Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 4

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Abstract:

Aim : canonical analysis of the 1st order action for 3D gravity.

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- GR. → constrained system

Constraint $\phi(p_i, q_i) = 0$ hold at every time

↳ conserved quantity

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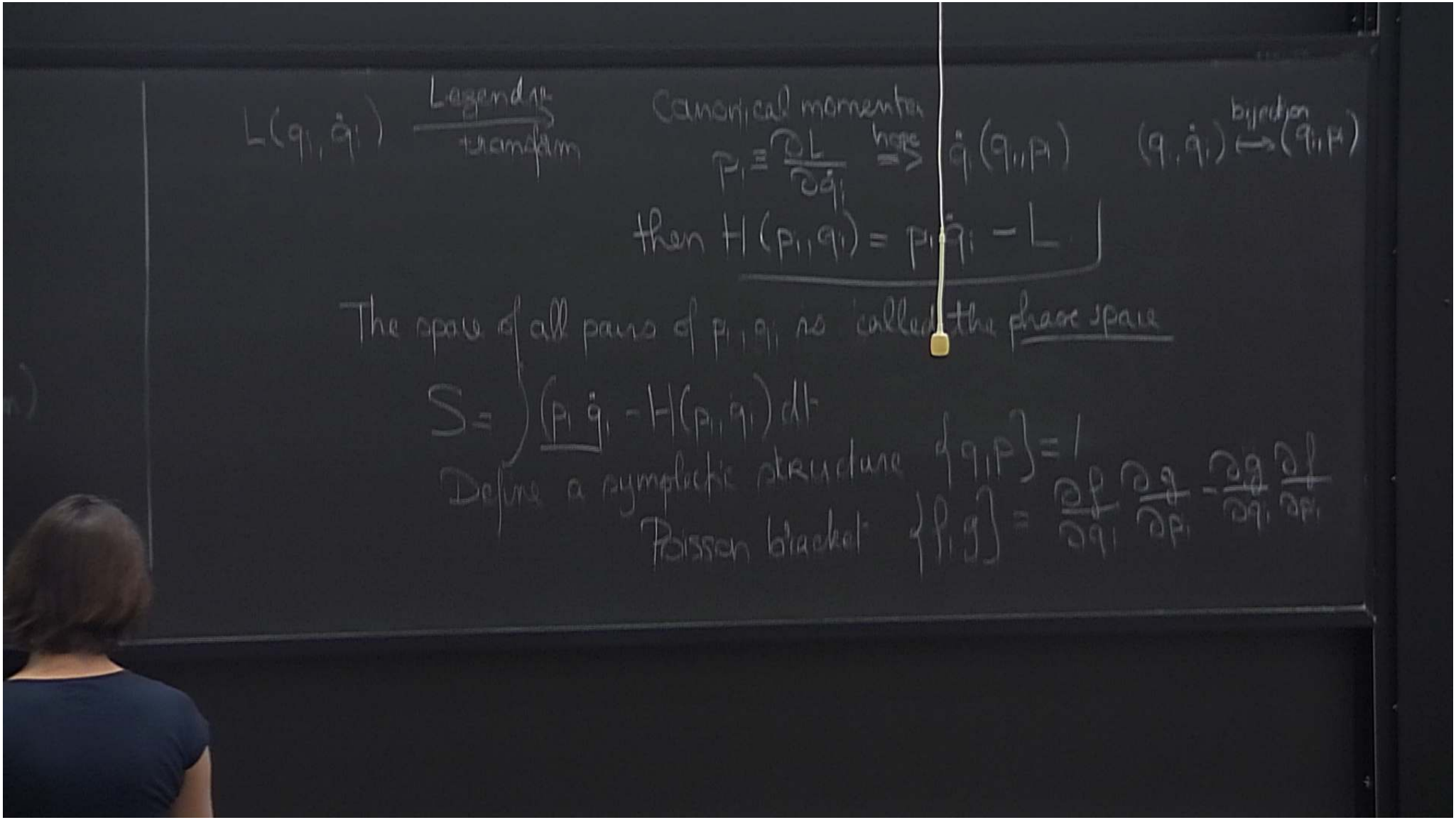
  ↳ conserved quantity → symmetries (Noether's theorem) see Tutorial 2.


$L(q_i, \dot{q}_i)$ Legendre transform
Canonical momenta
$p_i = \frac{\partial L}{\partial \dot{q}_i}$ hope $\Rightarrow$ $\dot{q}_i(q_i, p_i)$
$(q, \dot{q}_i) \xrightarrow{\text{bijection}} (q_i, p_i)$
then $H(p_i, q_i) = p_i \dot{q}_i - L$
The space of all pairs of p_i, q_i is called the phase space
$S = \int (p_i \dot{q}_i - H(p_i, q_i))\, dt$
Define a symplectic structure $\{q, p\} = 1$
Poisson bracket $\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}$

The time evolutions is defined as the flow of the Hamiltonian

$q_i(t)$
$p_i(t)$

$$\begin{cases} \dot{q}_i = \{q_i, H\} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \{p_i, H\} = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

If constraints $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ is not invertible $\Rightarrow \Psi_\alpha(p_i, q_i) = 0$

$$H = p_i \dot{q}_i - L + \lambda_\alpha \Psi_\alpha(p_i, q_i)$$

$$L = \frac{1}{2} m (v_1^2 + v_2^2) - \frac{1}{2} q_3 (q_1^2 + q_2^2 - r^2)$$

constraint $\Psi_1(p_i, q_i) = p_3$

ex 1 $L = \frac{1}{2} m (v_1^2 + v_2^2) - \frac{1}{2} q_3 (q_1^2 + q_2^2 - r^2)$ → Euler-Lagrange

- Primary constraint $\Psi_1(p_i, q_i) = p_3$
- Secondary constraint $\Psi_2 = \frac{1}{2}(q_1^2 + q_2^2 - r^2)$
- Tertiary constraint $\dot{\Psi}_2 = \{\Psi_2, H\} = m^{-1}(p_1 q_1 + p_2 q_2)$
- $\dot{\Psi}_3 = \{\Psi_3, H\} = -q_3(q_1^2 + q_2^2) + m^{-1}(p_1^2 + p_2^2)$ $\Rightarrow \Psi_4 = q_3 - m^{-1} r^{-2}(p_1^2 + p_2^2)$
- $\dot{\Psi}_4 = \{\Psi_4, H\} = u + 2 m^{-1} r^{-2} q_3 (p_1 q_1 + p_2 q_2)$

ex1 $L = \frac{1}{2}m(v_1^2 + v_2^2) - \frac{1}{2}q_3(q_1^2 + q_2^2 - r^2)$ → Euler-Lagrange

$\cdot p_i = \frac{\partial L}{\partial v_i}$ $\quad p_3 = 0$

· Primary constraint $\boxed{\Psi_1(p_i, q_i) = p_3}$ ⟶ $H = (p_i \dot{q}_i - L) + u\Psi$

· Secondary constraint $\Psi_2 = \frac{1}{2}(q_1^2 + q_2^2 - r^2)$ $\quad \{\Psi_1, H\} \neq 0$

· Tertiary constraint $\dot{\Psi}_2 = \{\Psi_2, H\} = m^{-1}(p_1 q_1 + p_2 q_2)$

· $\dot{\Psi}_3 = \{\Psi_3, H\} = -q_3(q_1^2 + q_2^2) + m^{-1}(p_1^2 + p_2^2)$ ⟹ $\Psi_4 = q_3 - m^{-1} r^{-2}(p_1^2 + p_2^2)$

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$\dot{\Psi}_3 = \{\Psi_3, H\}$

≈ 0

$L = \frac{1}{2}m(v_1^2 + v_2^2) - \frac{1}{2}q_3(q_1^2 + q_2^2 - r^2)$ → Euler-Lagrange

$p_i = \frac{\partial L}{\partial v_i}$ $\quad p_3 = 0$

mary constraint $\quad \Psi_1(p_i, q_i) = p_3$ $\quad \longrightarrow H = (p_i \dot{q}_i - L) + u\Psi$

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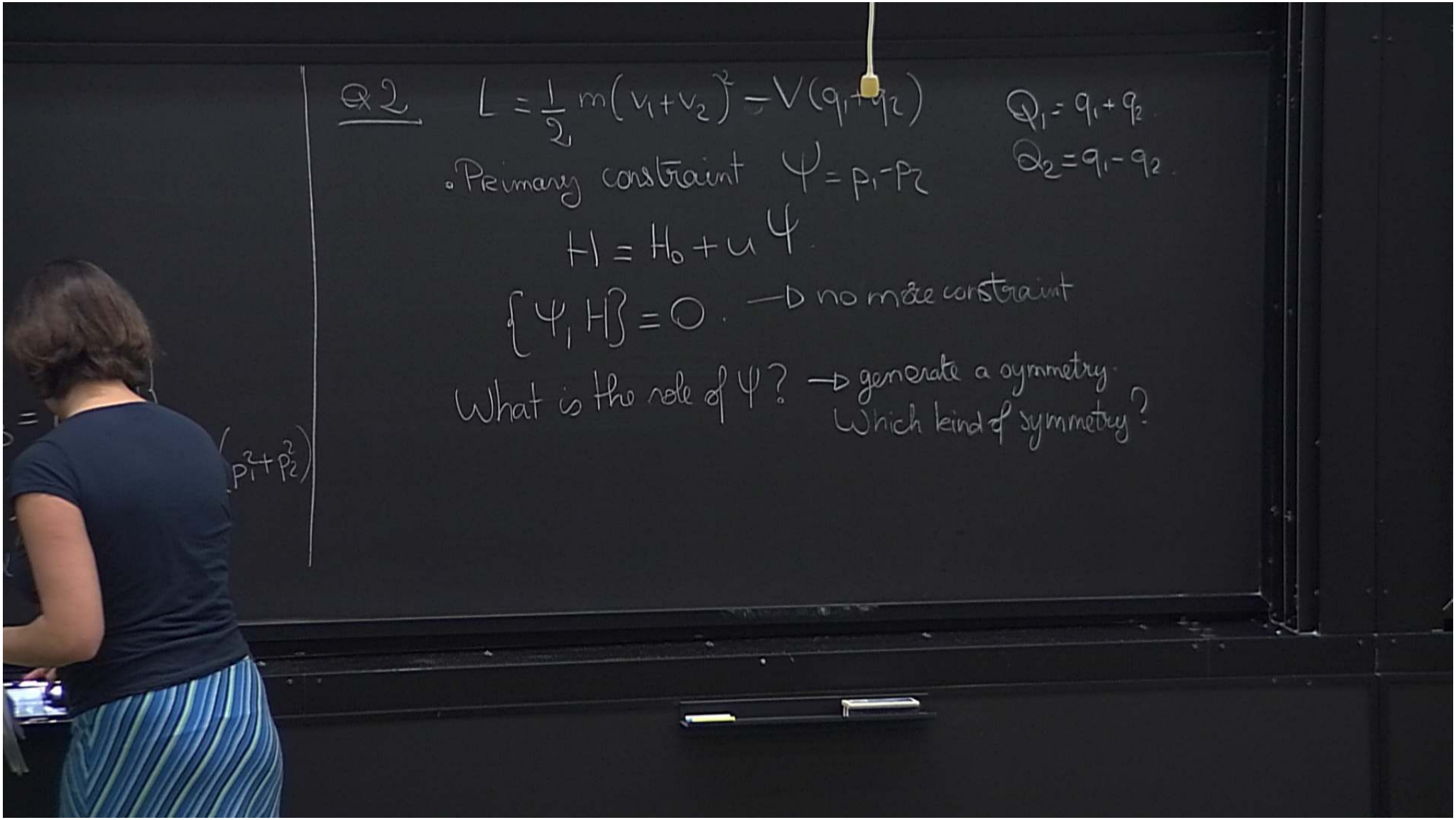
$\dot{\Psi}_4 = \{\Psi_4, H\} = u + 2m^{-1}r^{-2}q_3\underbrace{(p_1 q_1 + p_2 q_2)}_{\Psi_3} \approx u$

$\dot{\Psi}_3 = \{\Psi_3, H\}$

$\approx -q_3 + m^{-1}r^2(p_1^2 + p_2^2)$

Weakly


Q2 $L = \frac{1}{2} m(v_1+v_2)^2 - V(q_1+q_2)$
$Q_1 = q_1 + q_2$
$Q_2 = q_1 - q_2$
• Primary constraint $\Psi = p_1 - p_2$
$H_1 = H_0 + u\Psi$
$\{\Psi, H\} = 0$ → no more constraint
What is the role of Ψ? → generate a symmetry.
Which kind of symmetry?
$(p_1^2 + p_2^2)$

flow generated by ψ in the phase space

$$\delta q_1 = u\{q_1, \psi\} = u$$

$$\delta p_1 = u\{p_1, \psi\} = 0$$

$$\delta q_2 = u\{q_2, \psi\} = -u$$

$$\delta p_2 = u\{p_2, \psi\} = 0$$

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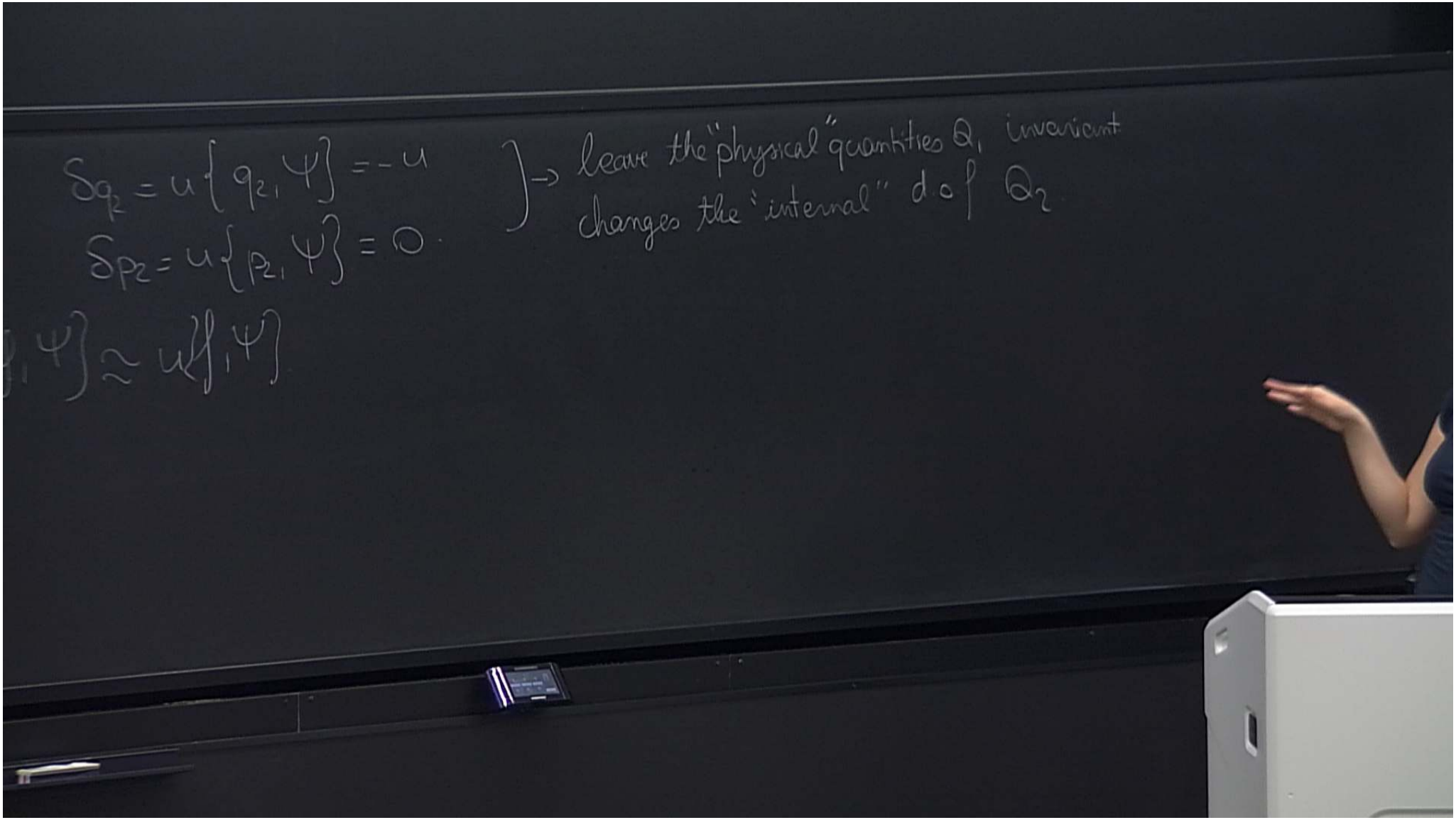
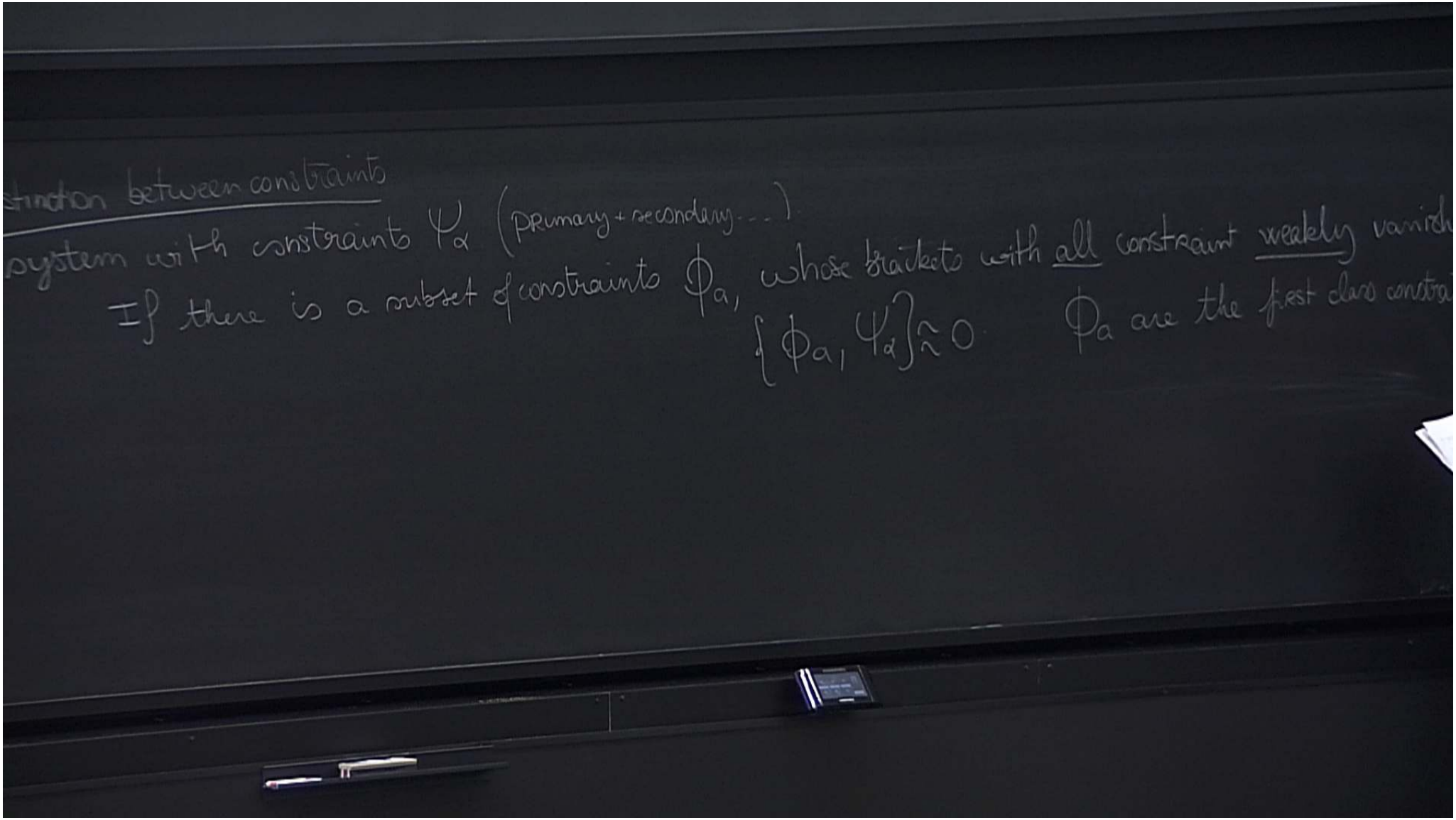
$$\{f, u\psi\} = \{f, u\}\psi + u\{f, \psi\} \approx u\{f, \psi\}$$


$\delta q_2 = u\{q_2, \Psi\} = -u$
$\delta p_2 = u\{p_2, \Psi\} = 0$
→ leave the "physical" quantities Q_1 invariant
changes the "internal" d.o.f Q_2
$\simeq u\{\,,\Psi\}$


stinction between constraints
system with constraints Ψ_α (primary + secondary...)
If there is a subset of constraints ϕ_a, whose brackets with all constraint weakly vanish
$\{\phi_a, \Psi_\alpha\} \approx 0$
ϕ_a are the first class constra

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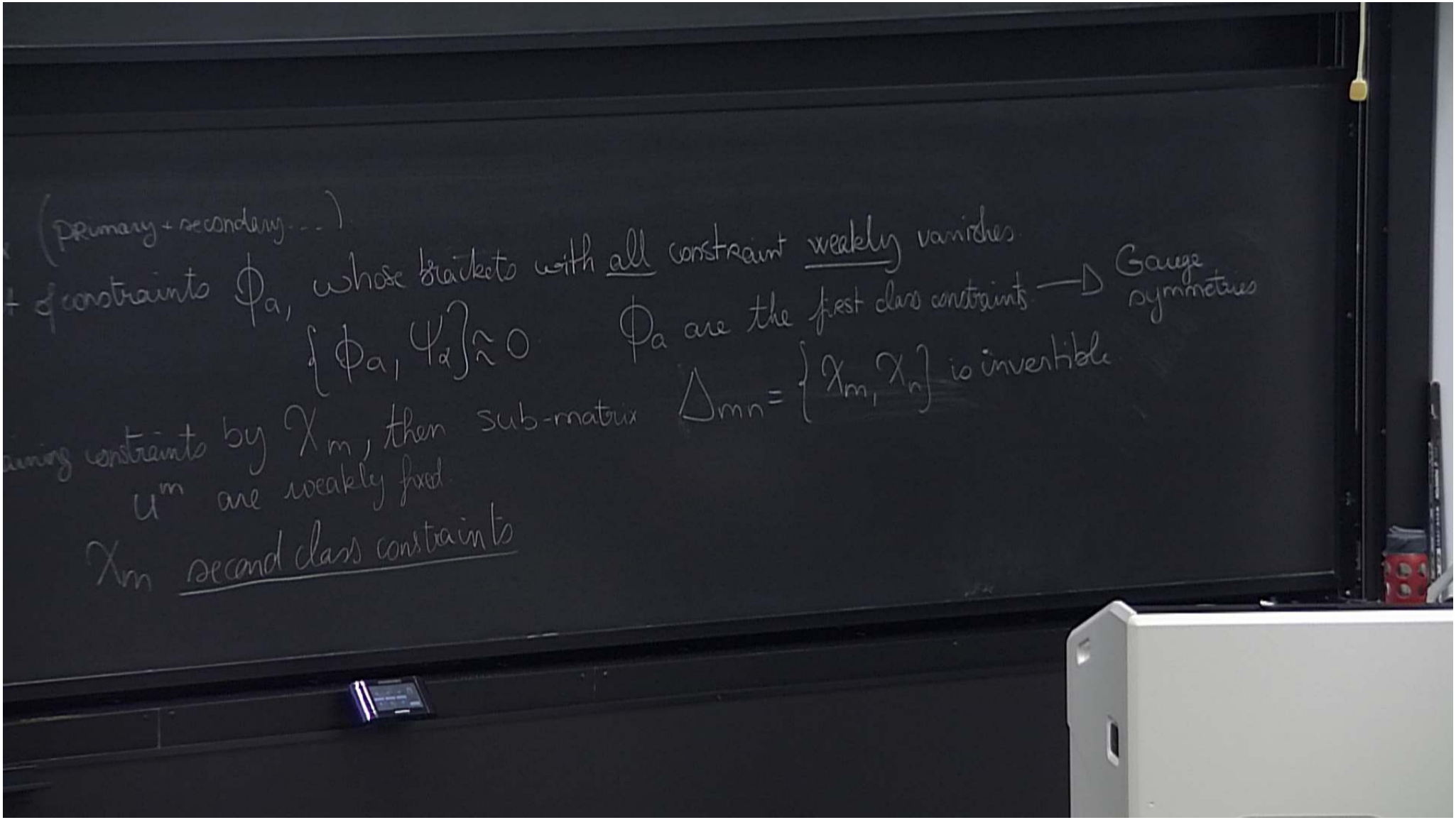
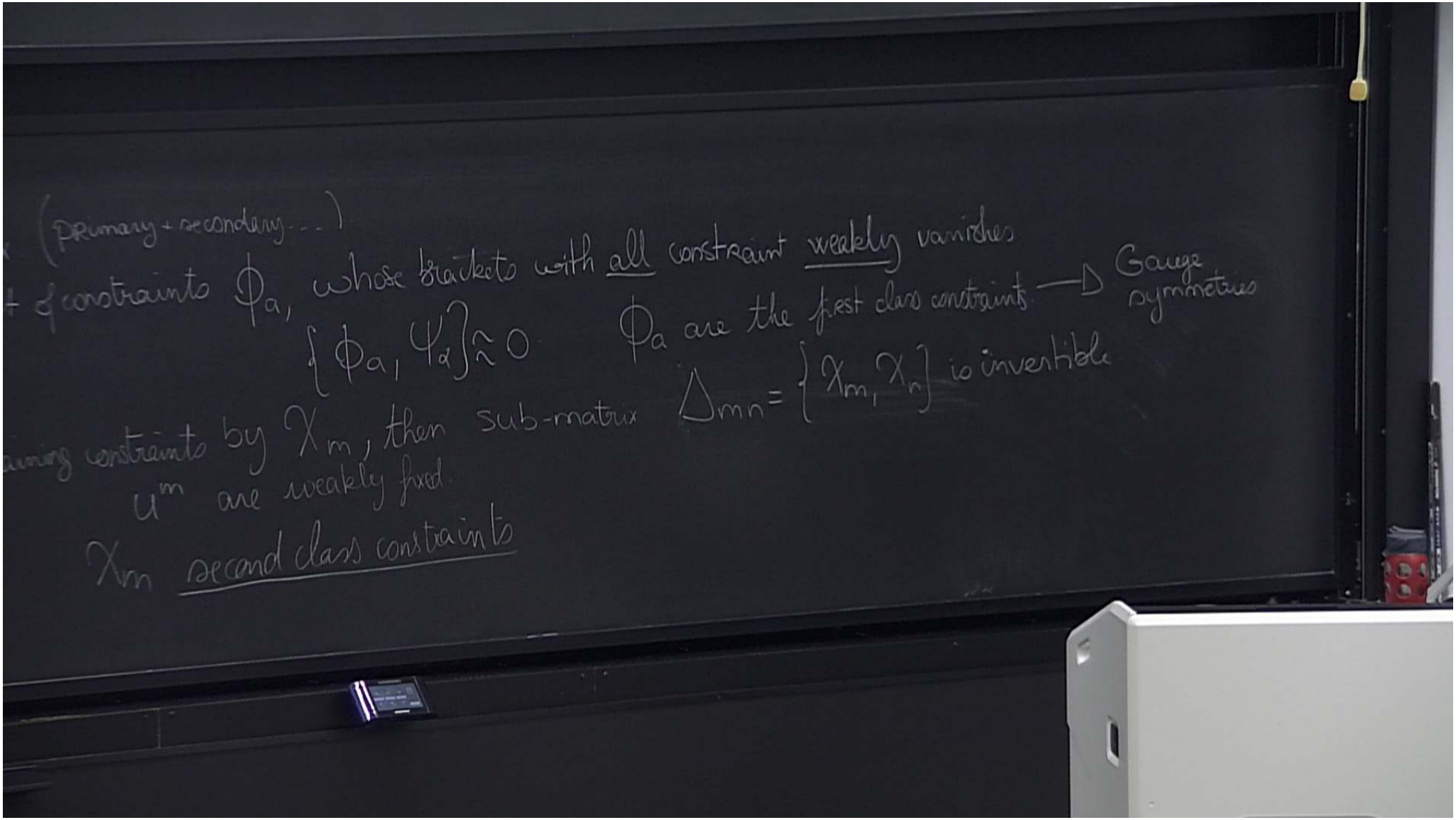
If we denote the remaining constraints by χ_m, then sub-matrix $\Delta_{mn} = \{\chi_m, \chi_n\}$ is in


(primary + secondary...)
+ f constraints ϕ_a, whose brackets with all constraint weakly vanishes
$\{\phi_a, \psi_\alpha\} \approx 0$
ϕ_a are the first class constraints → Gauge symmetries
aining constraints by χ_m, then sub-matrix $\Delta_{mn} = \{\chi_m, \chi_n\}$ is invertible
u^m are weakly fixed
χ_m second class constraints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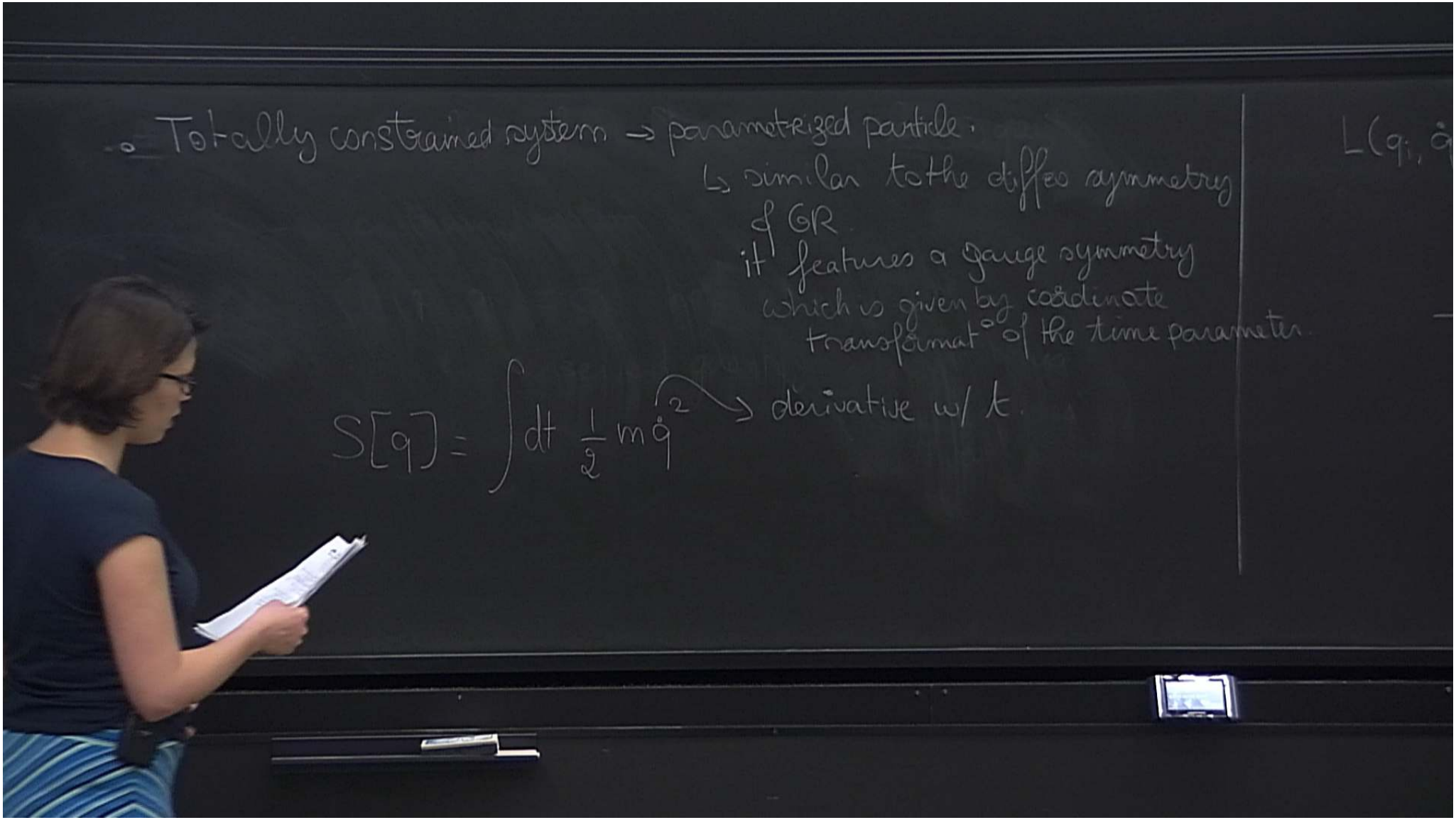
- Totally constrained system → parametrized particle.
  ↳ similar to the diffeo symmetry of GR
  it features a gauge symmetry which is given by coordinate transformat° of the time parameter.

$L(q_i, \dot{q}$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derivative w/ t.
$L(q_i, \dot{q}$

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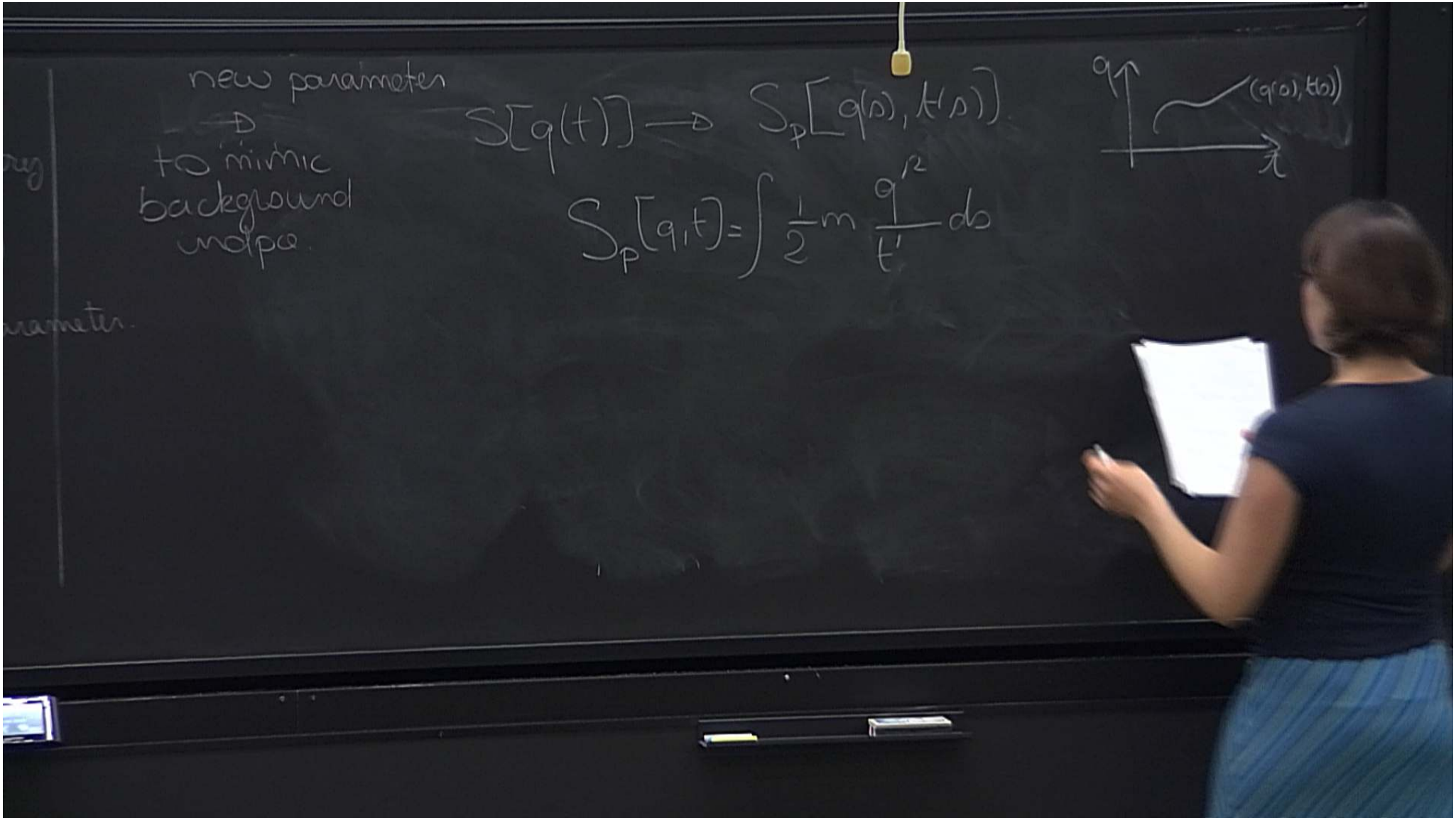
$$S[q] = \int dt \, \frac{1}{2} m \dot{q}^2$$ → derivative w/ t

↳ $q(z) = q + \frac{p}{m}(z - t)$

$L(q_i, \dot{q}$


new parameter
to mimic
background
wolpa
$S[q(t)] \longrightarrow S_p[q(\lambda), t(\lambda)]$
$S_p[q,t] = \int \frac{1}{2} m \frac{q'^2}{t'} d\lambda$
q
t
(q(0), t(0))

Canonical analysis - Hamiltonian picture.

$$P_t = \frac{\delta S}{\delta t'} = -m\frac{(q')^2}{2(t')^2} \qquad P_q = \frac{\delta S}{\delta q'} = m\frac{q'}{t'}$$

Canonical analysis - Hamiltonian picture.

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Hamiltonian $H = p_t t' + p_q q' - L \qquad L = -t' p_t = \frac{p_q^2}{2m} t'$

$= t' p_t + t' \frac{p_q^2}{2m} = t' C \equiv$ totally constrained.

↑ Lagrange multiplier