Title: PSI 2016/2017 Explorations in Quantum Gravity - Lecture 2

Date: Mar 21, 2017 10:15 AM

URL: http://pirsa.org/17030069

Abstract:

Actions for gravity

- Review of the 4D Einstein-Hilbert action
- 3D gravity
- First order formulation - triads
- Spin connections
- 3D first order action for gravity

$\boxed{\text{action}}$ ← Dirac program

↳ encodes the basic variables

↳ global and local symmetries

↳ gauge

a) Review of the 4D Einstein-Hilbert action

Einstein-Hilbert action: $S = \frac{1}{\kappa} \int R \sqrt{g}\, d^4x$ with $\kappa = \frac{16 \pi G}{c^3} = 1$

$= S[g_{\mu\nu}]$.

$g \; d^4x \quad$ with $\kappa = \frac{16\pi G}{c^3} = 1$

Equations of motion

↳ varying with respect to the inverse metric

(⟺ varition w/ metric)

$$g^{\mu\lambda} g_{\lambda\rho} = \delta^\mu{}_\rho$$

$$\Rightarrow \delta g_{\mu\nu} = - g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$$

$$= S[g_{\mu\nu}].$$

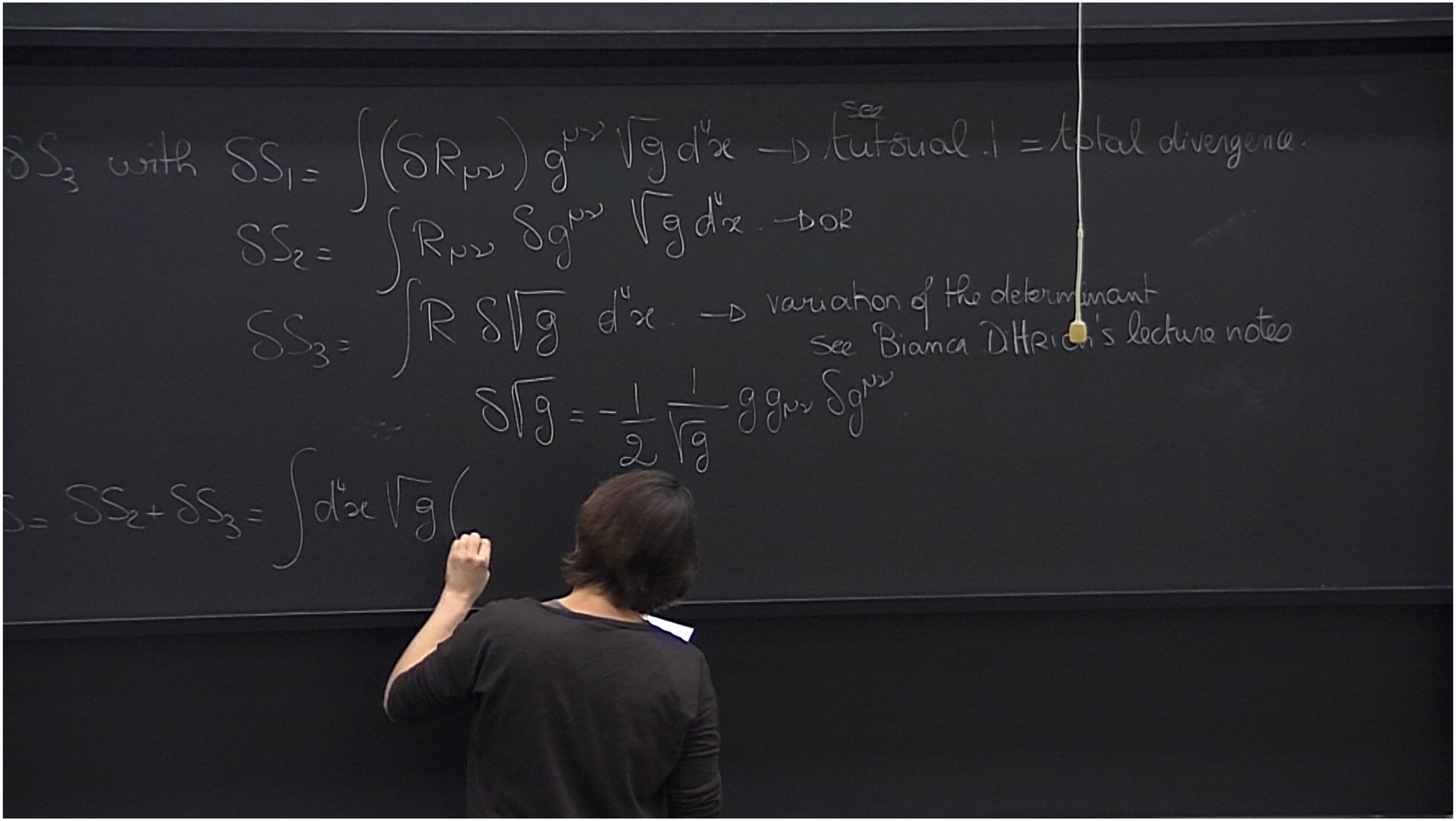
$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 \quad \text{with} \quad \delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x \longrightarrow \text{see tutorial!}$$

$= S[g_{\mu\nu}]$.

$\delta S = \delta S_1 + \delta S_2 + \delta S_3$ with $\delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x \rightarrow$ see tutorial!

$\delta S_2 = \int R_{\mu\nu}\, \delta g^{\mu\nu} \sqrt{g}\, d^4x$.

$\delta S_3 = \int R\, \delta\sqrt{g}\, d^4x$.



δS_3 with $\delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x$ → see tutorial 1 = total divergence.
$\delta S_2 = \int R_{\mu\nu}\, \delta g^{\mu\nu} \sqrt{g}\, d^4x$ → OK
$\delta S_3 = \int R\, \delta\sqrt{g}\, d^4x$ → variation of the determinant
see Bianca Dittrich's lecture notes
$\delta\sqrt{g} = -\frac{1}{2}\frac{1}{\sqrt{g}}\, g\, g_{\mu\nu}\, \delta g^{\mu\nu}$
$\delta S = \delta S_2 + \delta S_3 = \int d^4x \sqrt{g}\,($

$$\ldots_1 + \delta S_2 + \delta S_3 \quad \text{with} \quad \delta S_1 = \int (\delta R_{\mu\nu})\, g^{\mu\nu} \sqrt{g}\, d^4x \longrightarrow \overset{\text{see}}{\text{tutorial } 1} = \text{total divergen}\ldots$$

$$\delta S_2 = \int R_{\mu\nu}\, \delta g^{\mu\nu} \sqrt{g}\, d^4x \longrightarrow \text{OK}$$

$$\delta S_3 = \int R\, \delta\sqrt{g}\, d^4x \longrightarrow \text{variation of the determinant}$$

see Bianca Dittrich's lecture not…

$$\delta\sqrt{g} = -\frac{1}{2}\frac{1}{\sqrt{g}}\, g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\Longrightarrow \delta S = \delta S_2 + \delta S_3 = \int d^4x \sqrt{g} \underbrace{\left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}\right)}_{\equiv G_{\mu\nu}} \delta g^{\mu\nu}$$

$$\delta S_3 = \int R\, \delta\sqrt{g}\, d^4x \;\rightarrow\; \text{variation of the determinant}$$

see Bianca Dittrich's lecture notes

$$\delta\sqrt{g} = -\frac{1}{2}\frac{1}{\sqrt{g}}\, g\, g_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\int d^4x \sqrt{g}\, \underbrace{\left(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}\right)}_{\equiv\, G_{\mu\nu}} \delta g^{\mu\nu} \;\Rightarrow$$

Vacuum Einstein equations

$$G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

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Vacuum Einstein equations

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$$\Leftrightarrow R_{\mu\nu} = 0$$

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Vacuum Einstein equations

$$G_{\mu\nu} = 0 = R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}$$

$$\Leftrightarrow \boxed{R_{\mu\nu} = 0}$$

b) <u>3d gravity</u>

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Same eq of motions: $G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$

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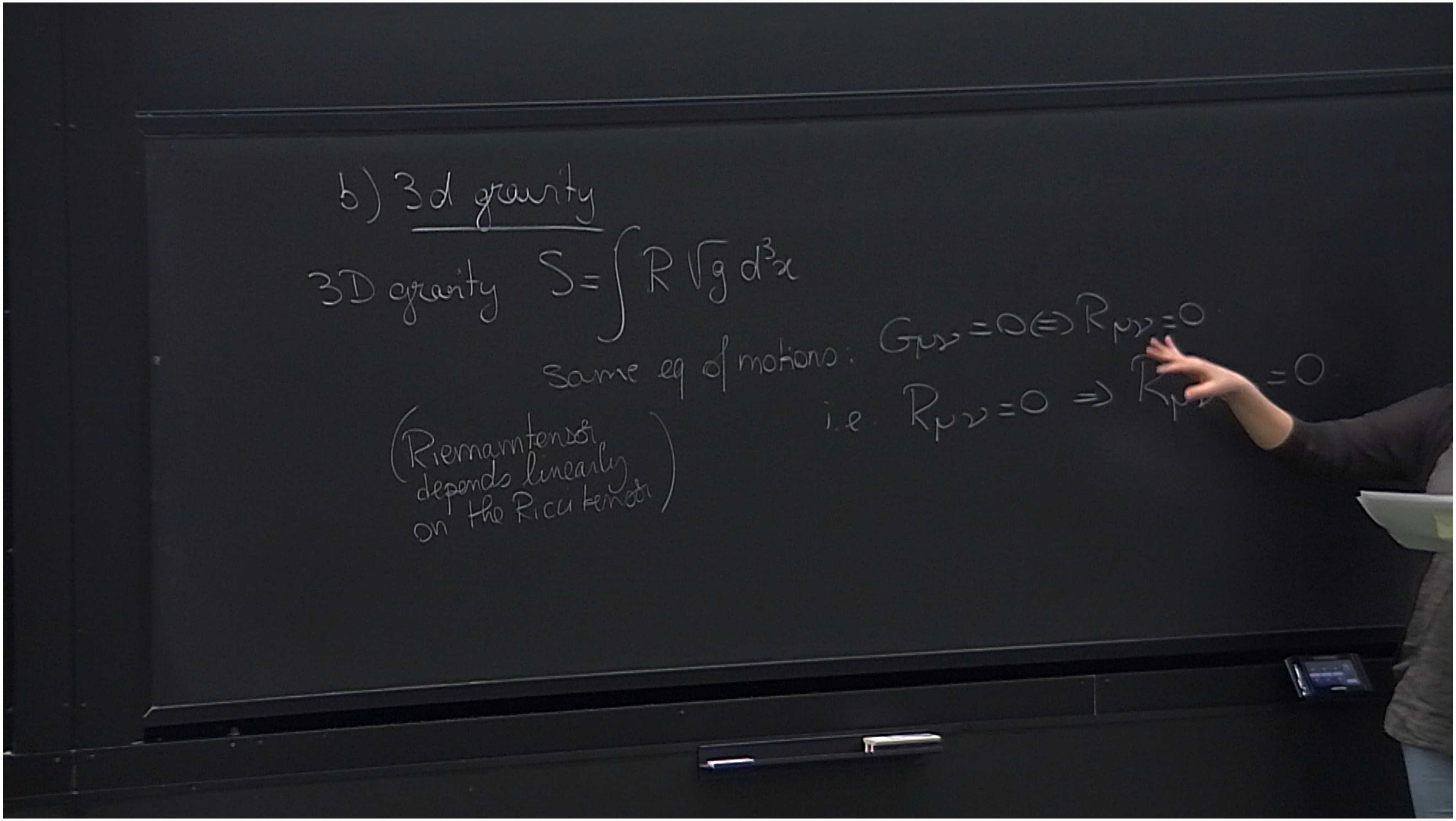
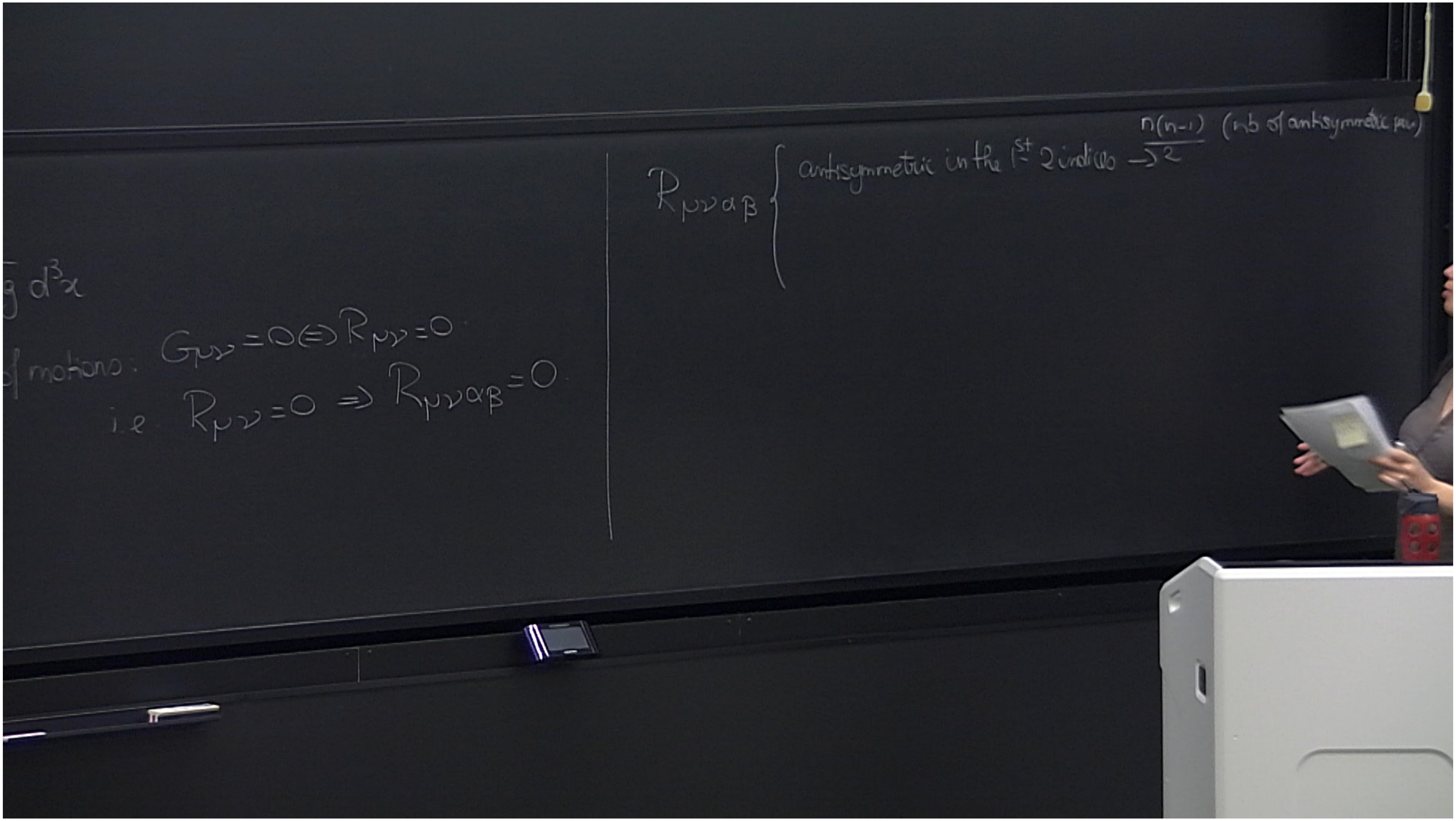
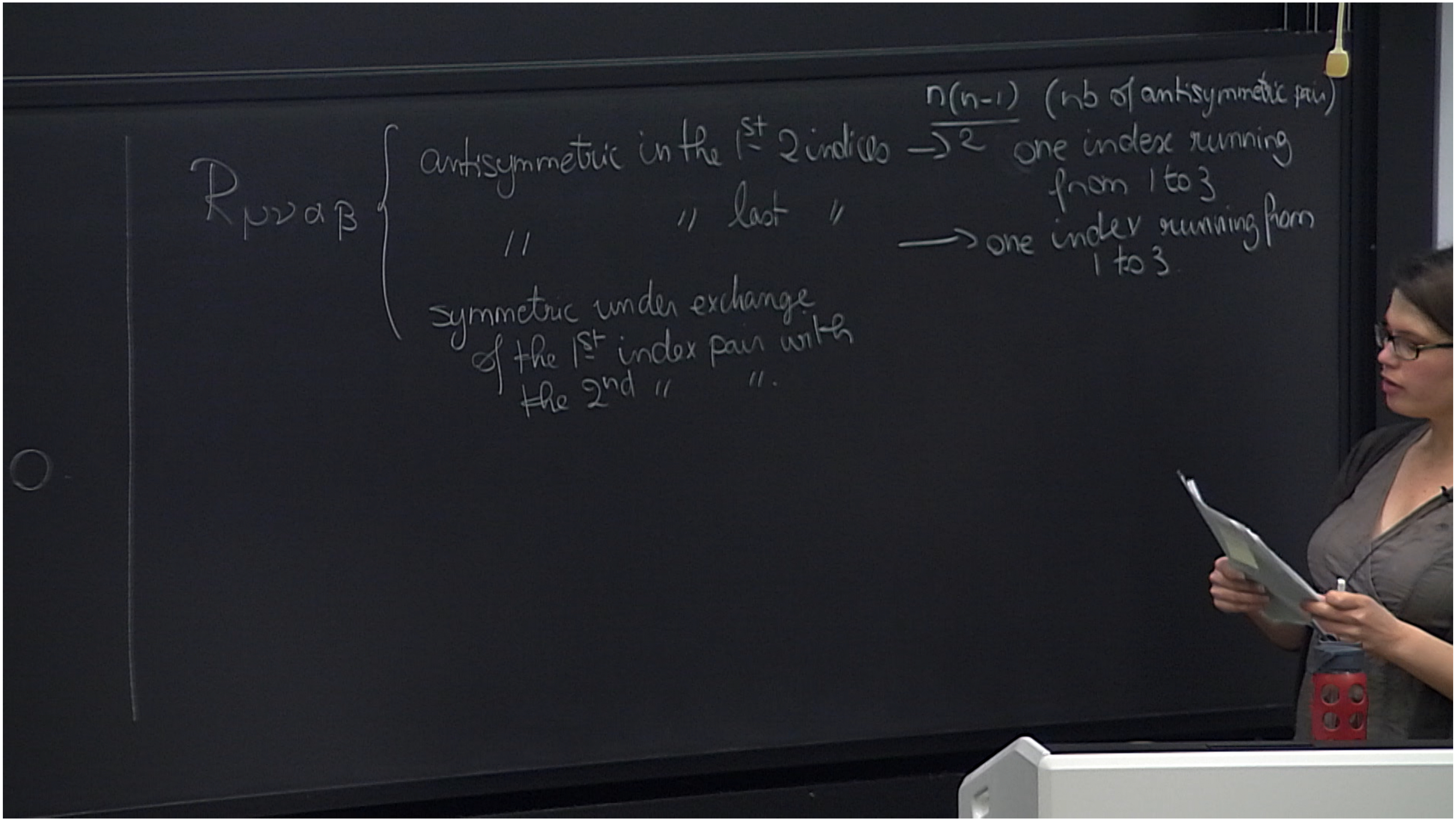
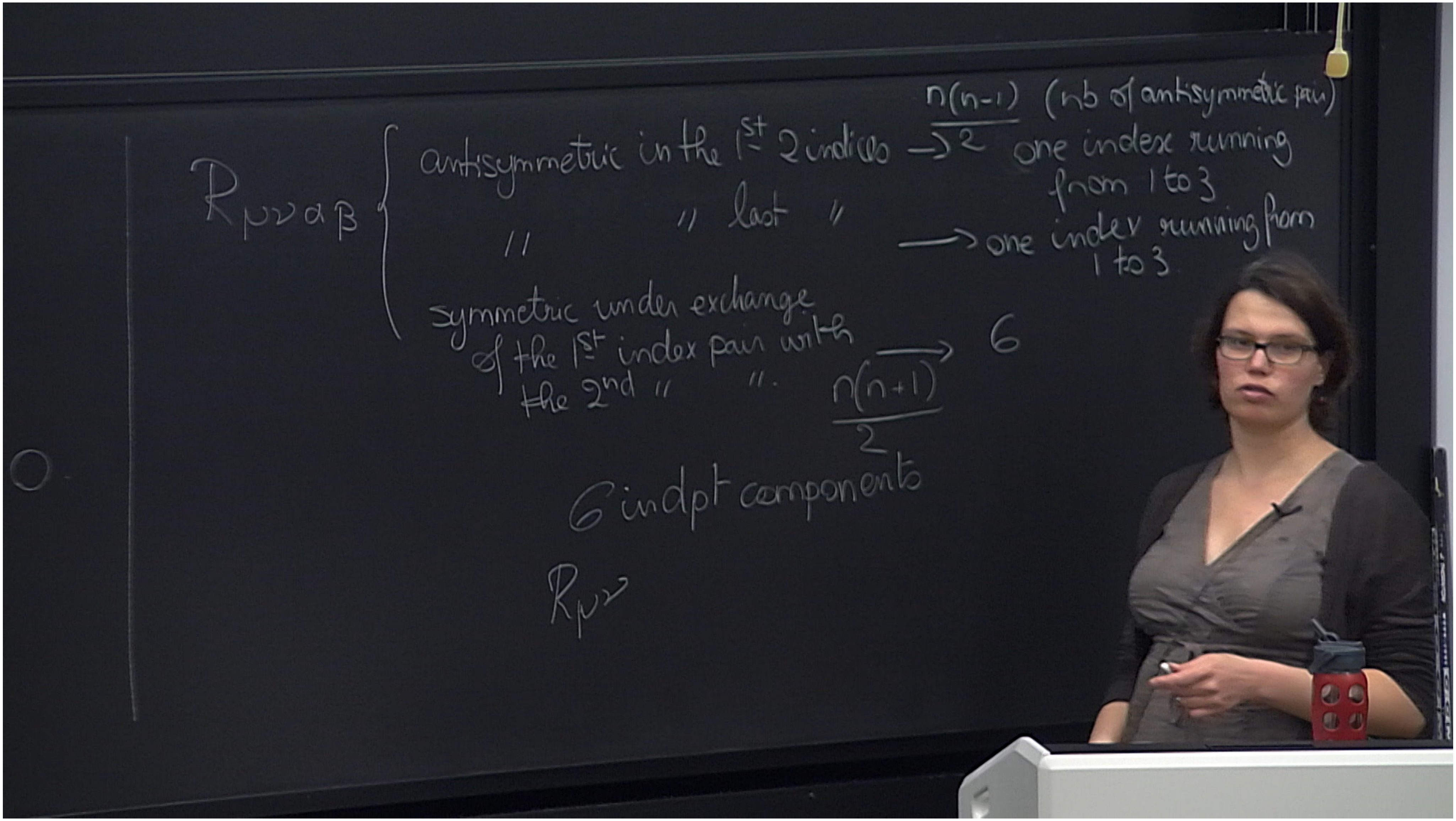
i.e. $R_{\mu\nu} = 0 \Rightarrow R_{\mu\nu\alpha\beta} = 0$


b) 3d gravity
3D gravity $S = \int R \sqrt{g}\, d^3x$
same eq of motions: $G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$
i.e. $R_{\mu\nu} = 0 \Rightarrow$
(Riemann tensor depends linearly on the Ricci tensor)


g d³x
of motions: Gμν = 0 ⟺ Rμν = 0.
i.e. Rμν = 0 ⇒ Rμναβ = 0.
Rμναβ
antisymmetric in the 1st 2 indices → n(n-1)/2 (nb of antisymmetric pairs)


$R_{\mu\nu\alpha\beta}$
antisymmetric in the 1st 2 indices → $\frac{n(n-1)}{2}$ (nb of antisymmetric pair) one index running from 1 to 3
" " last " → one index running from 1 to 3.
symmetric under exchange of the 1st index pair with the 2nd " ".



n(n-1)/2 (nb of antisymmetric pair)
Rμναβ
antisymmetric in the 1st 2 indices → one index running from 1 to 3
" " last " → one index running from 1 to 3
symmetric under exchange of the 1st index pair with the 2nd " " → 6
n(n+1)/2
6 indpt components
Rμν

$\Rightarrow$ every solution of the vacuum Einstein eq. with $\Lambda = 0$ is flat

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no local degree of freedom.

global d.o.f $\rightarrow$ topology

<u>Manifold</u>: $M \simeq \mathbb{R} \times \underset{2d}{\Sigma}$

$\Rightarrow$ every solution of the vacuum Einstein eq. with $\Lambda = 0$ is flat

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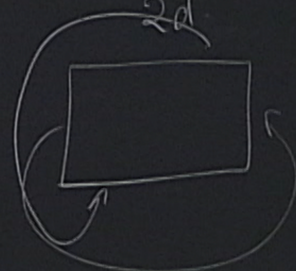
Manifold : $M \simeq \mathbb{R} \times \underbrace{\Sigma}_{2d}$

Torus



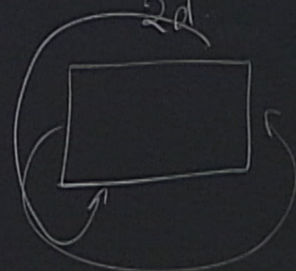
$\Rightarrow$ every solution of the vacuum Einstein eq. with $\Lambda = 0$ is flat

no local degree of freedom.

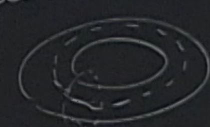
global d.o.f $\rightarrow$ topology

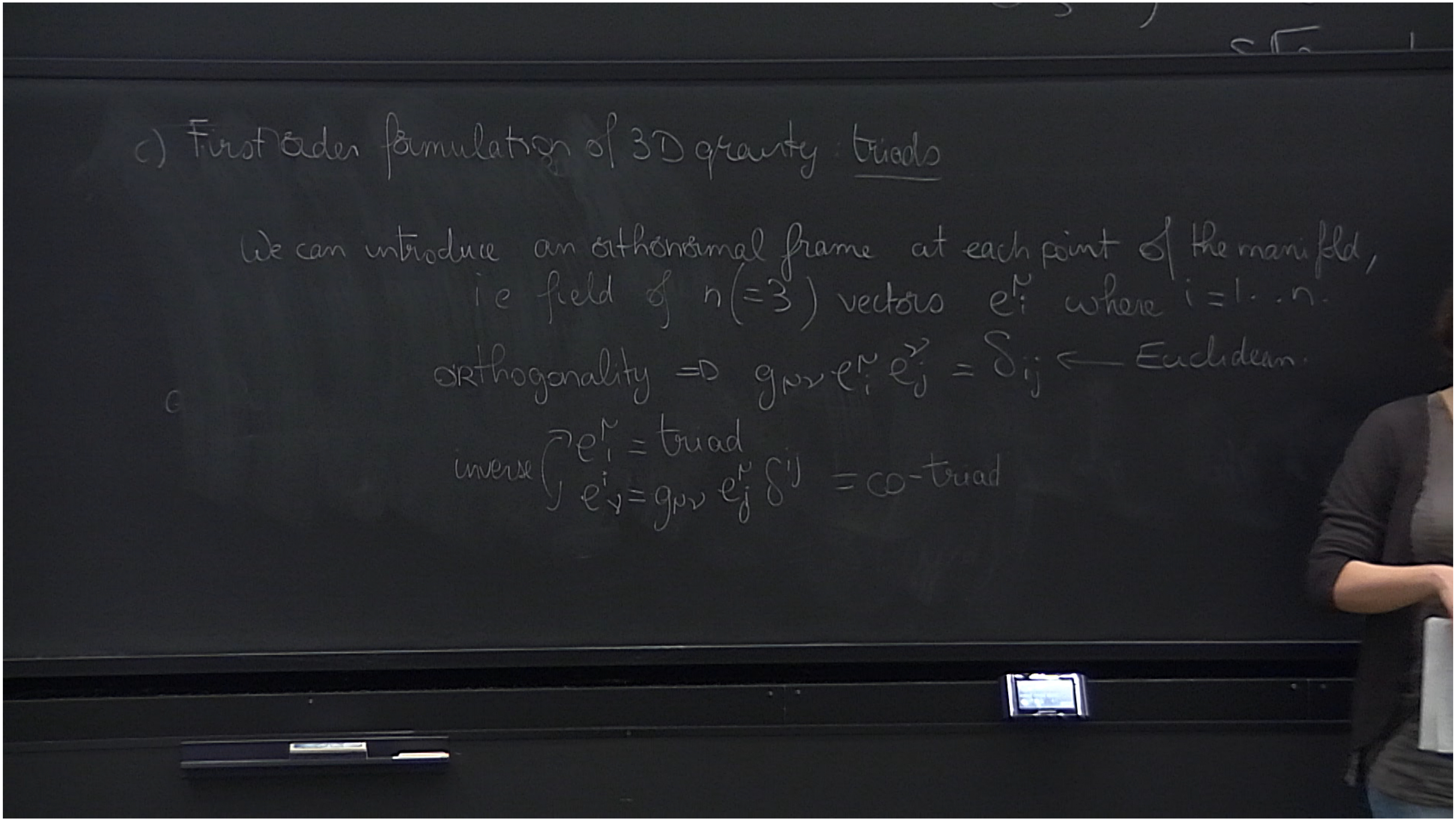
Manifold: $M \simeq \mathbb{R} \times \Sigma$ (2d)

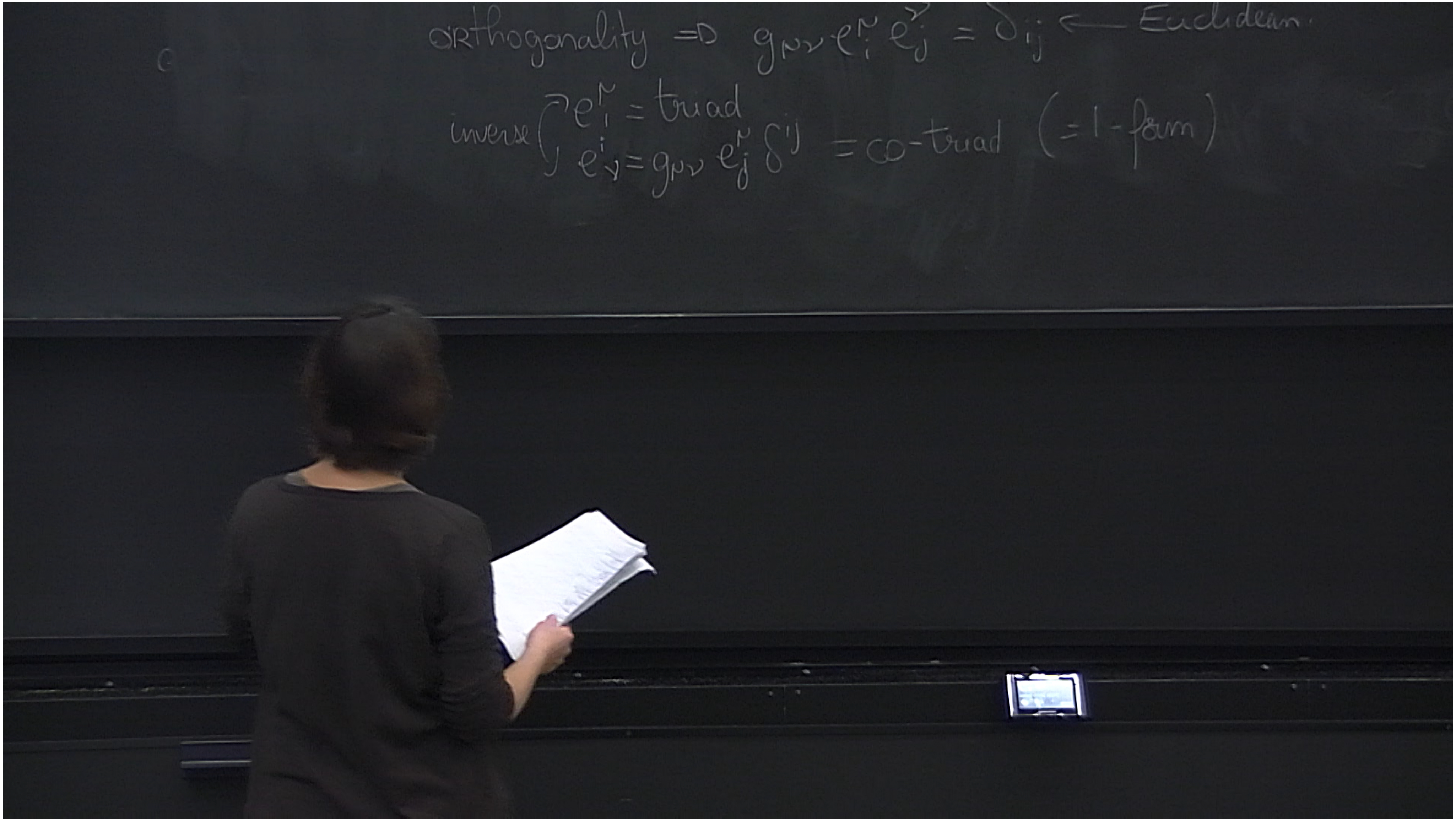
Torus



2 parameters: lengths of the curves




c) First order formulation of 3D gravity : triads
We can introduce an orthonormal frame at each point of the manifold,
i.e field of $n(=3)$ vectors e^μ_i where $i=1..n$.
orthogonality $\Rightarrow$ $g_{\mu\nu} e^\mu_i e^\nu_j = \delta_{ij}$ $\leftarrow$ Euclidean.
inverse
e^μ_i = triad
$e^i_\nu = g_{\mu\nu} e^\mu_j \delta^{ij}$ = co-triad


orthogonality ⇒ $g_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} = \delta_{ij}$ ← Euclidean.
inverse e^{μ}_{i} = triad
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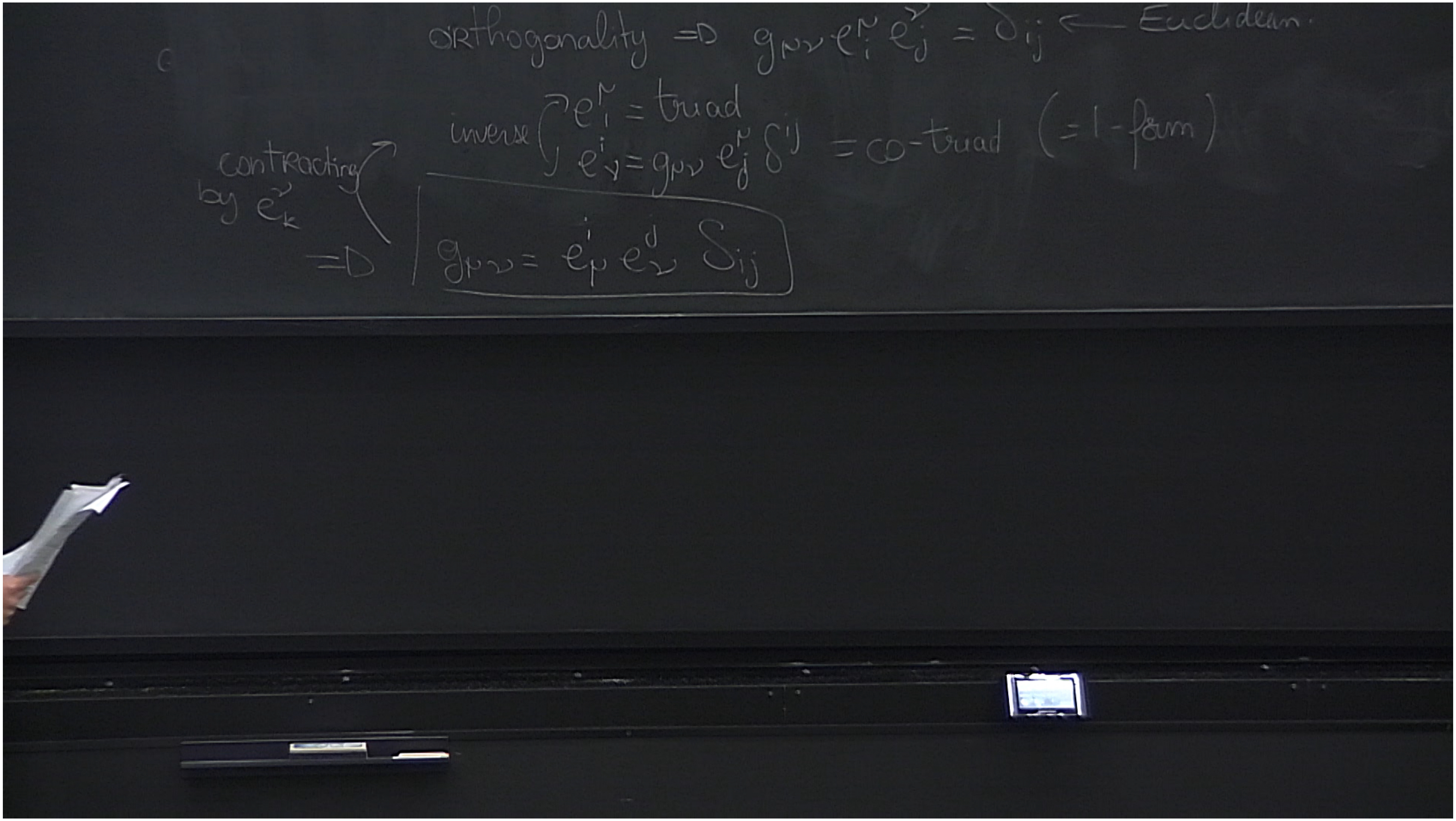
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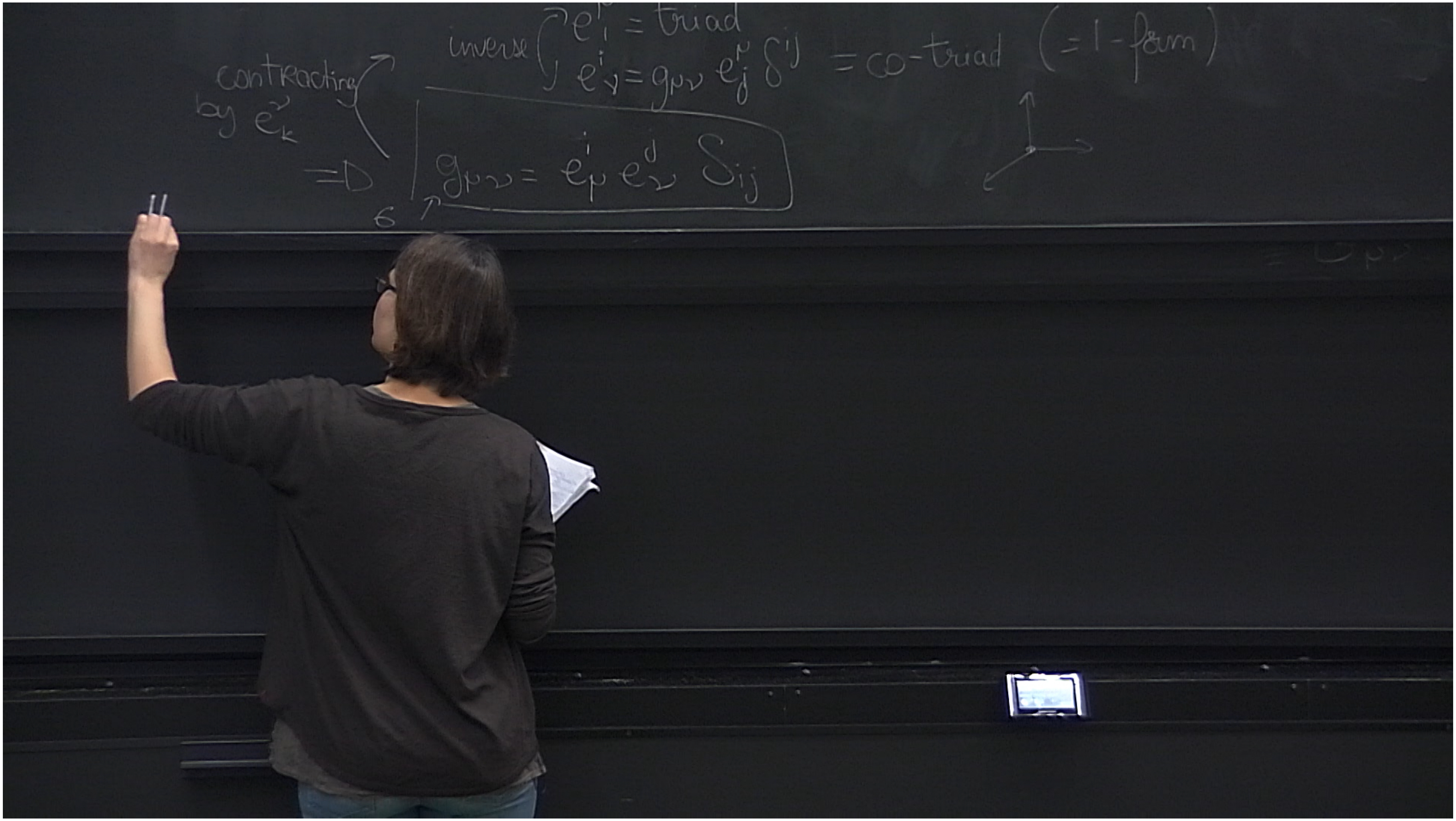
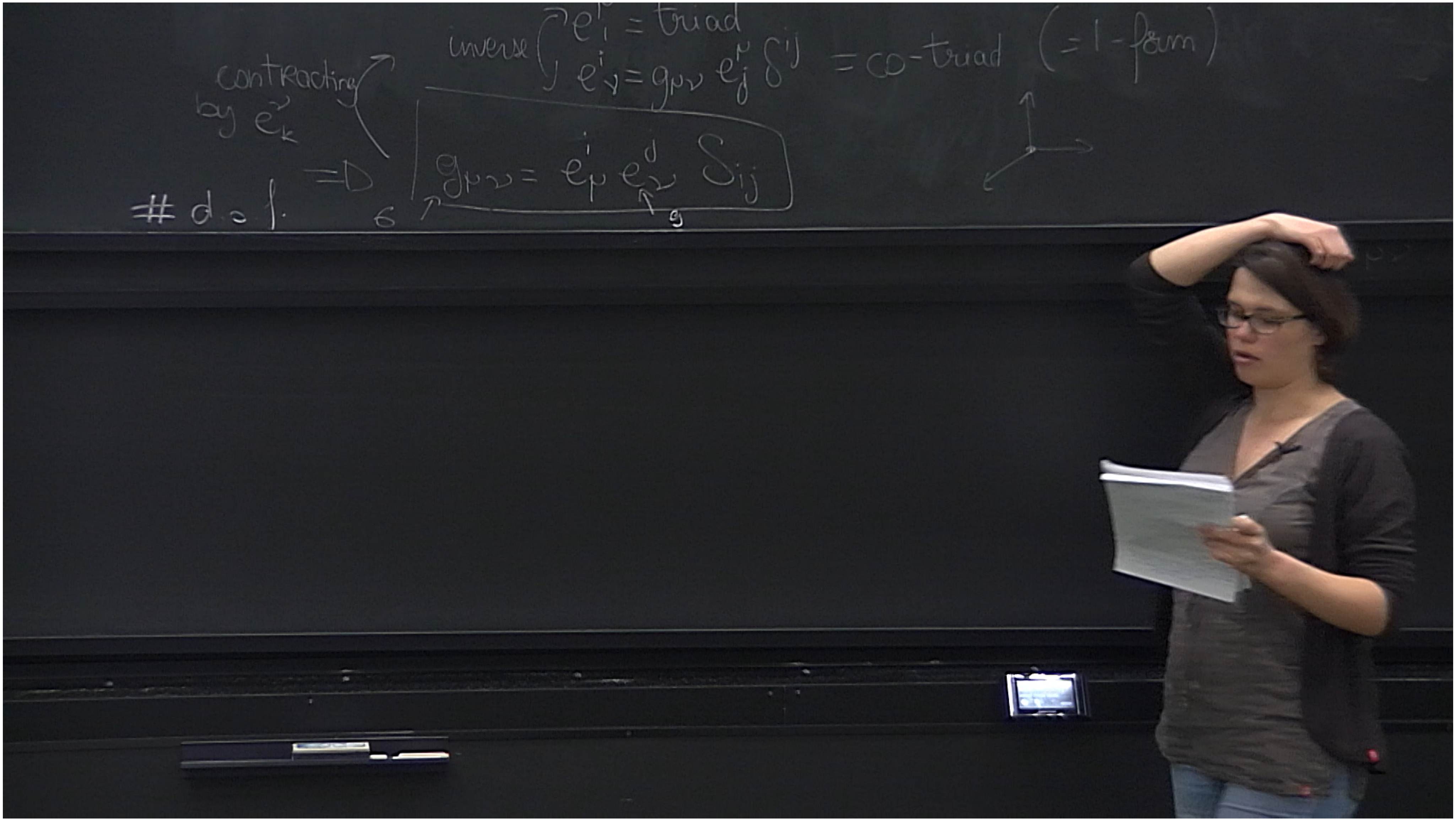
$e^{i}_{\nu} = g_{\mu\nu} e^{\mu}_{j} \delta^{ij}$ = co-triad (= 1-form)
contracting by e^{ν}_{k}
⟹ $g_{\mu\nu} = e^{i}_{\mu} e^{j}_{\nu} \delta_{ij}$


inverse
e = triad
e = gμν e δ = co-triad (= 1-form)
contracting by e
gμν = e e δij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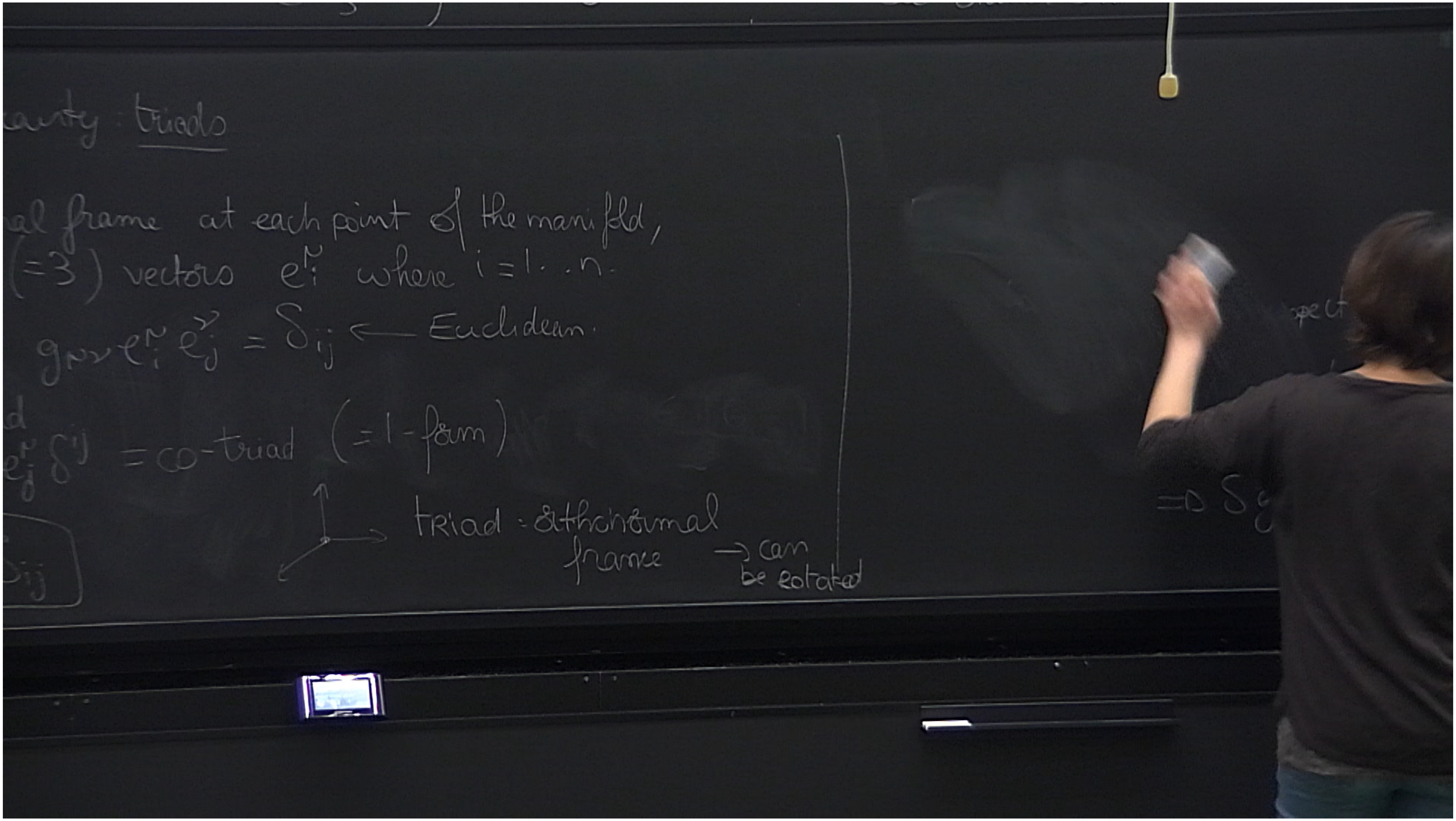
$e^i_\nu = g_{\mu\nu} e^\mu_j \delta^{ij}$ = co-triad (= 1-form)
contracting by $\vec{e}_k$
$g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$
d.o.f.
6
9


triads
frame at each point of the manifold,
(=3) vectors e^μ_i where $i = 1 .. n$.
$g_{\mu\nu} e^\mu_i e^\nu_j = \delta_{ij}$ ← Euclidean.
= co-triad (=1-form)
triad = orthonormal frame → can be rotated

...h point of the manifold,

e^r_i where $i = 1 \ldots n$

$\delta_{ij} \longleftarrow$ Euclidean

(= 1-form)

triad = orthonormal frame $\rightarrow$ can be rotated

additional SO(3)-gauge symmetry.

$\uparrow$ SU(2). $\quad su(2) \simeq so(3)$.

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($=$ 1-form)

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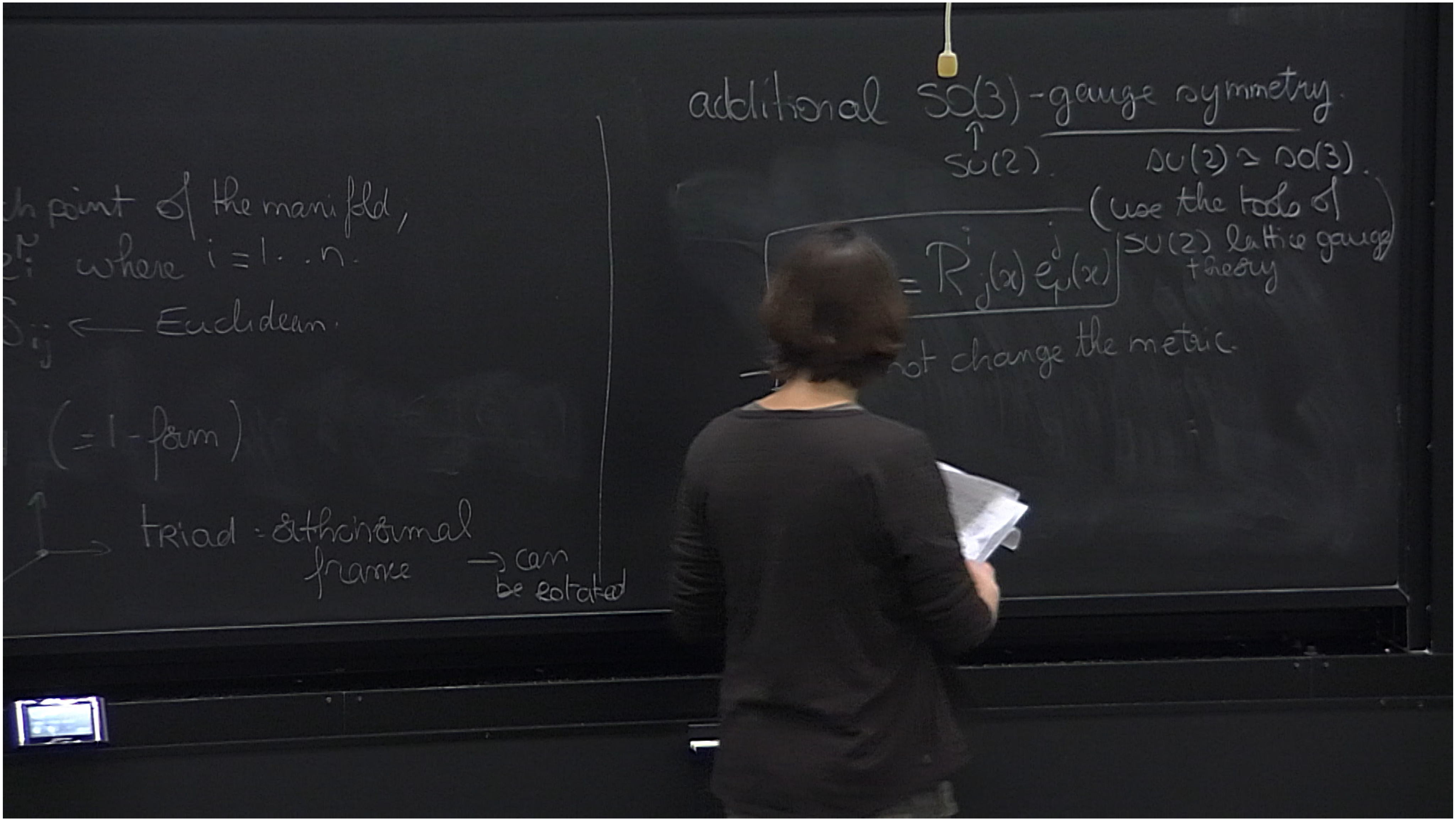
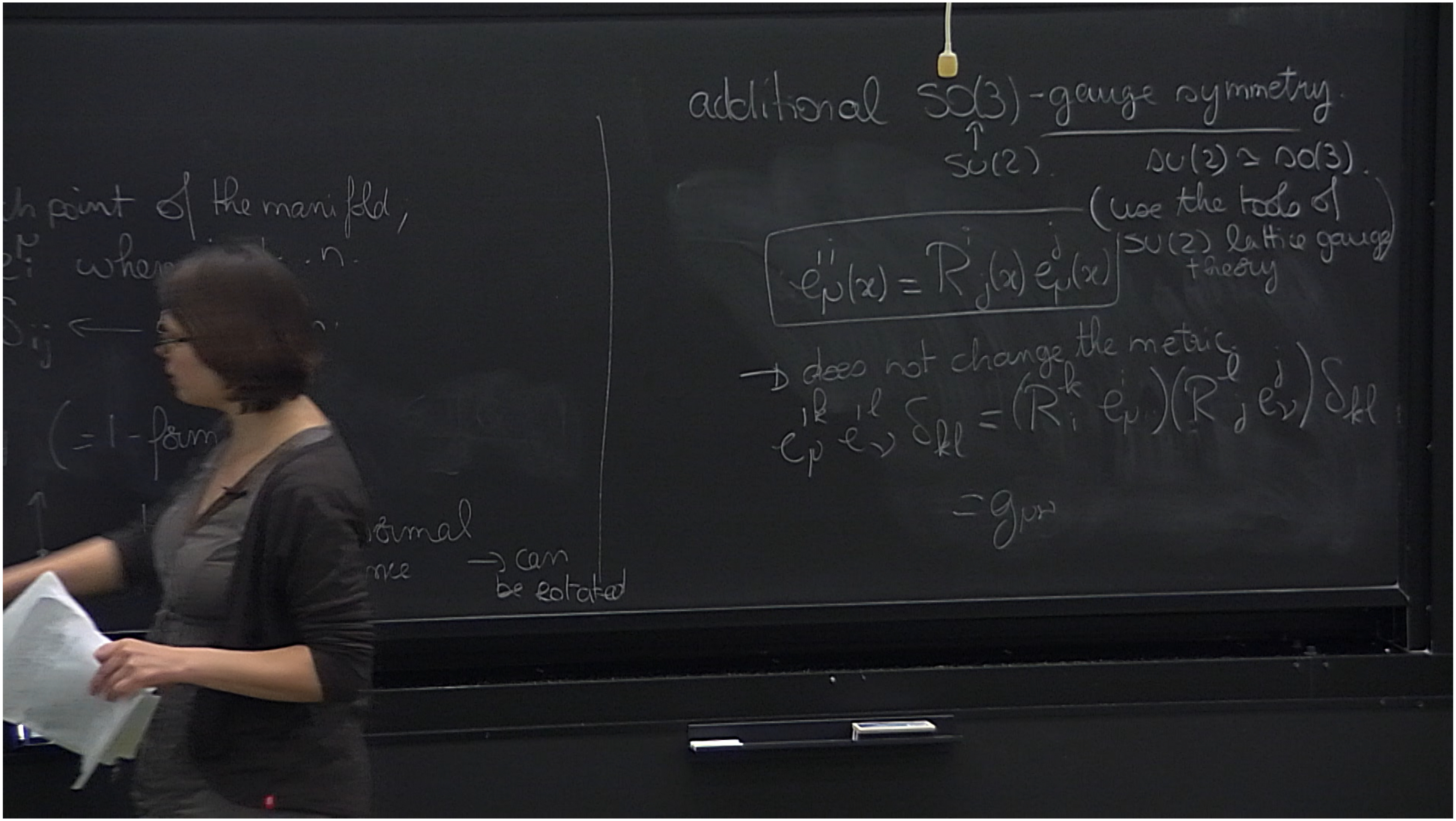
additional $SO(3)$-gauge symmetry.
$\uparrow$
$SU(2)$.

$SU(2) \simeq SO(3)$.
(use the tools of $SU(2)$ lattice gauge theory)


additional SO(3) -gauge symmetry.
SU(2).
SU(2) ≃ SO(3).
(use the tools of
SU(2) lattice gauge
theory
point of the manifold,
where i = 1 . . n.
Euclidean.
(= 1-form)
triad = orthonormal
frame
→ can
be rotated
change the metric.


additional SO(3) -gauge symmetry.
SU(2)
SU(2) ≃ SO(3).
(use the tools of SU(2) lattice gauge theory)
e'^i_μ(x) = R^i_j(x) e^j_μ(x)
→ does not change the metric
e'^k_μ e'^l_ν δ_kl = (R^k_i e^i_μ)(R^l_j e^j_ν) δ_kl
= g_μν
point of the manifold,
(= 1-form
normal
→ can be rotated

c) First order formulation of 3D gravity: triads

We can introduce an orthonormal frame at each point

i.e field of $n(=3)$ vectors e^μ_i where

orthogonality $\Rightarrow$ $g_{\mu\nu} e^\mu_i e^\nu_j = \delta_{ij} \longleftarrow$

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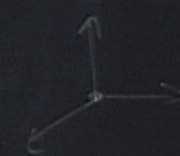
contracting by e^ν_k

$\Rightarrow$ $\boxed{g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}}$

d.o.f. 6 9



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additional SO(3) - gauge symmetry.

↑ SU(2).

SU(2) ≃ SO(3).

(use the tools of SU(2) lattice gauge theory)

$$\boxed{e'^{i}_{\mu}(x) = R^{i}{}_{j}(x)\, e^{j}_{\mu}(x)}$$

→ does not change the metric

$$e'^{k}_{\mu}\, e'^{l}_{\nu}\, \delta_{kl} = (R^{k}{}_{i}\, e^{i}_{\mu})(R^{l}{}_{j}\, e^{j}_{\nu})\, \delta_{kl}$$

$= g_{\mu\nu}$ $\quad R^{T} R = \mathrm{id}$.

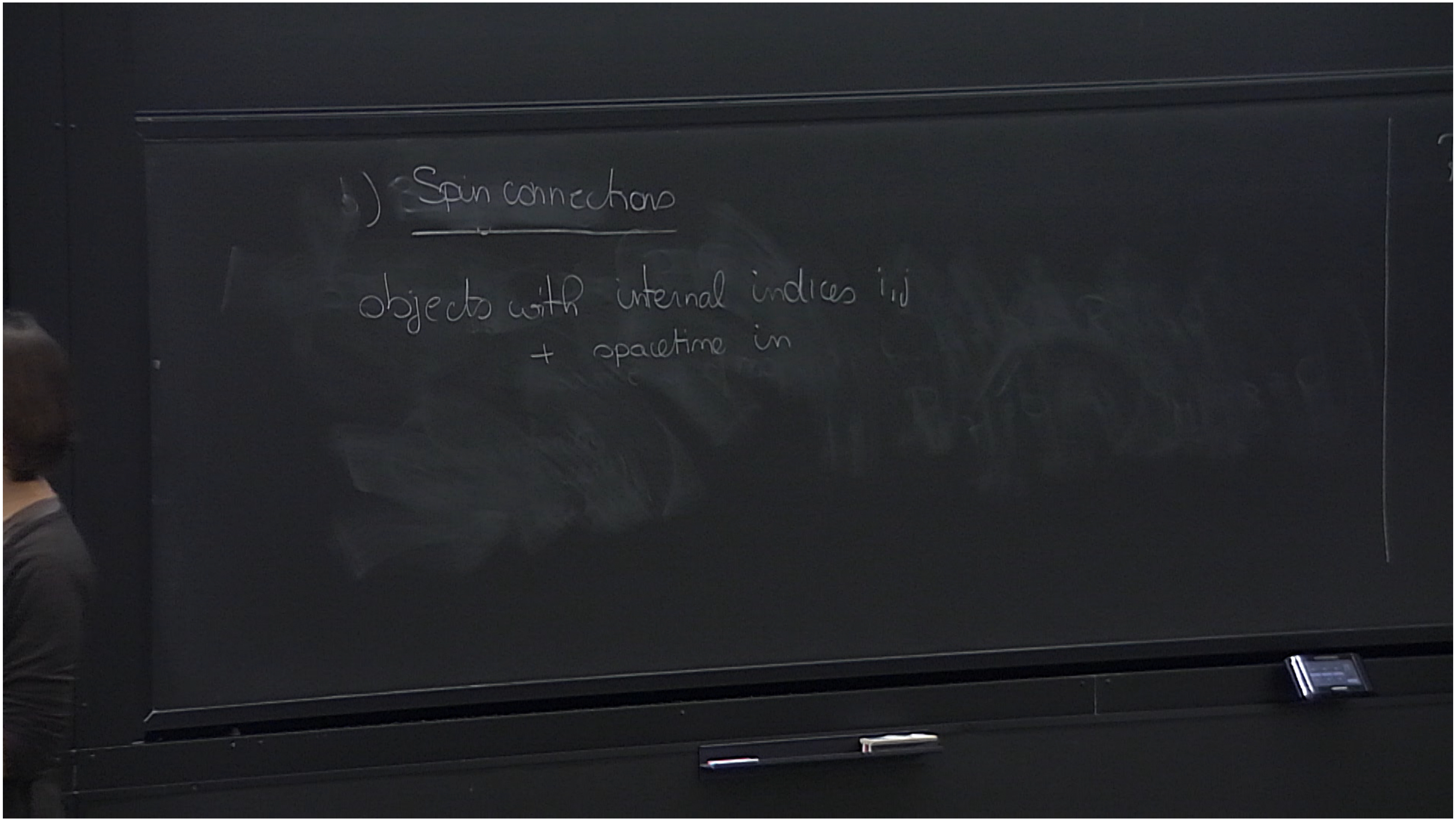
...h point of the manifold,

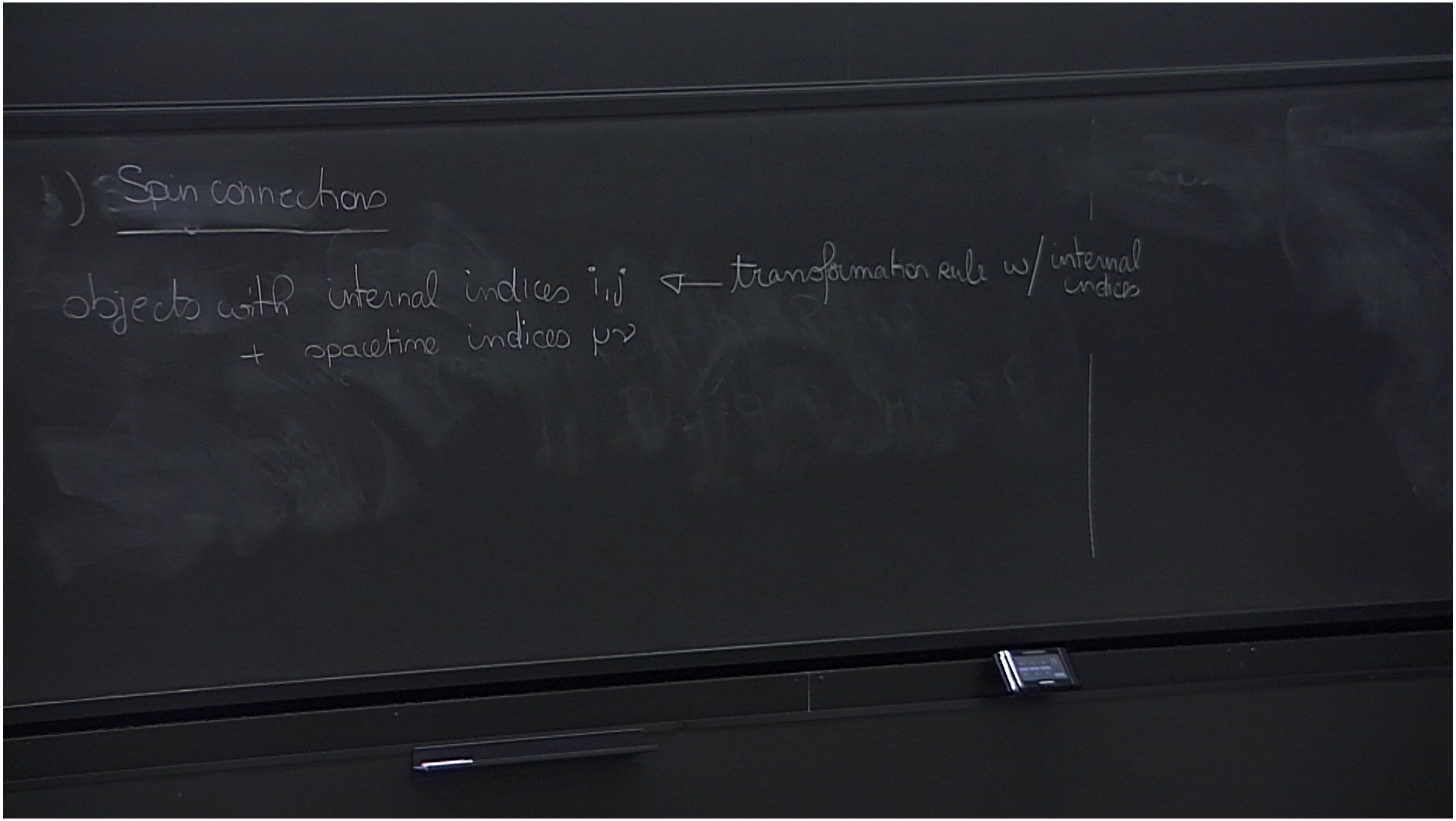
e^{μ}_{i} where $i = 1 \ldots n$.

δ_{ij} ← Euclidean.

(= 1-form)

triad = orthonormal frame → can be rotated


b) Spin connections
objects with internal indices i,j
+ spacetime in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objects with internal indices i, j ← transformation rule w/ internal indices
+ spacetime indices μ ν

) Spin connections

objects with internal indices i,j $\leftarrow$ transformation rule w/ internal indices

$+$ spacetime indices $\mu\nu$ $\leftarrow$ spacetime tensor

b) Spin connections

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Spacetime indices : covariant derivative? ∇_μ

b) <u>Spin connections</u>

objects with internal indices i, j ← transformation rule w/ internal indices

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<u>Spacetime indices</u> : covariant derivative? ∇_μ (with respect to $\Gamma^{\sigma}_{\mu\nu}$)

b) Spin connections

objects with internal indices i, j $\leftarrow$ transformation rule w/ internal indices
+ spacetime indices $\mu\nu$ $\leftarrow$ spacetime tensor

Spacetime indices : covariant derivative? ∇_μ (with respect to $\Gamma^{\sigma}_{\mu\nu}$)

$\hookrightarrow$ need to // transport spacetime tensors.

$\hookrightarrow$ If v_μ is a vector, then $\nabla_\nu v_\mu$ tensor

any solution of the vacuum Einstein eq. with $\Lambda = 0$ is flat

+ ⚠ internal indices

covariant derivative w/ internal indices: $\underline{\omega_{\mu k}^{i}}$: spin connection

+ ⚠ internal indices

covariant derivative w/ internal indices: $\omega^{i}_{\mu k}$: spin connection

Covariant derivative w/ internal and spacetime indices

$$D_\mu \phi^j = \partial_\mu \phi^j + \omega^{j}_{\mu k} \phi^k$$

$$D_\mu \phi_j = \partial_\mu \phi_j - \omega^{k}_{\mu j} \phi_k$$

$$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^{e}_{\mu\nu} v^j_e + \omega^{j}_{\mu k} v^k.$$

1. internal indices

covariant derivative w/ internal indices: $\underline{\omega^{i}_{\mu k}}$: spin connection

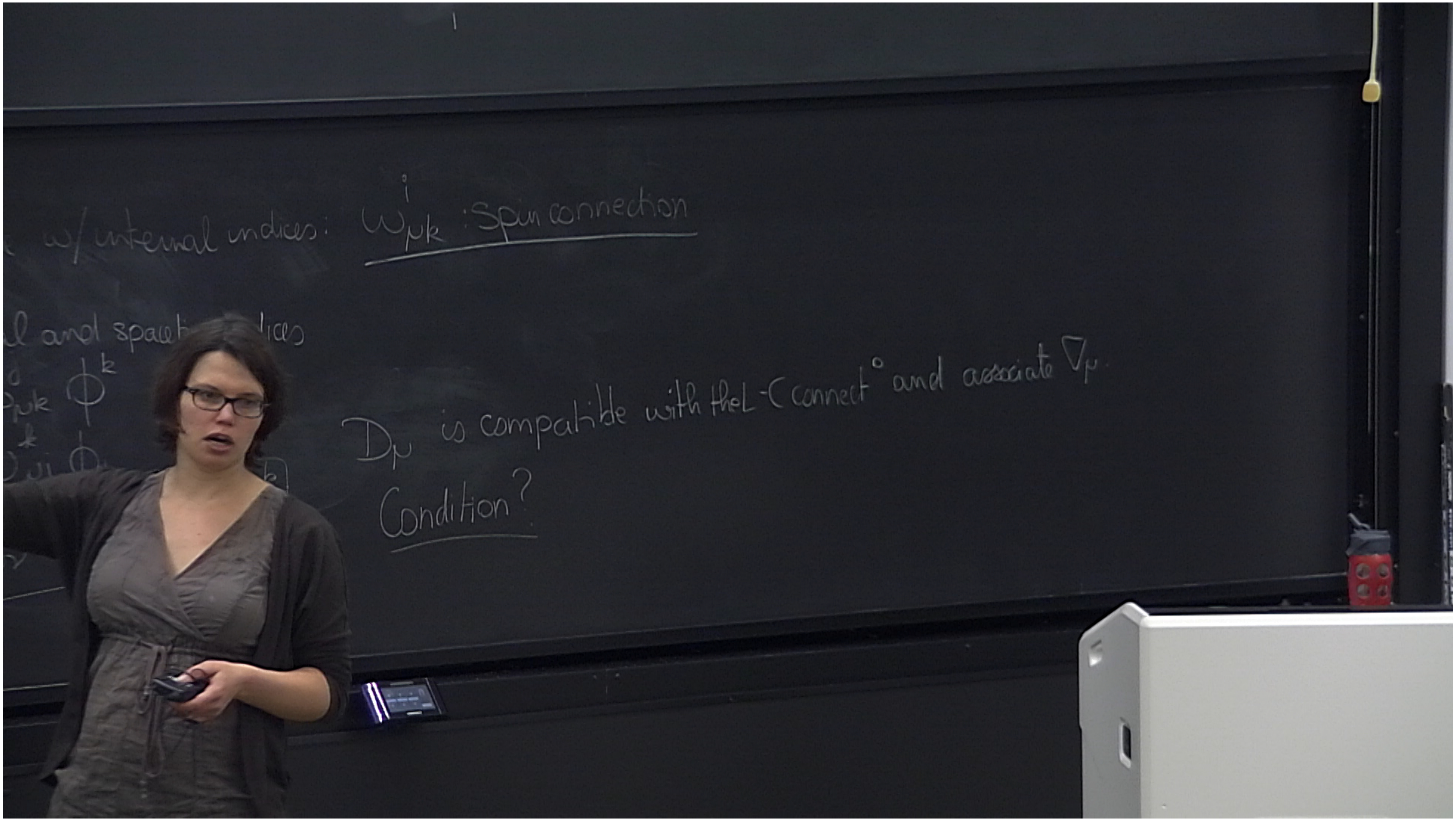
covariant derivative w/ internal and spacetime indices

$$D_\mu \phi^j = \partial_\mu \phi^j + \omega^{j}_{\mu k} \phi^k$$

$$D_\mu \phi_j = \partial_\mu \phi_j - \omega^{k}_{\mu j} \phi_k$$

$$\boxed{D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^{e}_{\mu\nu} v^j_e + \omega^{j}_{\mu k} v^k}$$

D_μ is compatible with the L-C connect° an


w/ internal indices: $\omega_{\mu k}^{\ i}$: spin connection
and spacetime indices
D_μ is compatible with the L-C connect° and associate ∇_μ.
Condition?