Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 3

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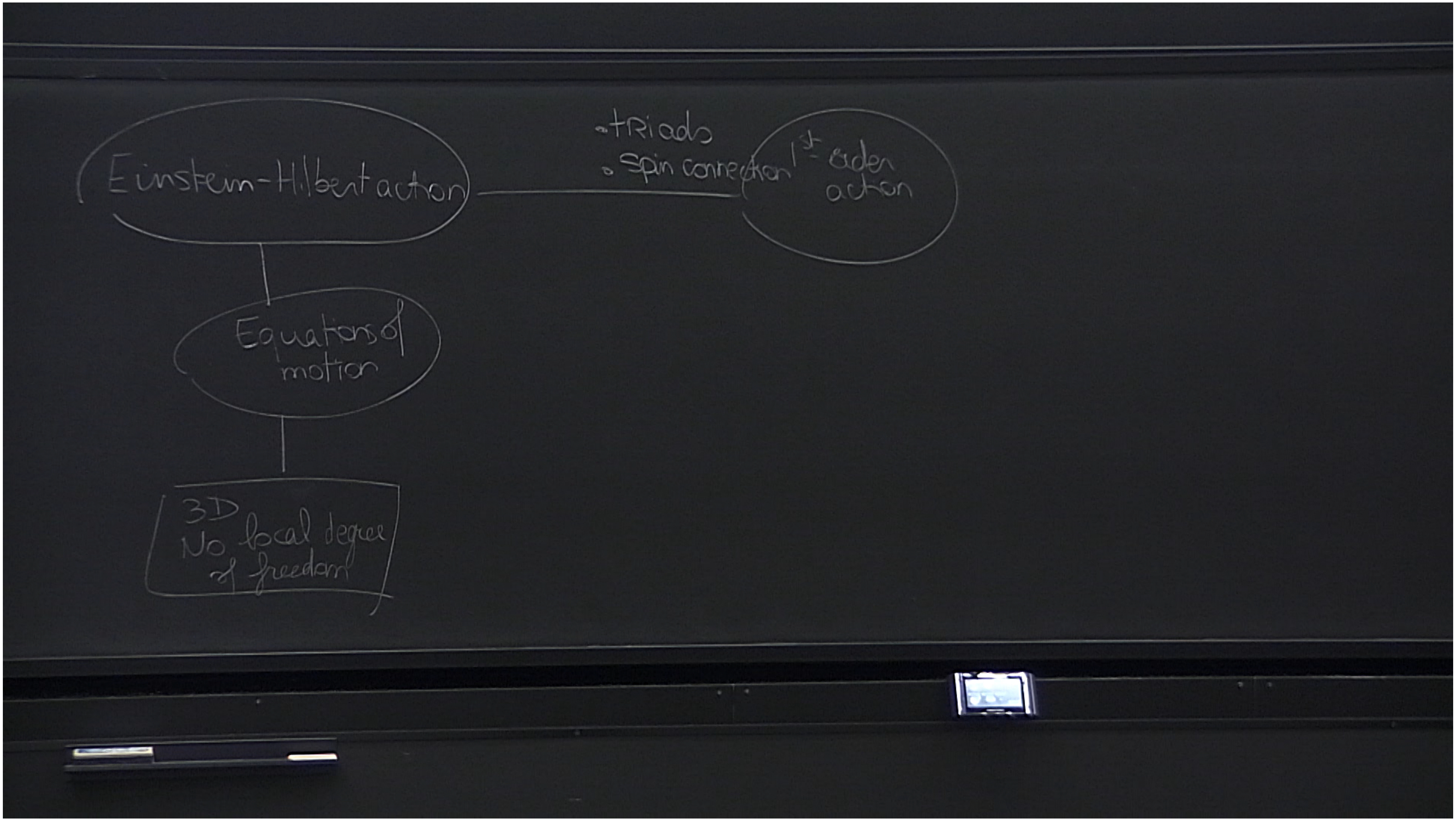
Abstract:


Einstein-Hilbert action
Equations of motion
3D


Einstein-Hilbert action
triads
1st order action
Equations of motion
3D
No local degree of freedom


Einstein-Hilbert action
• triads
• spin connection
1st order action
Equations of motion
3D
No local degree of freedom

connection 1^{st}-order action

Spin connection ∇_ν (L-C connection)

+ Covariant derivative D_ν

Compatibility condition $D_\mu e^j_\nu = 0$

covariant derivative D_μ

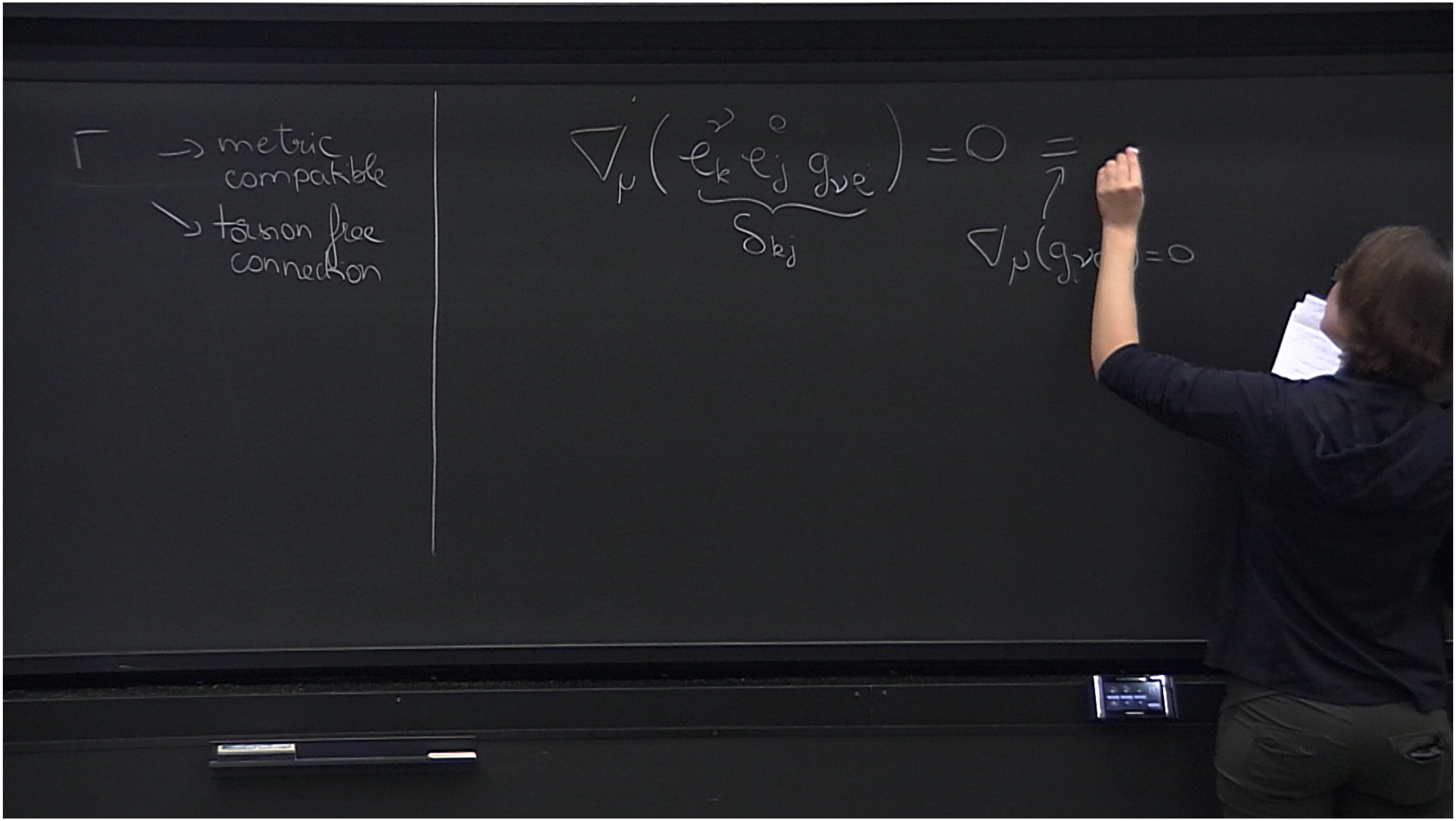
Compatibility condition $\boxed{D_\mu e^j_\nu = 0}$

(to make sure that the derivative commutes with the contraction)

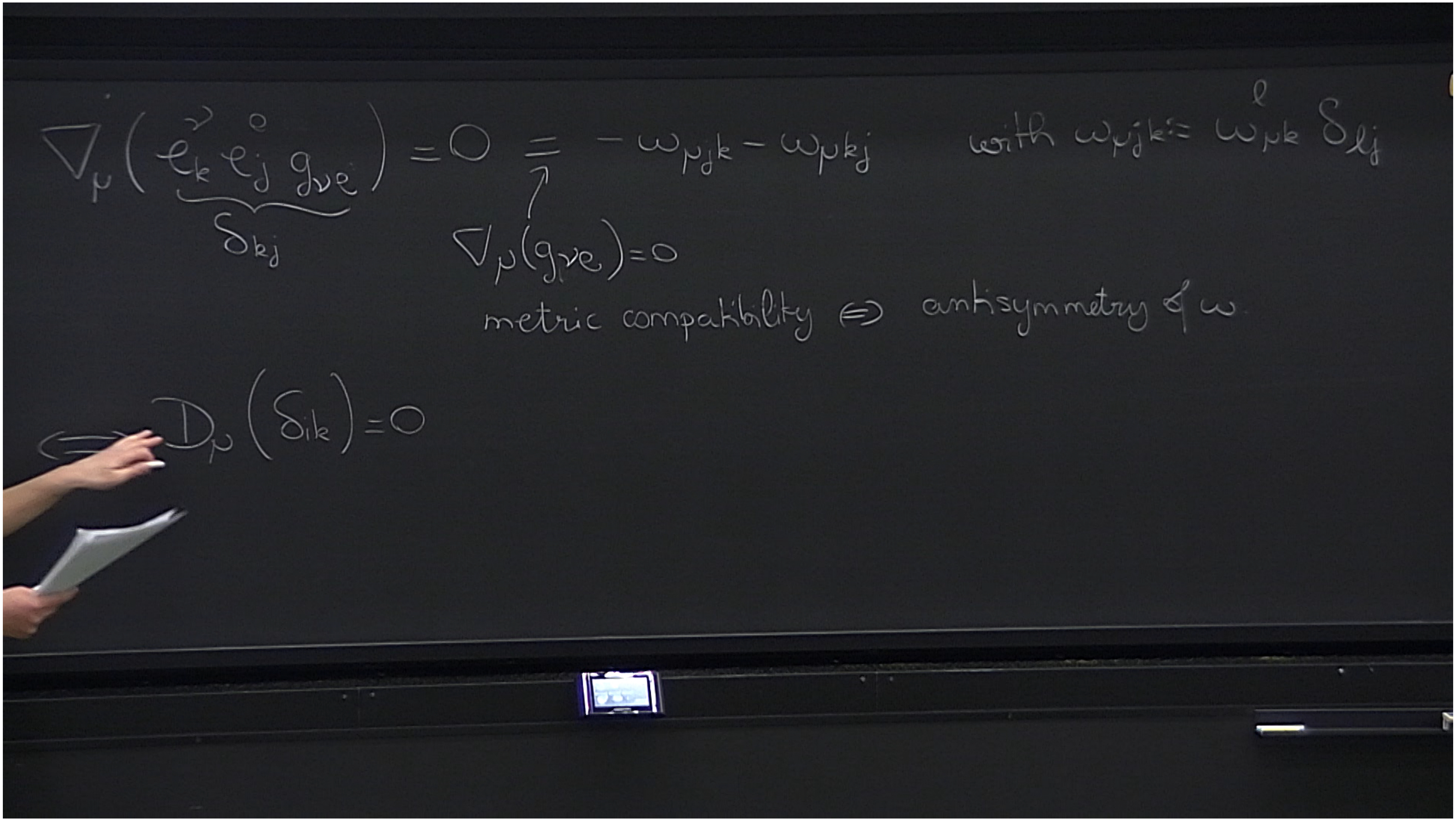
$$D_\mu e^j_\nu = \partial_\mu e^j_\nu - \Gamma^\rho_{\mu\nu} e^j_\rho + \omega^j_{\mu k} e^k_\nu = 0$$

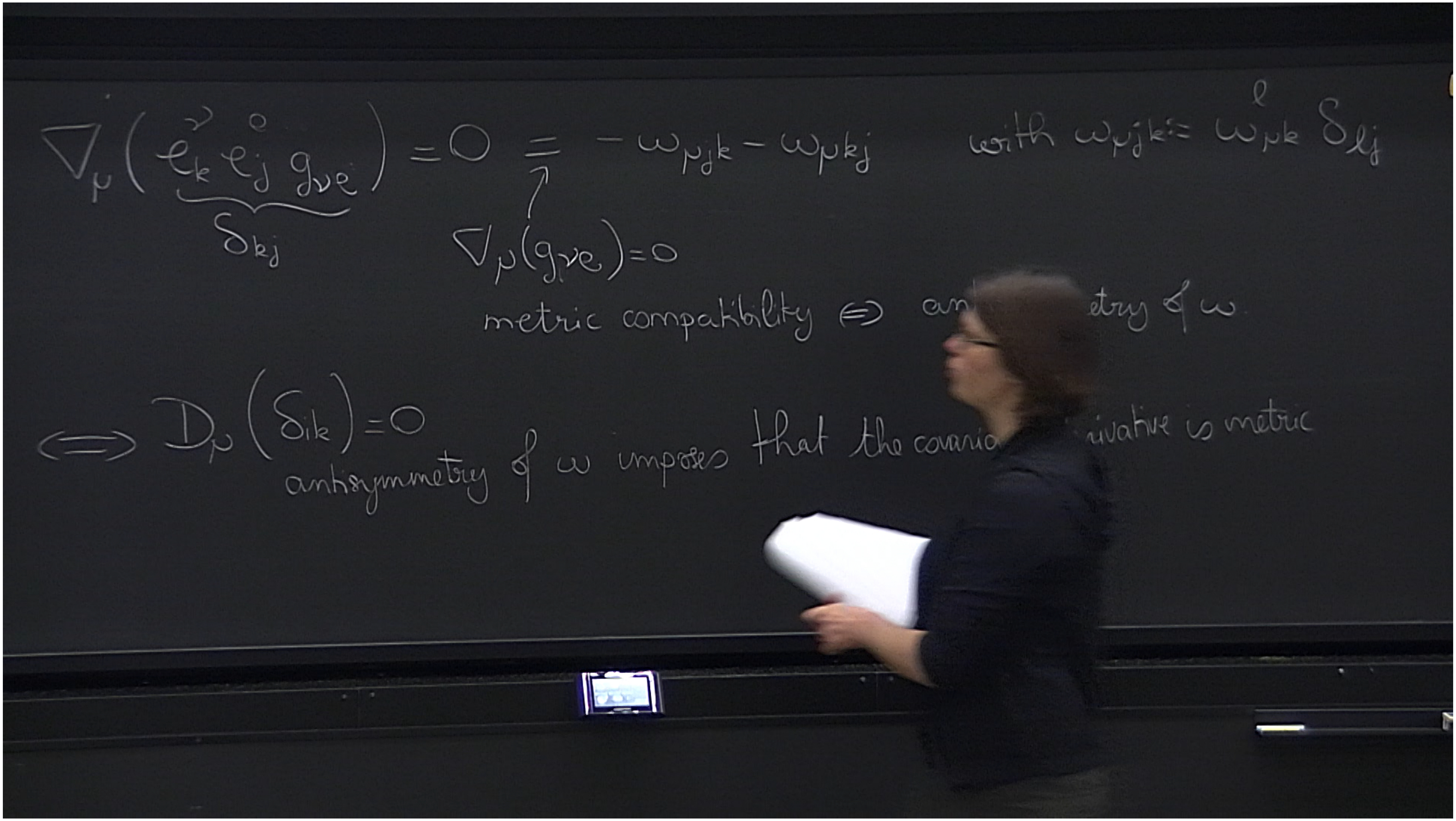
contraction by e^ν_i ($e^k_\nu e^\nu_i = \delta^k_i$)

$$\Rightarrow \boxed{\omega^j_{\mu k} = - e^\nu_k \nabla_\mu e^j_\nu}$$


Γ → metric compatible
→ torsion free connection
$\nabla_\mu \big(\underbrace{e^{\mu}_{k}\, e^{e}_{j}\, g_{\nu e}}_{\delta_{kj}} \big) = 0 =$
$\nabla_\mu (g_{\nu e}) = 0$

$$\nabla_\mu \big(\underbrace{e^{\nu}_{k}\, e^{\ell}_{j}\, g_{\nu\ell}}_{\delta_{kj}} \big) = 0 \;\overset{\nabla_\mu (g_{\nu\ell}) = 0}{=}\; -\,\omega_{\mu jk} - \omega_{\mu kj} \qquad \text{with } \omega_{\mu jk} := \omega^{\ell}_{\mu k}\, \delta_{\ell j}$$


$\nabla_\mu(\underbrace{e^\nu_k e^\ell_j g_{\nu\ell}}_{\delta_{kj}}) = 0 = -\omega_{\mu jk} - \omega_{\mu kj}$ with $\omega_{\mu jk} := \omega_{\mu k}^{\ell}\,\delta_{\ell j}$
$\nabla_\mu(g_{\nu\ell}) = 0$
metric compatibility $\Leftrightarrow$ antisymmetry of ω.
$\Longleftrightarrow D_\mu(\delta_{ik}) = 0$


$\nabla_\mu (\underbrace{e^\nu_k \, e^\rho_j \, g_{\nu\rho}}_{\delta_{kj}}) = 0 = -\omega_{\mu jk} - \omega_{\mu kj}$ with $\omega_{\mu jk} := \omega^{\ l}_{\mu k} \delta_{lj}$
$\nabla_\mu (g_{\nu\rho}) = 0$
metric compatibility $\Leftrightarrow$ an etry of ω
$\Leftrightarrow D_\mu(\delta_{ik}) = 0$
antisymmetry of ω imposes that the covaria ivative is metric

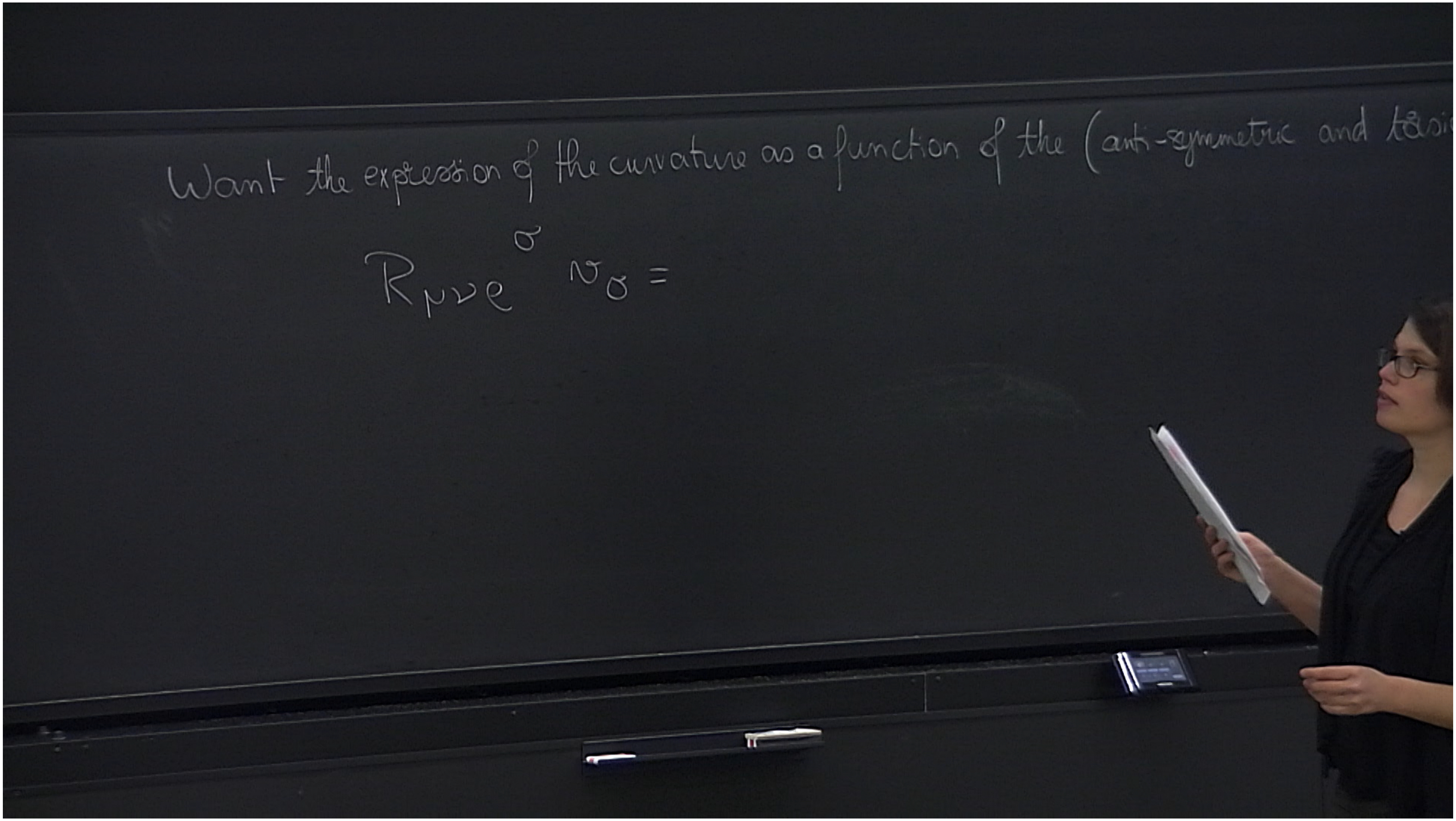
$$\underbrace{D_\mu e^j_\nu}_{=0} - \underbrace{D_\nu e^j_\mu}_{=0} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu - \Gamma^\rho_{\mu\nu} e^j_\rho + \Gamma^\rho_{\nu\mu} e^j_\rho + \omega_\mu{}^j{}_k e^k_\nu - \omega$$

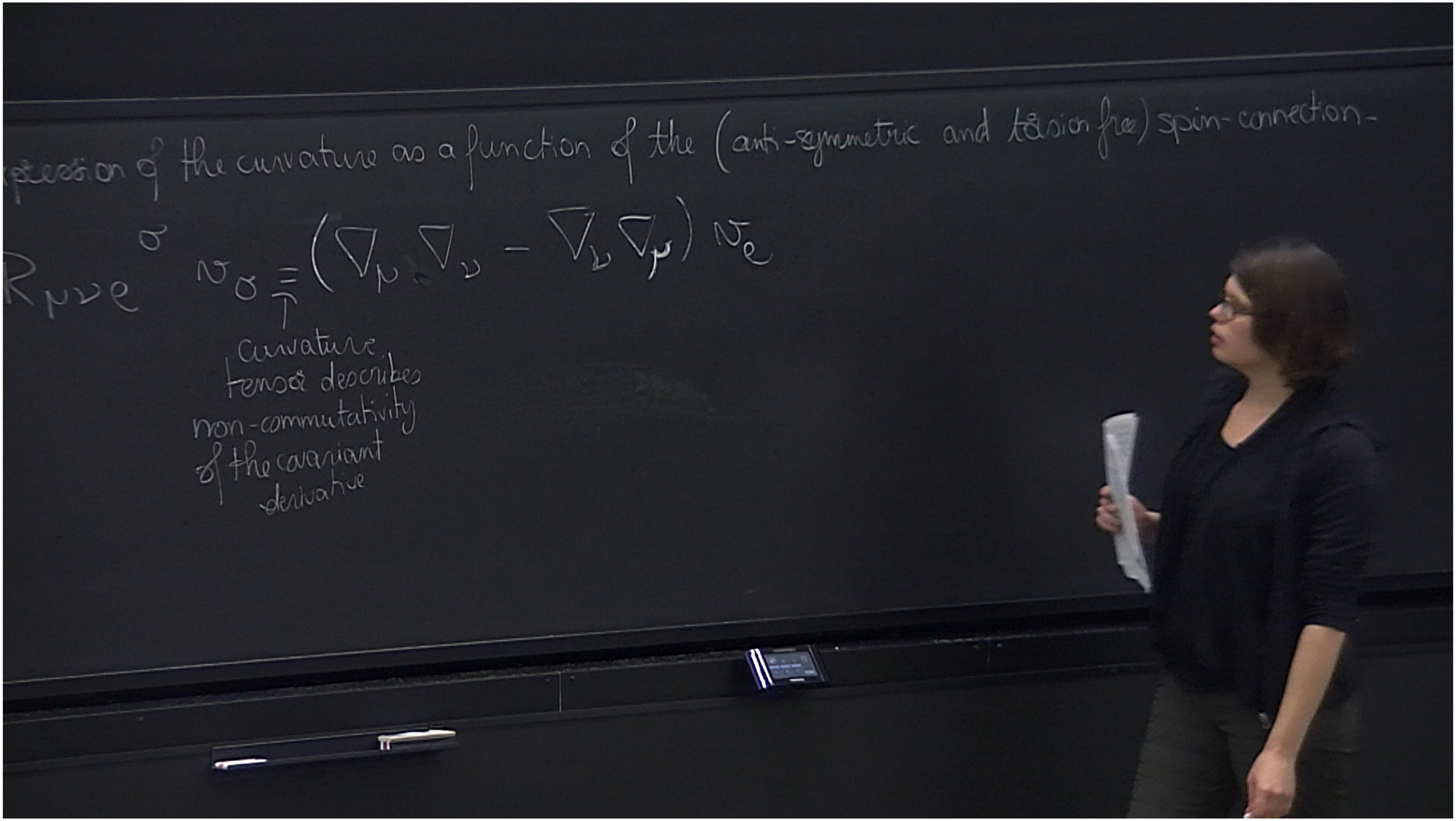
$$\underbrace{\partial_\mu e^j_\nu - \partial_\nu e^j_\mu} - \Gamma^e_{\mu\nu} e^j_e + \Gamma^e_{\nu\mu} e^j_e + \underbrace{\omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu} = 0$$

$$\Rightarrow \quad T^j_{\mu\nu} = \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$$

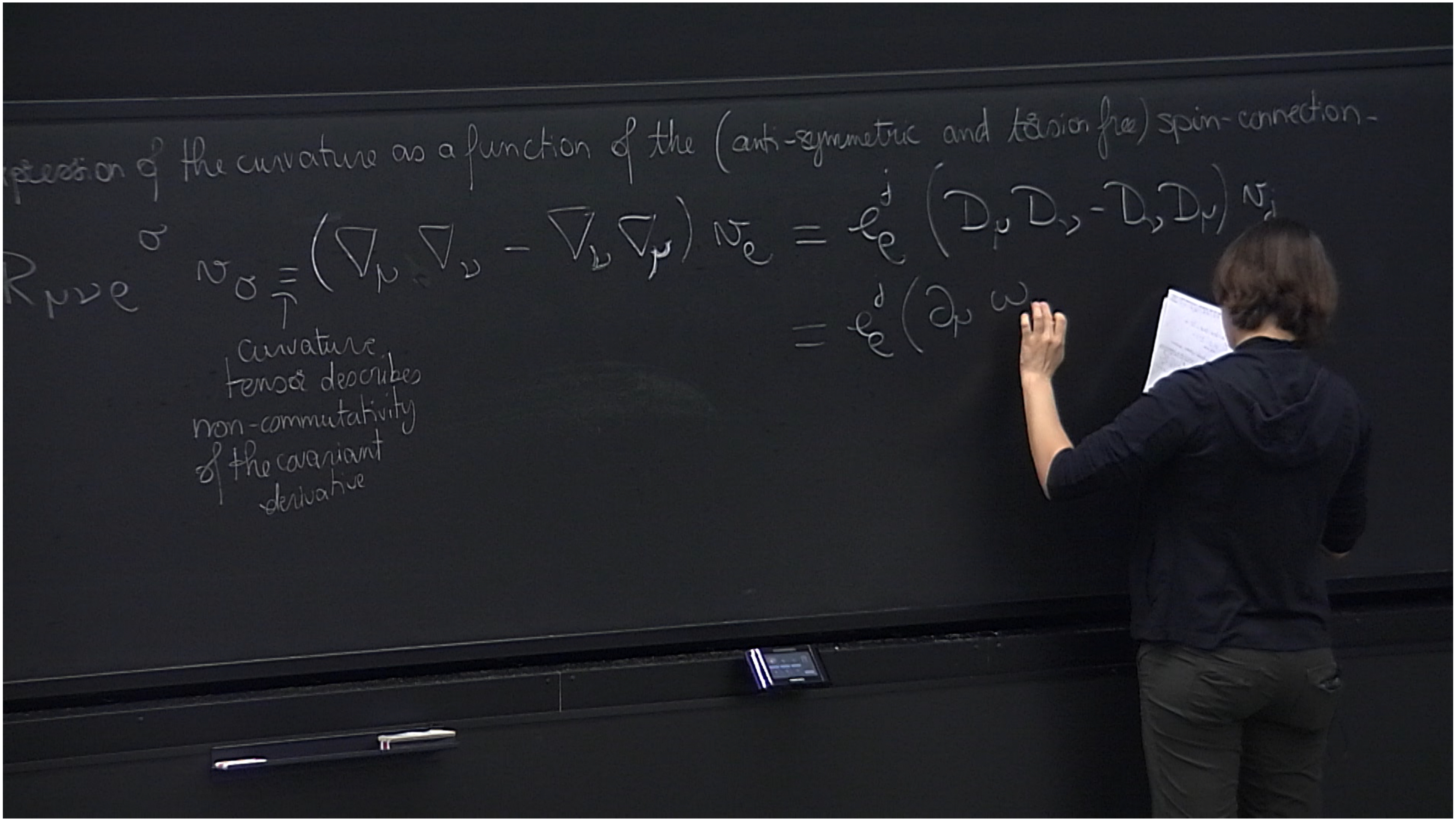
$$= \big(\underbrace{\Gamma^e_{\mu\nu} - \Gamma^e_{\nu\mu}}_{=0}\big) e^j_e \Leftrightarrow T_{\mu\nu} = 0$$

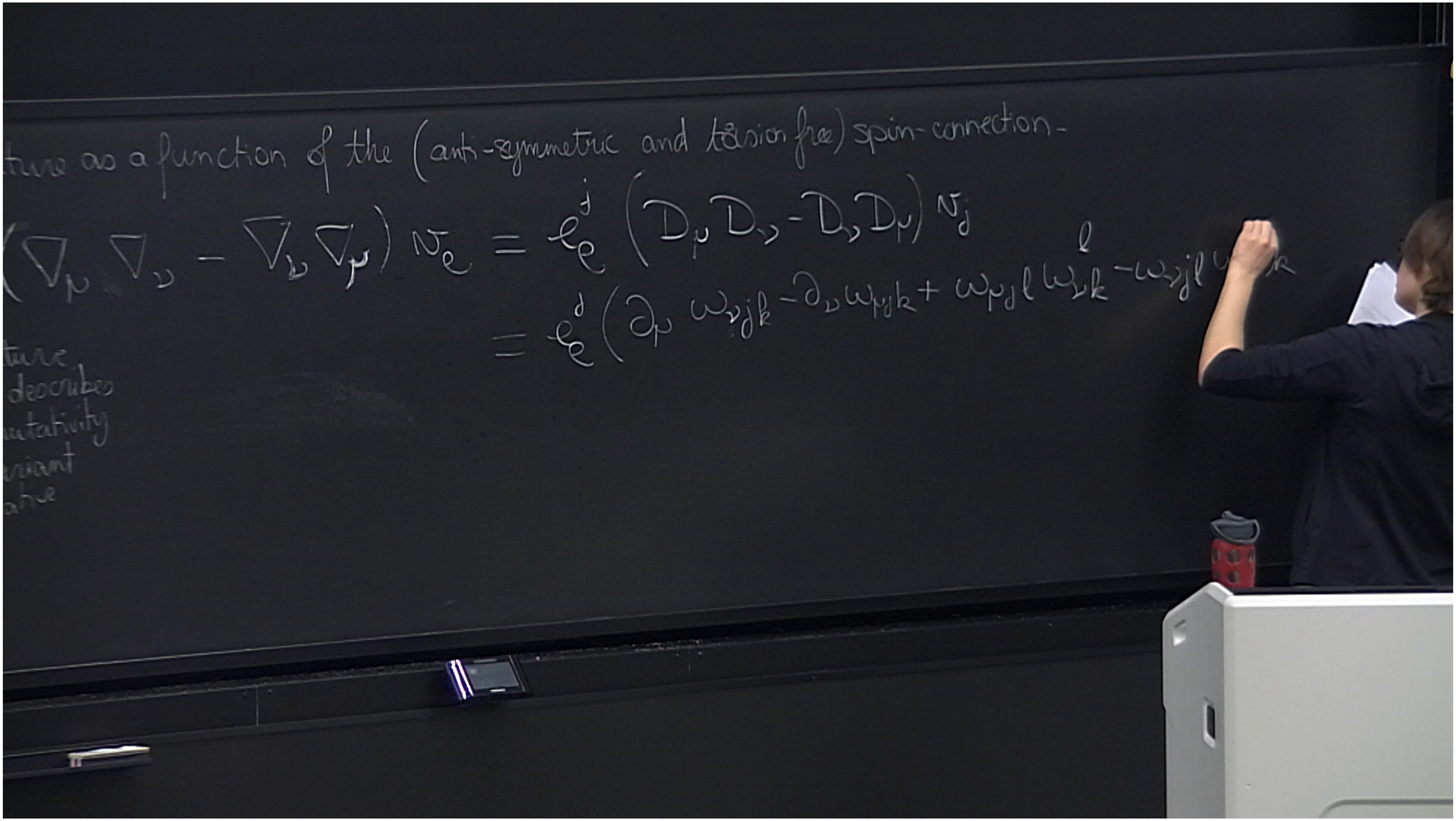
Want the expression of the curvature as a function of the (anti-symmetric and torsion free)


Want the expression of the curvature as a function of the (anti-symmetric and
$R_{\mu\nu\rho}{}^{\sigma} v_{\sigma} =$



...pression of the curvature as a function of the (anti-symmetric and torsion free) spin-connection -
$R_{\mu\nu e}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_e$
curvature
tensor describes
non-commutativity
of the covariant
derivative


pression of the curvature as a function of the (anti-symmetric and torsion free) spin-connection -
$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_\rho = e_\rho^j (D_\mu D_\nu - D_\nu D_\mu) v_j$
$= e_\rho^j (\partial_\mu \omega$
curvature tensor describes non-commutativity of the covariant derivative


ture as a function of the (anti-symmetric and torsion free) spin-connection.
$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_e = e_e^{\,j} (D_\mu D_\nu - D_\nu D_\mu) v_j$
$= e_e^{\,j} (\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l} \omega_{\nu}{}^{l}{}_{k} - \omega_{\nu j l}$
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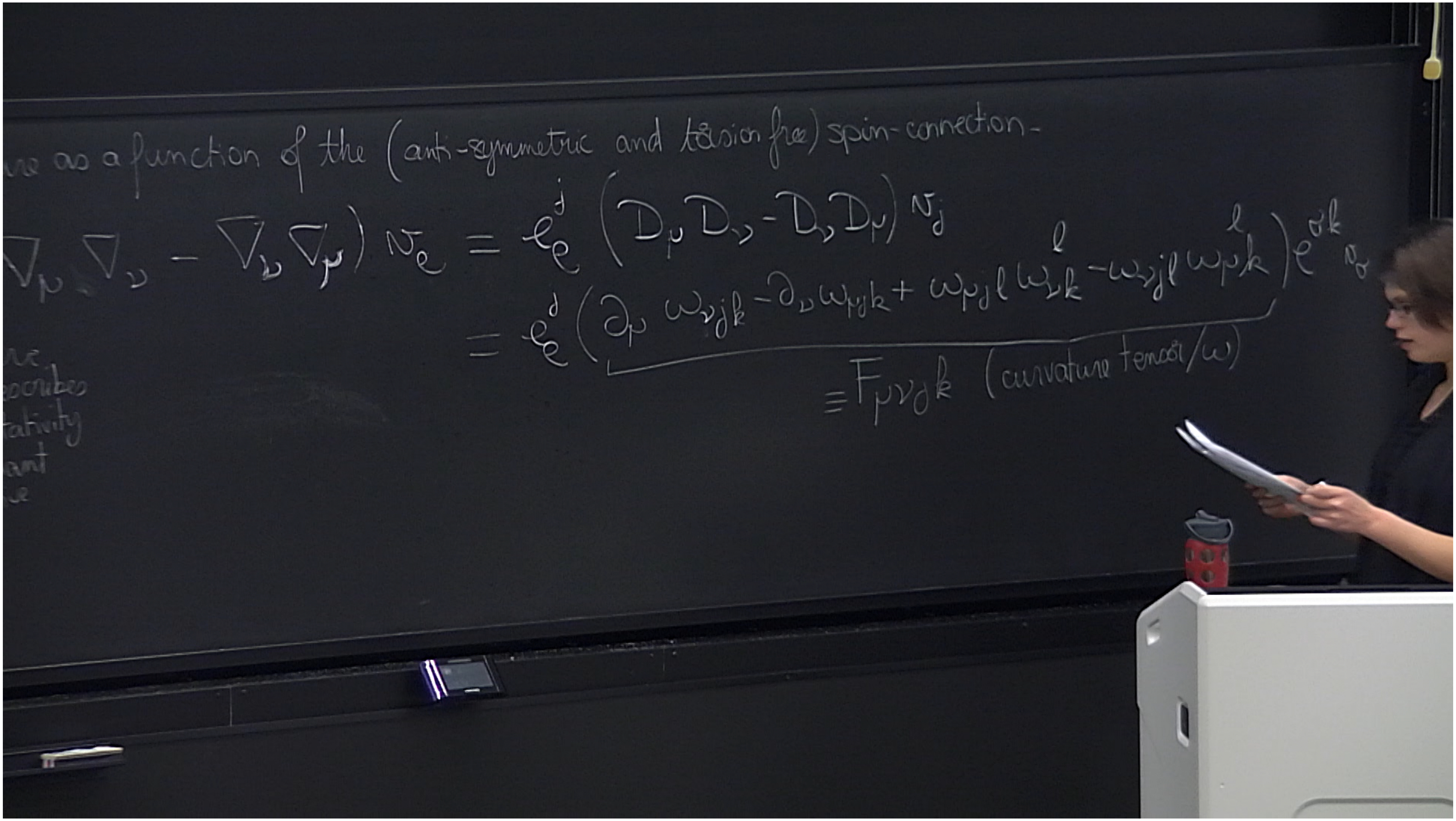

re as a function of the (anti-symmetric and torsion free) spin-connection.
$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_e = e_e^j (D_\mu D_\nu - D_\nu D_\mu) v_j$
$= e_e^j \big(\underbrace{\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl} \omega_\nu{}^l{}_k - \omega_{\nu jl} \omega_\mu{}^l{}_k}_{\equiv F_{\mu\nu jk}}\big) e^{\sigma k} v_\sigma$
$\equiv F_{\mu\nu jk}$ (curvature tensor/ω)
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$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) v_e = e_e^{\ j} (D_\mu D_\nu - D_\nu D_\mu) v_j$$

$$= e_e^{\ j} \Big(\underbrace{\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l}\, \omega_{\nu\ k}^{\ l} - \omega_{\nu j l}\, \omega_{\mu\ k}^{\ l}}_{\equiv F_{\mu\nu jk} \ \text{(curvature tensor/}\omega\text{)}} \Big) e^{\sigma k} v_\sigma$$

$$R_{\mu\nu e}{}^{\sigma} = e_e^{\ j} e^{\sigma k} F_{\mu\nu jk}$$

Want the expression of the curvature as a function of the (anti-symmetric and t

$$R_{\mu\nu e}{}^{\sigma}\, v_\sigma \equiv (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, v_e = e_e^{j}\,(D_\mu D$$

curvature tensor describes non-commutativity of the covariant derivative

$$= e_e^{j}\,(\partial_\mu \omega$$

$$R_{\mu\nu e}{}^{\sigma} = e_e^{j}\, e^{\sigma k}\, F_{\mu\nu}$$

- First order action for 3D gravity

→ Ricci scalar $R = R_{\mu\nu}\, g^{\mu\nu} = e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\sigma}(\omega)$

→ 3D

to enforce the antisymmetry of $\omega_{\mu jk}$

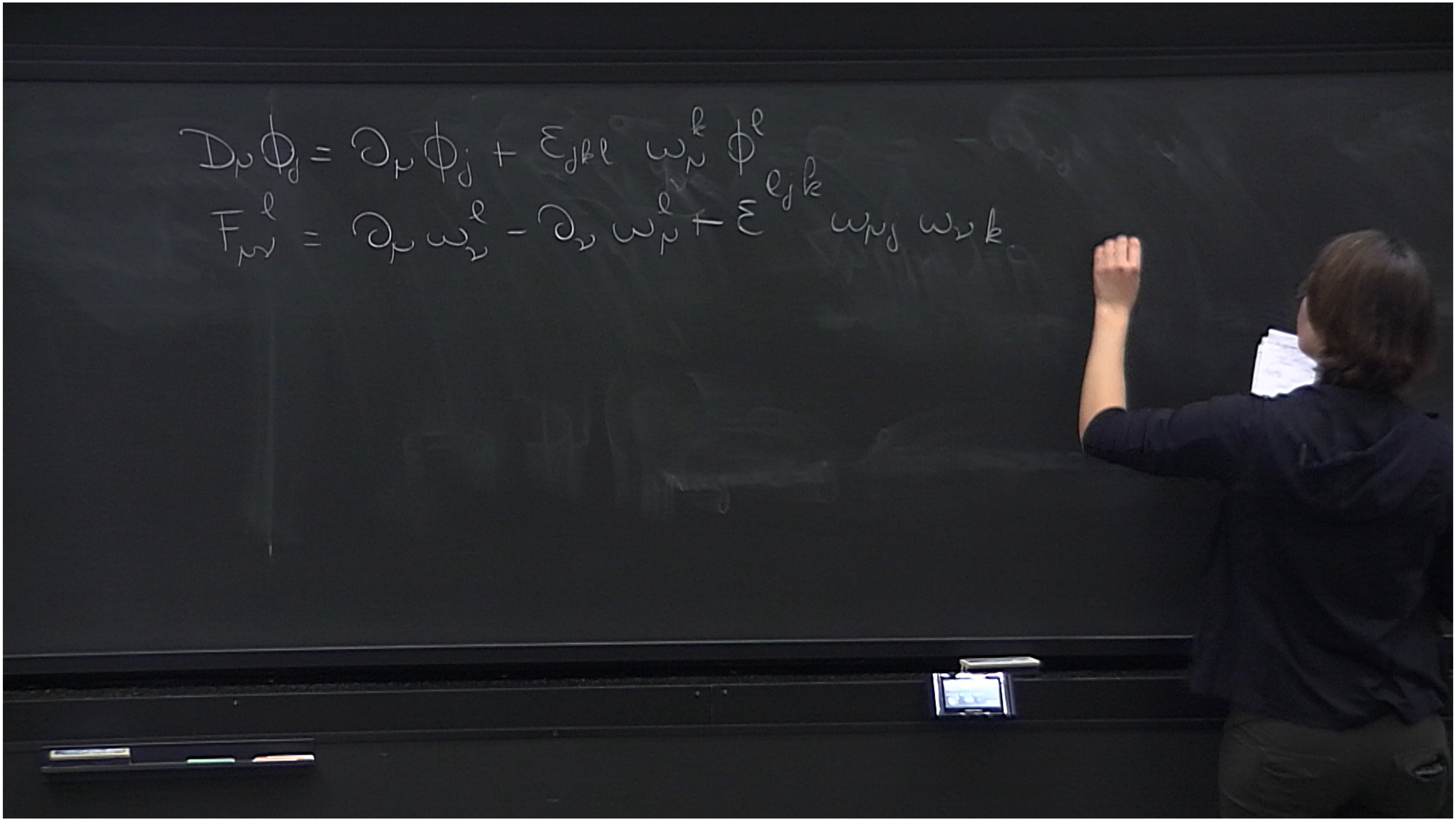
$$\boxed{\omega_{\mu jk} = \varepsilon_{jlk}\, \omega^{l}_{\mu}}$$

$$\boxed{F_{\mu\nu jk} = \varepsilon_{jlk}\, F^{l}_{\mu\nu}}$$

→ to introduce easily

ε_{jlk} the structure

cst of

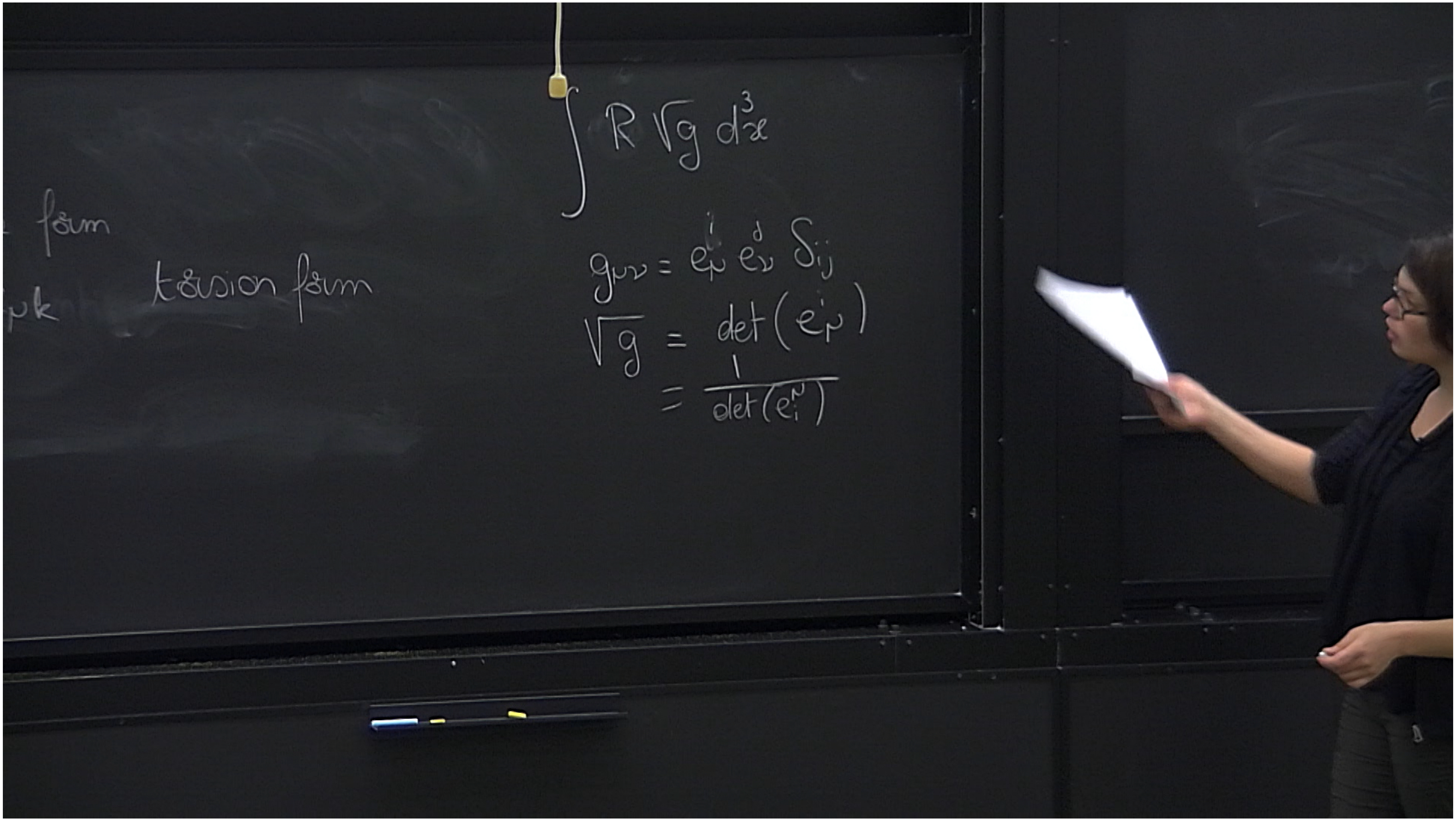


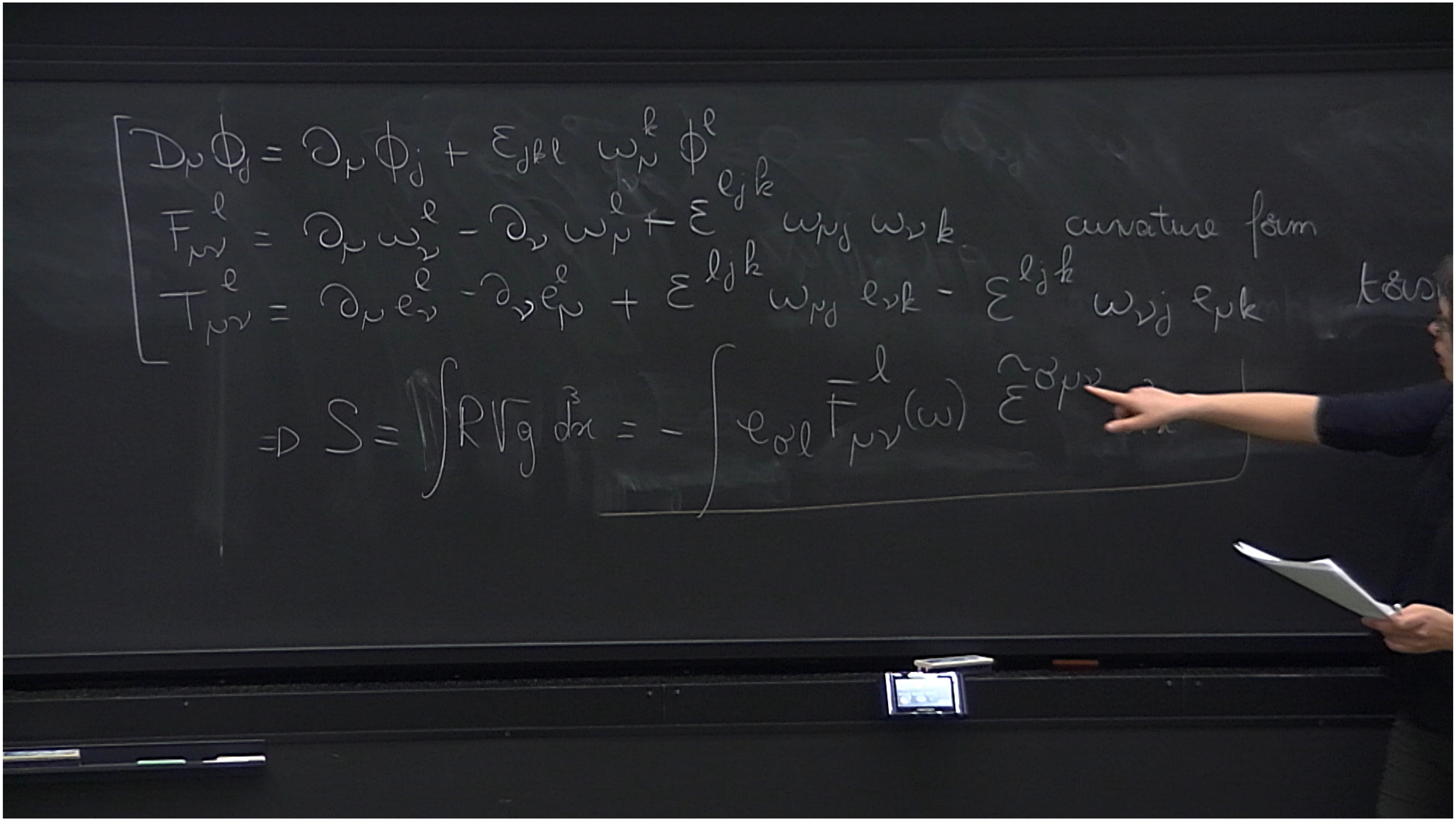
$D_\mu \phi_j = \partial_\mu \phi_j + \epsilon_{jkl} \, \omega_\mu^k \phi^l$
$F_{\mu\nu}^l = \partial_\mu \omega_\nu^l - \partial_\nu \omega_\mu^l + \epsilon^{ljk} \omega_{\mu j} \omega_{\nu k}$

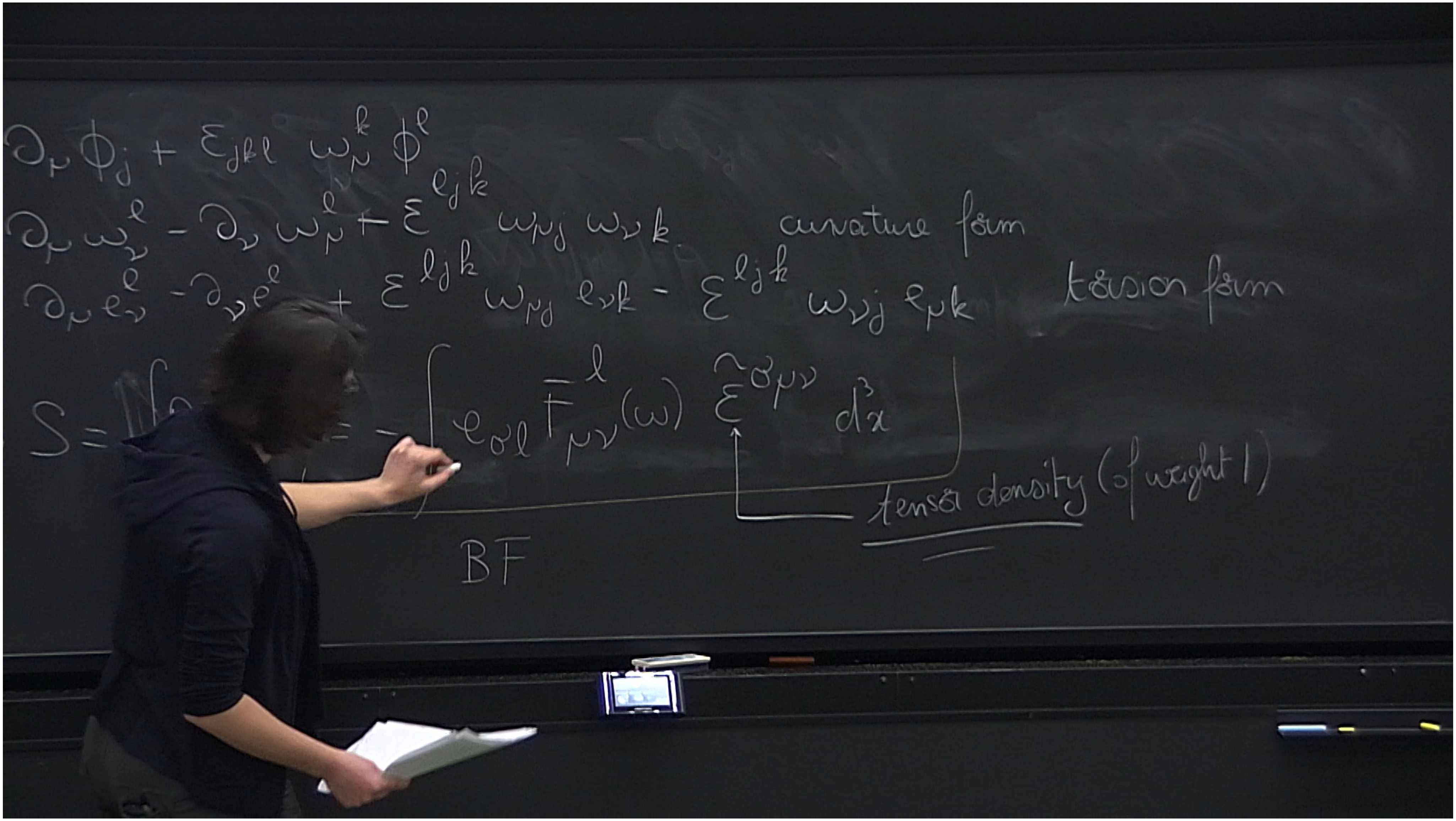
$$D_\nu \phi_j = \partial_\nu \phi_j + \varepsilon_{jkl}\, \omega_\nu^k \, \phi^l$$

$$F_{\mu\nu}^l = \partial_\mu \omega_\nu^l - \partial_\nu \omega_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$$ curvature form

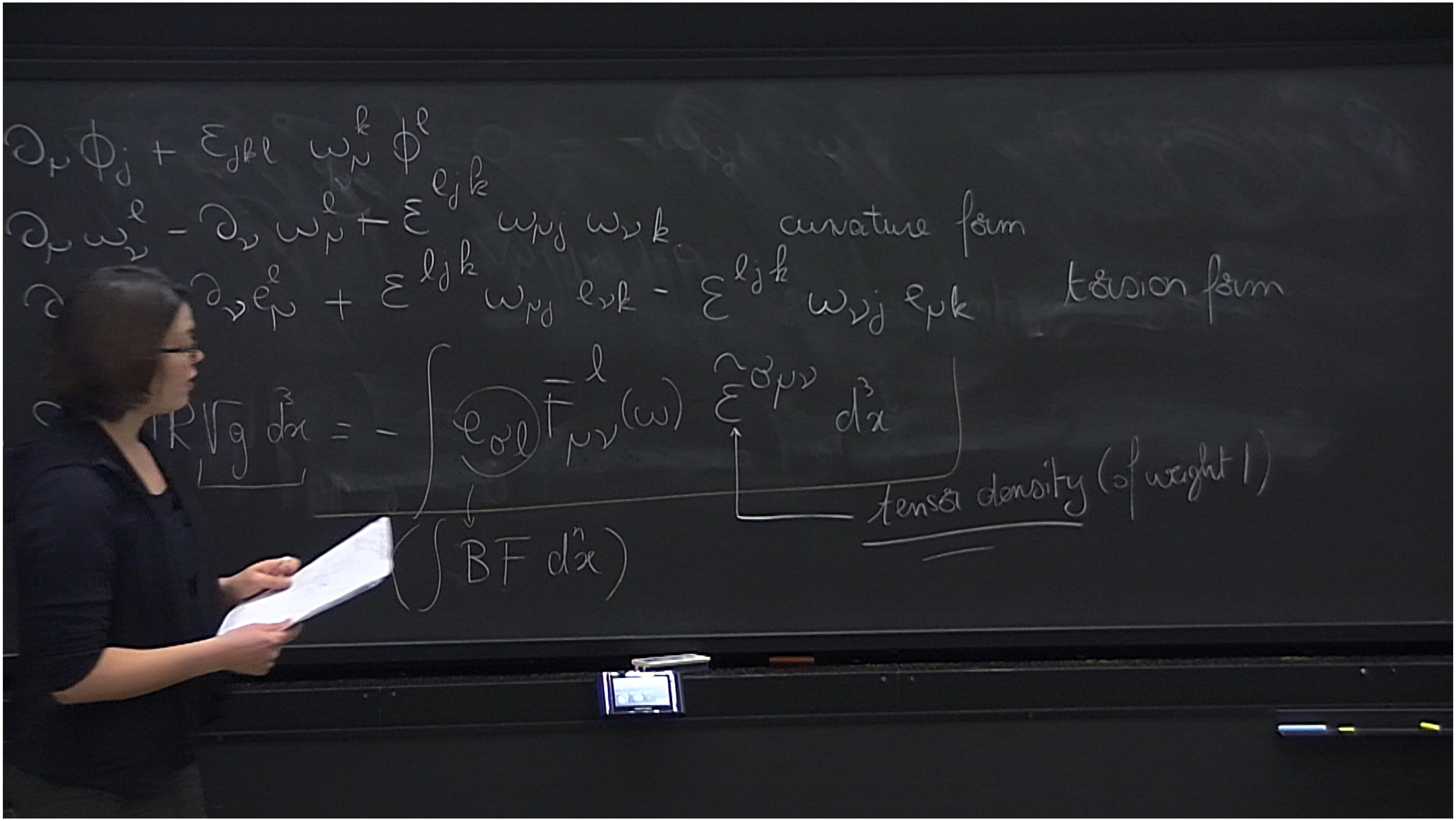
$$T_{\mu\nu}^l = \partial_\mu e_\nu^l - \partial_\nu e_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \varepsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$$ torsion


form
μk
torsion form
$\int R \sqrt{g}\, d^3x$
$g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$
$\sqrt{g} = \det(e^i_\mu)$
$= \frac{1}{\det(e^\mu_i)}$


$D_\nu \phi_j = \partial_\nu \phi_j + \varepsilon_{jkl}\, \omega_\nu^k\, \phi^l$
$F_{\mu\nu}^l = \partial_\mu \omega_\nu^l - \partial_\nu \omega_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$
curvature form
$T_{\mu\nu}^l = \partial_\mu e_\nu^l - \partial_\nu e_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \varepsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$
$\Rightarrow S = \int R \sqrt{g}\, d^3x = -\int e_{\sigma l}\, \bar{F}_{\mu\nu}^l(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}$



$\partial_\mu \phi_j + \varepsilon_{jkl}\, \omega_\nu^k \phi^l$
$\partial_\mu \omega_\nu^l - \partial_\nu \omega_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$
curvature form
$\partial_\mu e_\nu^l - \partial_\nu e_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \varepsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$
torsion form
$S = \ldots = -\int e_{\sigma l}\, F^l_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$
tensor density (of weight 1)
BF


$\partial_\mu \phi_j + \varepsilon_{jkl}\, \omega_\nu^k \phi^l$
$\partial_\mu \omega_\nu^l - \partial_\nu \omega_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, \omega_{\nu k}$
curvature form
$\partial_\nu e_\mu^l + \varepsilon^{ljk}\, \omega_{\mu j}\, e_{\nu k} - \varepsilon^{ljk}\, \omega_{\nu j}\, e_{\mu k}$
torsion form
$R\sqrt{g}\, d^3x = -\int e_{\sigma l}\, F^l_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$
tensor density (of weight 1)
$\left(\int BF\, d^nx\right)$

$R\sqrt{g}\,d^3x$

$\omega_{\nu k}$ curvature form

$e_{\nu k} - \varepsilon^{ljk}\,\omega_{\nu j}\,e_{\mu k}$ torsion form

$F^{l}_{\mu\nu}(\omega)\,\tilde{\varepsilon}^{\sigma\mu\nu}\,d^3x$

tensor density (of weight 1)

$d^n x)$

$g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$

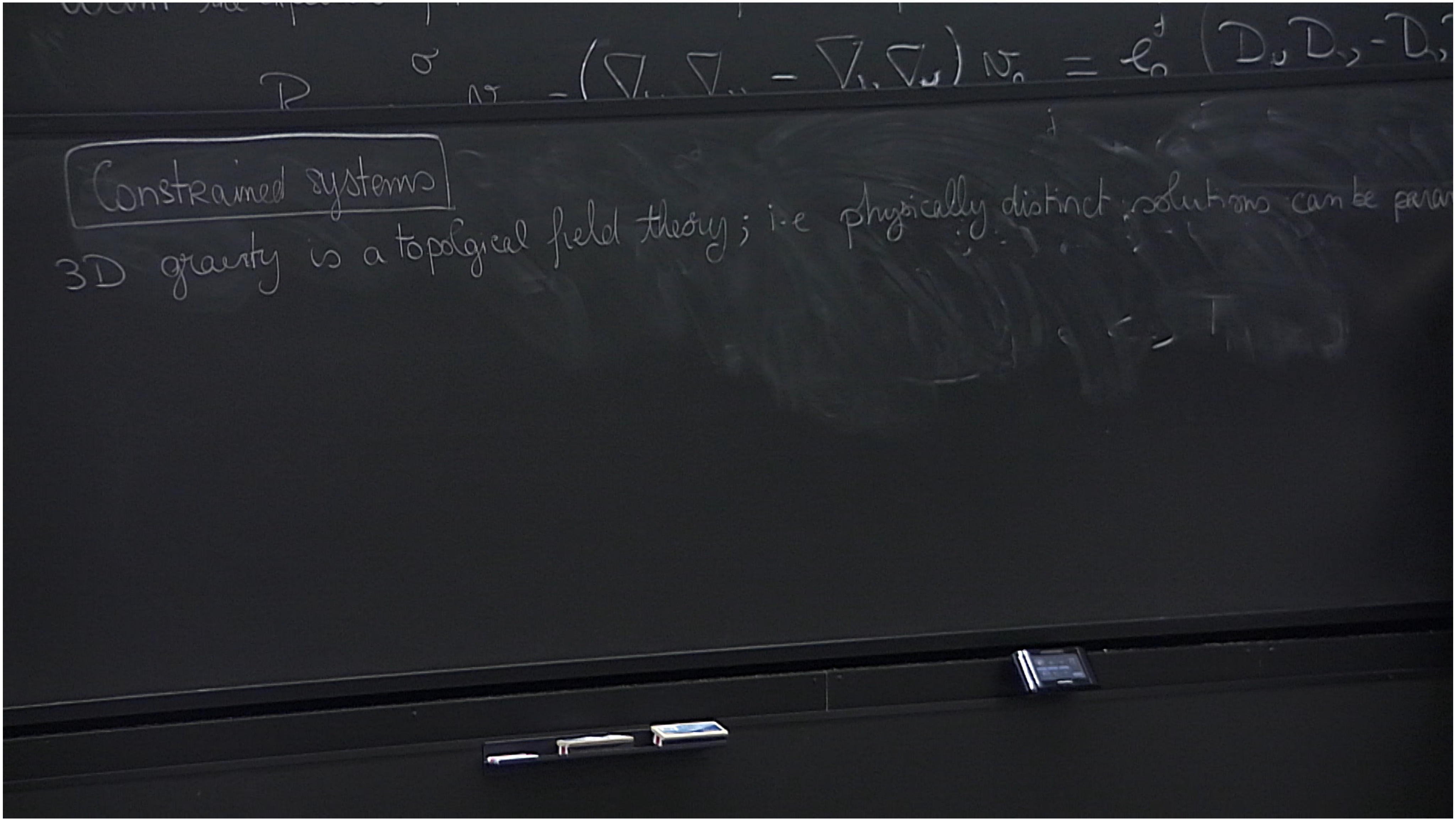
$\sqrt{g} = \det(e^i_\mu)$

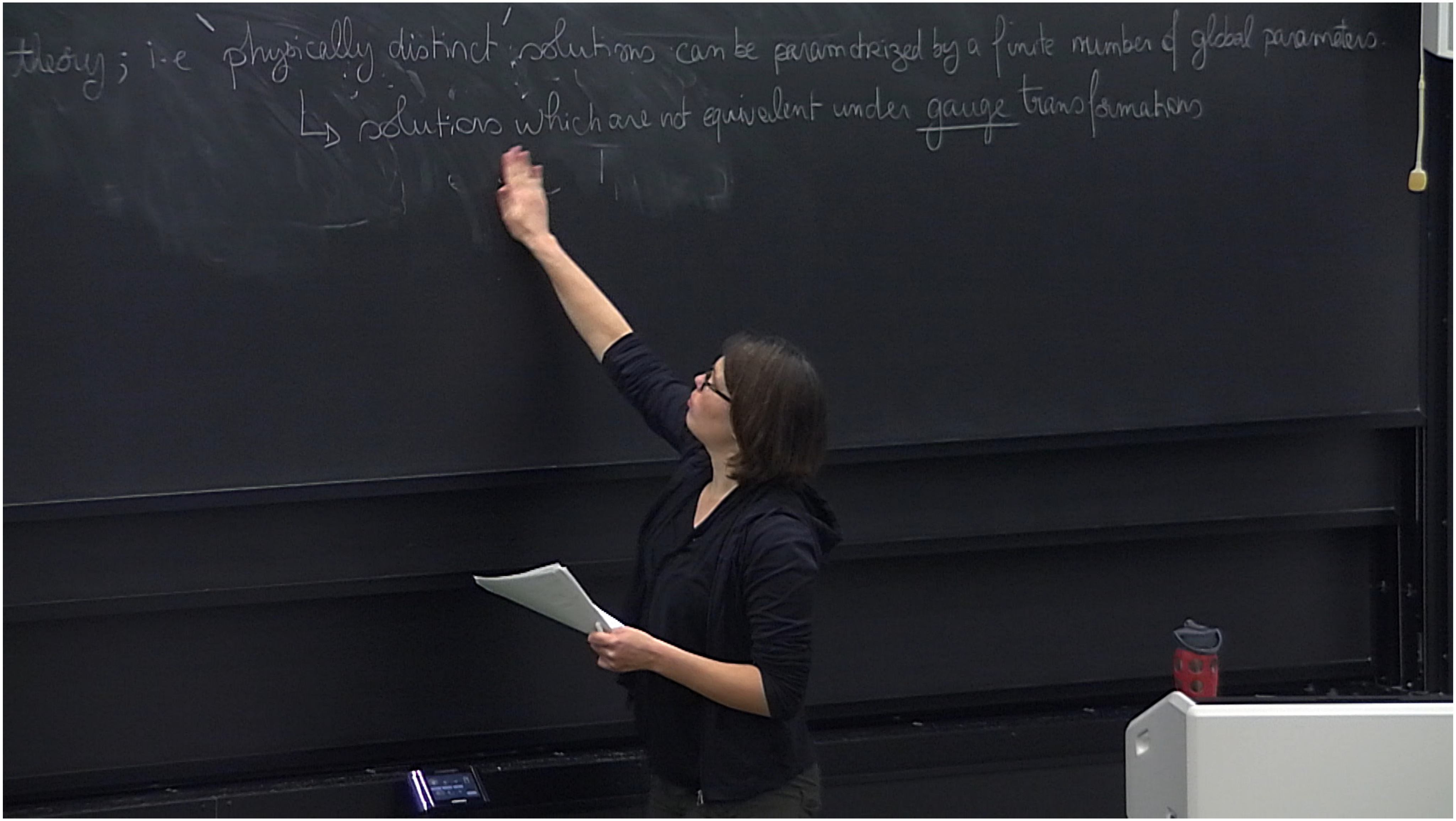
$= \dfrac{1}{\det(e^\mu_i)}$

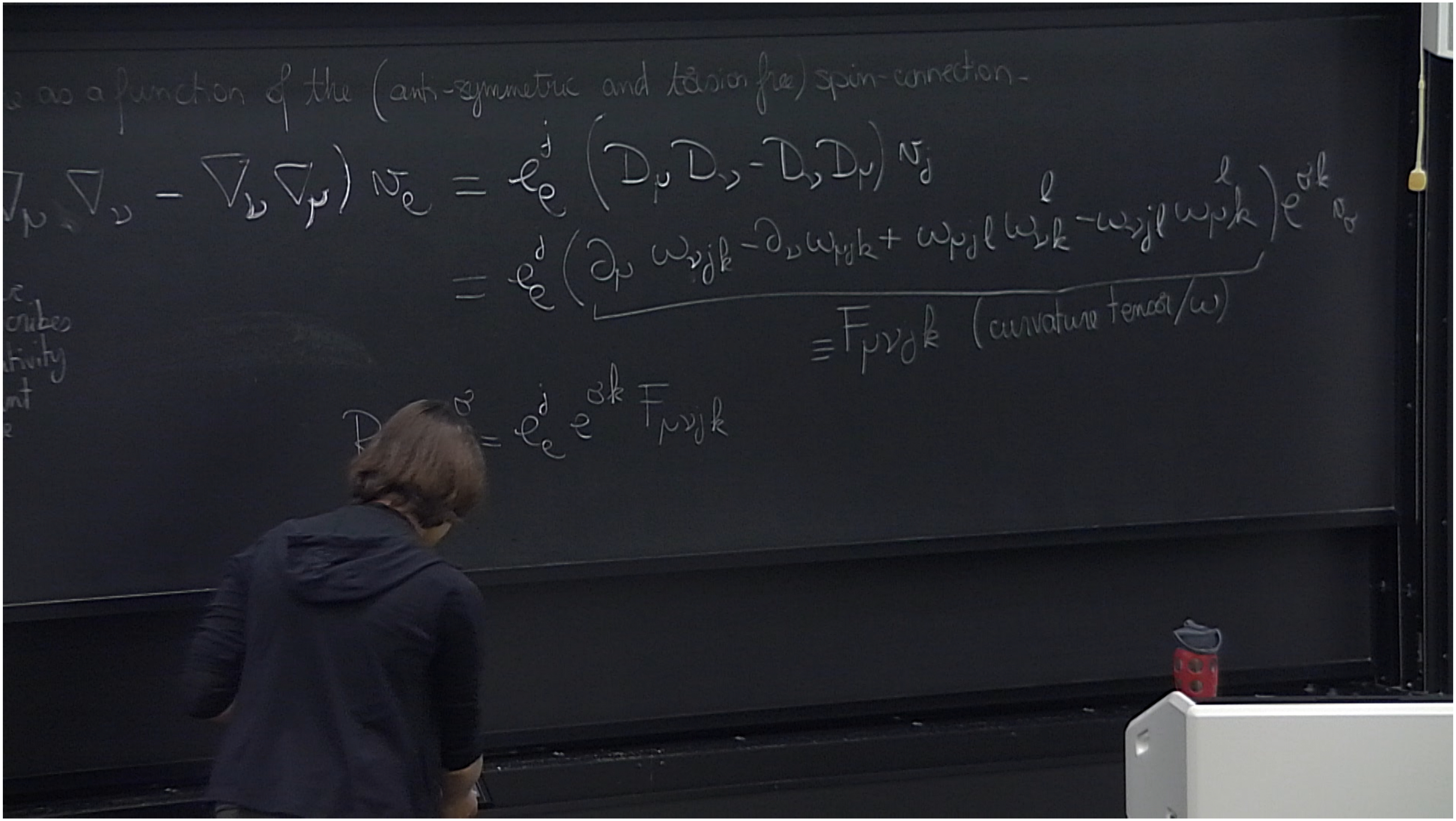
$\boxed{\text{EOM}}$

$\delta e^k_\sigma \Rightarrow F^j_{\mu\nu} = 0$

$\delta\omega\ (1\ I^\sigma) \Rightarrow T^j_{\mu\nu} = 0$


Constrained systems
3D gravity is a topolgical field theory; i.e physically distinct solutions can be


theory; i.e 'physically distinct' solutions can be parametrized by a finite number of global parameters.
↳ solutions which are not equivalent under gauge transformations


...e as a function of the (anti-symmetric and torsion free) spin-connection -
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$= e_e^{\ j} (\partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l}\, \omega_{\nu\ k}^{\ l} - \omega_{\nu j l}\, \omega_{\mu\ k}^{\ l}) e^{\sigma k} v_\sigma$
$\equiv F_{\mu\nu jk}$ (curvature tensor/ω)
$R \ldots^{\sigma} = e_e^{\ j} e^{\sigma k} F_{\mu\nu jk}$
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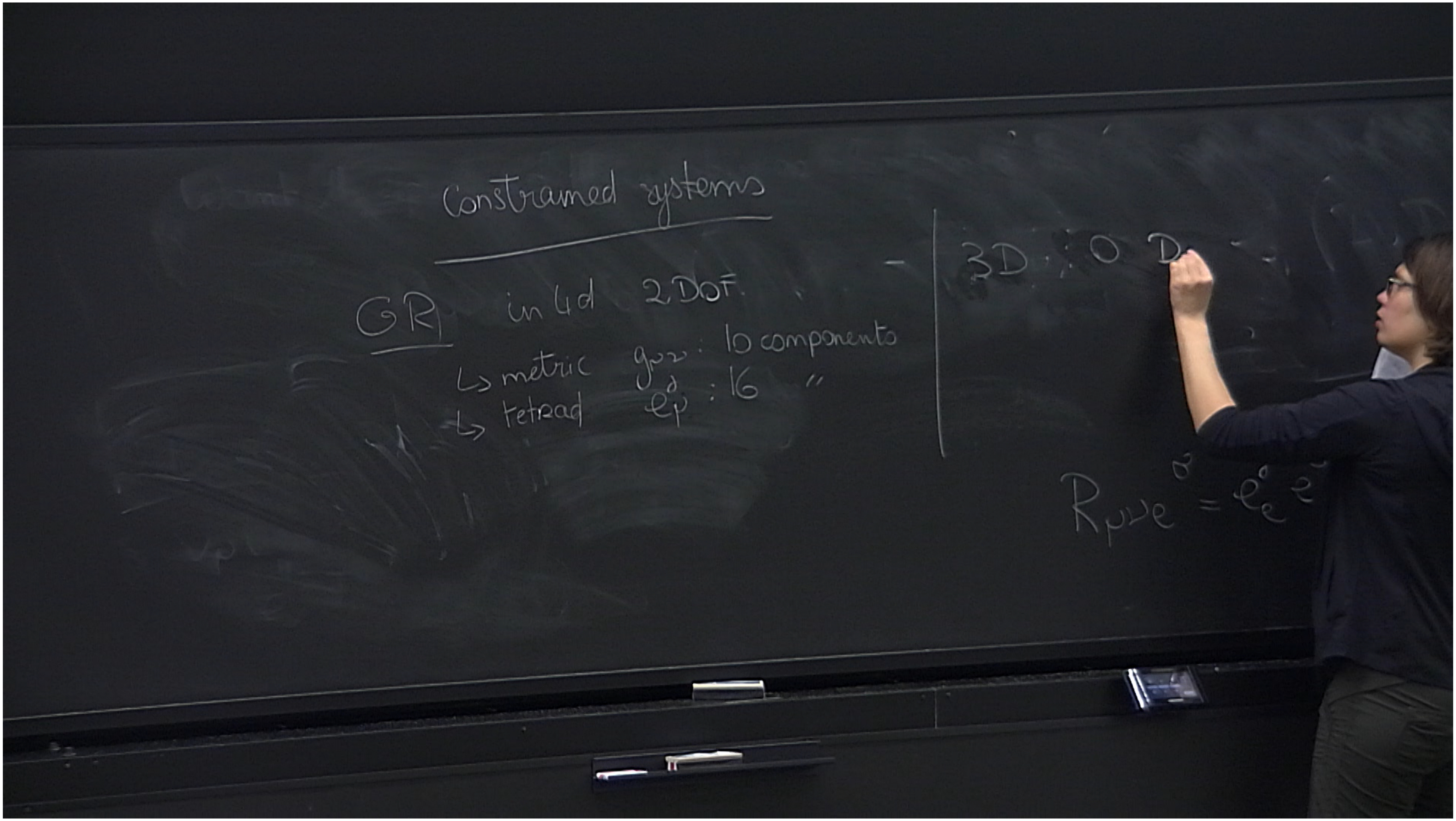
Constrained systems
GR
in 4d 2 DOF
↳ metric $g_{\mu\nu}$: 10 components
↳ tetrad e^{I}_{μ} : 16 "
3D : 0 D
$R_{\mu\nu\rho}{}^{\sigma} =$