$

Canonical analysis - Hamiltonian picture.

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Hamiltonian $\quad H = p_t t' + p_q q' - L$

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$\Rightarrow$ constraint $C = p_t + \frac{p_q^2}{2m} = 0$

<u>Hamiltonian</u> $H = p_t t' + p_q q' - L \qquad L = -t' p_t = \frac{p_q^2}{2m} t'$

$$= t' p_t + t' \frac{p_q^2}{2m} = t' C \equiv \text{totally constrained.}$$

Lagrange multiplier