Title: Arithmetic Structures in Spectral Models of Gravity

Date: Mar 03, 2017 03:00 PM

URL: http://pirsa.org/17030049

Abstract: <p>The spectral action functional of noncommutative geometry provides a model of Euclidean (modified) gravity, possibly coupled to matter. The terms in the large energy asymptotic expansion of the spectral action can be computed via pseudodifferential calculus. In the case of highly symmetric spacetimes, like Robertson-Walker metrics and Bianchi IX gravitational instantons, there is a richer arithmetic structure in the spectral action, and the terms in the asymptotic expansion are expressiblein terms of periods of motives and of modular forms. This reveals a new occurrence of interesting periods and motives in high-energy physics.</p>

Arithmetic Structures in Spectral Models of Gravity

Matilde Marcolli

Perimeter Institute for Theoretical Physics, March 2017

References:

- Farzad Fathizadeh, Matilde Marcolli, *Periods and motives in the spectral action of Robertson-Walker spacetimes*, arXiv:1611.01815

- Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Spectral Action for Bianchi type-IX cosmological models*, arXiv:1506.06779, J. High Energy Phys. (2015) 85, 28 pp.

- Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Modular forms in the spectral action of Bianchi IX gravitational instantons*, arXiv:1511.05321

Spectral action models of gravity (modified gravity)

- Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$
  1. unital associative algebra $\mathcal{A}$
  2. represented as bounded operators on a Hilbert space $\mathcal{H}$
  3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$
- prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$
- extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)

Action functional

- Suppose *finitely summable* $ST = (\mathcal{A}, \mathcal{H}, D)$

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) < \infty, \quad \Re(s) >> 0$$

- Spectral action (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

f = smooth approximation to (even) cutoff

Asymptotic expansion (Chamseddine–Connes) for (almost) commutative geometries:

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta\in\Sigma^+_{ST}} f_\beta \Lambda^\beta \int\!\!\!\!- |D|^{-\beta} + f(0)\,\zeta_D(0)$$

- Residues

$$\int\!\!\!\!- |D|^{-\beta} = \frac{1}{2}\mathrm{Res}_{s=\beta}\, \zeta_D(s)$$

- Momenta $f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$
- Dimension Spectrum Σ_{ST} poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$
- positive dimension spectrum $\Sigma^+_{ST} = \Sigma_{ST} \cap \mathbb{R}^*_+$

Matilde Marcolli Arithmetic Spectral Action

Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

Pseudo-differential Calculus: (manifold case)
to obtain *full* asymptotic expansion of the Spectral Action

- Dirac operator D and pseudodifferential symbol of D^2

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

each p_k homogeneous of order k in ξ

- Cauchy integral formula

$$e^{-tD^2} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}\,(D^2-\lambda)^{-1}\,d\lambda$$

- Seeley de-Witt coefficients ($m = \dim M$)

$$\mathrm{Tr}(e^{-tD^2}) \sim_{t\to 0+} \quad t^{-m/2}\sum_{n=0}^{\infty} a_{2n}(D^2)\,t^n$$

Parametrix Method

- D^2 order 2 elliptic differential operator: exists a parametrix R_λ with

$$\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x,\xi,\lambda)$$

- $r_j(x,\xi,\lambda)$ pseudodifferential symbol order $-2-j$

$$r_j(x,t\xi,t^2\lambda) = t^{-2-j} r_j(x,\xi,\lambda)$$

- R_λ approximates $(D^2-\lambda)^{-1}$ with $\sigma((D^2-\lambda)R_\lambda) \sim 1$
- recursive equation:

$$\sigma((D^2-\lambda)R_\lambda) \sim ((p_2(x,\xi)-\lambda)+p_1(x,\xi)+p_0(x,\xi))\circ\left(\sum_{j=0}^{\infty} r_j(x,\xi,\lambda)\right) \sim 1$$

- solution for R_λ constructed recursively:

$$r_0(x,\xi,\lambda) = (p_2(x,\xi) - \lambda)^{-1}$$

$$r_n(x,\xi,\lambda) = -\sum \frac{1}{\alpha!} \partial_\xi^\alpha r_j(x,\xi,\lambda)\, D_x^\alpha p_k(x,\xi)\, r_0(x,\xi,\lambda),$$

summation over all $\alpha \in \mathbb{Z}^4_{\geq 0}, j \in \{0,1,\ldots,n-1\}, k \in \{0,1,2\}$, with $|\alpha| + j + 2 - k = n$

Seeley-deWitt coefficients and Parametrix Method

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i} \int \int_\gamma e^{-\lambda}\, \mathrm{tr}\,(r_{2n}(x,\xi,\lambda))\, d\lambda\, d^m\xi$$

- odd j coefficients vanish: $r_j(x,\xi,\lambda)$ odd function of ξ

A different method: Wodzicki residue

• Wodzicki residue: unique trace functional on algebra of pseudodifferential operators on smooth sections of vector bundle over smooth manifold

• classical pseudodifferential operator P_σ of order $d \in \mathbb{Z}$ local symbol

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi) \qquad (\xi \to \infty),$$

σ_{d-j} positively homogeneous order $d-j$ in ξ

• Residue:

$$\mathrm{Res}(P_\sigma) = \int_{S^*M} \mathrm{Tr}\,(\sigma_{-m}(x,\xi))\; d^{m-1}\xi\, d^m x,$$

$S^*M = \{(x,\xi) \in T^*M; ||\xi||_g = 1\}$ cosphere bundle

• spectral formulation of residue: pseudodifferential operator P_σ, Laplacian Δ

$$P_\sigma \mapsto \mathrm{Res}_{s=0} \mathrm{Tr}(P_\sigma \Delta^{-s})$$

same up to a constant $c_m = 2^{m+1}\pi^m$

• Mellin transform (for simplicity $\mathrm{Ker}(\Delta) = 0$):

$$\mathrm{Tr}(\Delta^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^s \, \frac{dt}{t}$$

• heat kernel expansion

$$\mathrm{Tr}\left(e^{-t\Delta}\right) = t^{-m/2} \sum_{n=0}^{N} a_{2n} t^n + O(t^{-m/2+N+1})$$

- find for any non-negative integer $n \leq m/2 - 1$:

$$\mathrm{Res}_{s=m/2-n}\mathrm{Tr}(\Delta^{-s}) = \frac{a_{2n}(\Delta)}{\Gamma(m/2-n)},$$

- in particular

$$\mathrm{Res}_{s=1}\mathrm{Tr}(\Delta^{-s}) = a_{m-2}(\Delta)$$

- in terms of Wodzicki residue:

$$a_{m-2}(\Delta) = \frac{1}{c_m}\mathrm{Res}(\Delta^{-1}) = \frac{1}{2^{m+1}\pi^m}\mathrm{Res}(\Delta^{-1})$$

applied to $\Delta = D^2$

- coefficient $a_2(D^2)$

$$a_2(D^2) = \frac{1}{c_4}\mathrm{Res}(D^{-2}) = \frac{1}{32\pi^4}\int_{S^*M} \mathrm{Tr}\,(\sigma_{-4}(D^{-2}))\; d^3\xi\, d^4x$$

- for other coefficients, introduce an auxiliary product space for correct counting of dimensions: use flat r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$

$$\Delta = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

$\Delta_{\mathbb{T}^r}$ flat Laplacian on $\mathbb{T}^r$

$$a_{2+r}(D^2) = \frac{1}{2^5\,\pi^{4+r/2}}\mathrm{Res}(\Delta^{-1})$$

because Künneth formula gives

$$a_{2+r}((x,x'),\Delta) = a_{2+r}(x,D^2)a_0(x',\mathbb{T}^r) = 2^{-r}\pi^{-r/2}a_{2+r}(x,D^2)$$

with volume term only non-zero heat coefficient for flat metric

Robertson–Walker spacetime

- Topologically $S^3 \times \mathbb{R}$
- Metric (Euclidean)

$$ds^2 = dt^2 + a(t)^2 d\sigma^2$$

scaling factor $a(t)$, round metric $d\sigma^2$ on S^3

- Hopf coordinates on S^3

$$x = (t, \eta, \phi_1, \phi_2) \mapsto (t, \sin\eta\cos\phi_1, \sin\eta\sin\phi_2, \cos\eta\cos\phi_1, \cos\eta\sin\phi_2),$$

$$0 < \eta < \frac{\pi}{2}, \qquad 0 < \phi_1 < 2\pi, \qquad 0 < \phi_2 < 2\pi.$$

- Robertson-Walker metric in Hopf coordinates

$$ds^2 = dt^2 + a(t)^2 \left(d\eta^2 + \sin^2(\eta)\, d\phi_1^2 + \cos^2(\eta)\, d\phi_2^2\right)$$

Dirac operator

• orthonormal coframe $\{\theta^a\}$

$$D = \sum_a \theta^a \nabla^S_{\theta_a}$$

• spin connection ∇^S with matrix of 1-forms $\omega = (\omega^a_b)$ with

$$\nabla \theta^a = \sum_b \omega^a_b \otimes \theta^b$$

• metric-compatibility and torsion-freeness (Levi–Civita connection)

$$\omega^a_b = -\omega^b_a, \qquad d\theta^a = \sum_b \omega^a_b \wedge \theta^b$$

- Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b$$

with $\omega^b_a = \sum_c \omega^b_{ac} \theta^c$

- matrices γ^a Clifford action of θ^a on spin bundle:

$(\gamma^a)^2 = -I$

$\gamma^a \gamma^b + \gamma^b \gamma^a = 0$ for $a \neq b$

Pseudodifferential Symbol $\sigma_D(x,\xi)$ of Dirac operator D sum $q_1(x,\xi)+q_0(x,\xi)$ with $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)\in T^*_xM\simeq\mathbb{R}^4$ cotangent fiber at $x=(t,\eta,\phi_1,\phi_2)$

$$q_1(x,\xi) = \begin{pmatrix} 0 & 0 & \frac{i\sec(\eta)\xi_4}{a(t)}-\xi_1 & \frac{i\xi_2}{a(t)}+\frac{\csc(\eta)\xi_3}{a(t)} \\ 0 & 0 & \frac{i\xi_2}{a(t)}-\frac{\csc(\eta)\xi_3}{a(t)} & -\xi_1-\frac{i\sec(\eta)\xi_4}{a(t)} \\ -\xi_1-\frac{i\sec(\eta)\xi_4}{a(t)} & -\frac{i\xi_2}{a(t)}-\frac{\csc(\eta)\xi_3}{a(t)} & 0 & 0 \\ \frac{\csc(\eta)\xi_3}{a(t)}-\frac{i\xi_2}{a(t)} & \frac{i\sec(\eta)\xi_4}{a(t)}-\xi_1 & 0 & 0 \end{pmatrix},$$

$$q_0(\xi) = \begin{pmatrix} 0 & 0 & \frac{3ia'(t)}{2a(t)} & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} \\ 0 & 0 & \frac{\cot(\eta)-\tan(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} \\ \frac{3ia'(t)}{2a(t)} & \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & 0 & 0 \\ \frac{\tan(\eta)-\cot(\eta)}{2a(t)} & \frac{3ia'(t)}{2a(t)} & 0 & 0 \end{pmatrix}. \quad (1)$$

Pseudodifferential symbol of square D^2 of Dirac operator:

$$\sigma_{D^2}(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

$$p_2(x,\xi) = q_1(x,\xi)\, q_1(x,\xi) = \left(\sum g^{\mu\nu}\xi_\mu\xi_\nu\right) I_{4\times 4}$$

$$= \left(\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\csc^2(\eta)\xi_3^2}{a(t)^2} + \frac{\sec^2(\eta)\xi_4^2}{a(t)^2}\right) I_{4\times 4},$$

$$p_1(x,\xi) = q_0(x,\xi)\, q_1(x,\xi) + q_1(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_1}{\partial x_j}(x,\xi),$$

$$p_0(x,\xi) = q_0(x,\xi)\, q_0(x,\xi) + \sum_{j=1}^{4} -i\frac{\partial q_1}{\partial \xi_j}(x,\xi)\, \frac{\partial q_0}{\partial x_j}(x,\xi).$$

- computer calculation of $\mathrm{tr}(\sigma_{-4}(x,\xi))$ takes a couple of pages to write out (sum of fractions involving trigonometric functions and powers of ξ_i, scaling factor $a(t)$ and derivative)
- important properties of resulting expression:
  - each term with an odd power of ξ_j in numerator will integrate to 0 in integration of 1-density
  - numerical coefficients of all terms in integrand are *rational numbers*
  - treat scaling factor $a(t)$ and derivative $a'(t)$, $a''(t)$ as affine variables $\alpha, \alpha_1, \alpha_2$ (integration without performing time integration)
  - there is a natural change of coordinates replacing trigonometric functions by polynomials: rational function

change of coordinates

$$u_0 = \sin^2(\eta), \qquad u_1 = \xi_1, \qquad u_2 = \xi_2,$$
$$u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4,$$

Then have

$$\xi_1^2 + \frac{\xi_2^2}{a(t)^2} + \frac{\xi_3^2 \csc^2(\eta)}{a(t)^2} + \frac{\xi_4^2 \sec^2(\eta)}{a(t)^2} = u_1^2 + \frac{1}{a(t)^2}(u_2^2 + u_3^2 + u_4^2),$$
$$\cot^2(\eta) = \frac{1-u_0}{u_0},$$
$$\csc^2(\eta) = \frac{1}{u_0},$$
$$\sec^2(\eta) = \frac{1}{1-u_0},$$
$$\cot(\eta)\cot(2\eta) = \frac{\cot^2(\eta)}{2} - \frac{1}{2},$$
$$\csc^2(2\eta) = \frac{1}{4}\csc^2(\eta)\sec^2(\eta),$$
$$\tan^2(\eta) = \sec^2(\eta) - 1,$$
$$\tan(\eta)\cot(2\eta) = \frac{1}{2} - \frac{\tan^2(\eta)}{2},$$
$$\cot^2(2\eta) = \frac{\tan^2(\eta)}{8} + \frac{\cot^2(\eta)}{8} + \frac{1}{8}\csc^2(\eta)\sec^2(\eta) - \frac{3}{4}.$$

Also exponents of the variables ξ_j are even positive integers

a_2-term as a period integral $C \cdot \int_{A_4} \Omega^{\alpha}_{(\alpha_1,\alpha_2)}$ with $C \in \mathbb{Q}[(2\pi i)^{-1}]$

- Algebraic differential form

$$\Omega = f\, \tilde{\sigma}_3,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2) \in \mathbb{A}^2$

- functions $f(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = f_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2))^\ell}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2) = P_{(\alpha_1,\alpha_2)}(u_0, u_1, u_2, u_3, u_4, \alpha)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, \alpha, \alpha_1, \alpha_2]$
with r, k, m and ℓ non-negative integers

- algebraic differential form $\widetilde{\sigma}_3 = \widetilde{\sigma}_3(u_0, u_1, u_2, u_3, u_4)$

$$\frac{1}{2}\left(u_1\, du_0\, du_2\, du_3\, du_4 - u_2\, du_0\, du_1\, du_3\, du_4 + u_3\, du_0\, du_1\, du_2\, du_4 - u_4\, du_0\, du_1\, du_2\, du_3\right)$$

- forms $\Omega^\alpha = \Omega^\alpha_{(\alpha_1,\alpha_2)}$ restricting to fixed value of $\alpha \in \mathbb{A}^1 \setminus \{0\}$: two parameter family
- defined on the complement in $\mathbb{A}^5$ of union of two affine hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and hypersurface $\widehat{CZ}_\alpha$ defined by vanishing of the quadratic form

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2)$$

- $\mathbb{Q}$-semialgebraic set: subset S of some $\mathbb{R}^n$

$$S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, P(x_1, \ldots, x_n) \geq 0\},$$

for some polynomial $P \in \mathbb{Q}[x_1, \ldots, x_n]$, and complements, intersections, unions
- domain of integration $\mathbb{Q}$-semialgebraic set

$$A_4 = \left\{ (u_0, u_1, u_2, u_3, u_4) \in \mathbb{A}^5(\mathbb{R}) \;:\; \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0) u_4^2 = 1, \\ 0 < u_i < 1, \text{ for } i = 0,1,2 \end{array} \right\}$$

a_4-term and Wodzicki Residue

$$a_4 = \frac{1}{2^5\,\pi^5} \mathrm{Res}(\Delta_4^{-1})$$

need $\mathrm{tr}(\sigma_{-6}(\Delta_4^{-1}))$ of order -6 in expansion of symbol of Δ_4^{-1}

- general recursive procedure with auxiliary flat tori T^r

$$\Delta_{r+2} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r}$$

$$\sigma_{-2}(\Delta_{r+2}^{-1}) = \left(p_2(x, \xi_1, \xi_2, \xi_3, \xi_4) + (\xi_5^2 + \cdots + \xi_{4+r}^2) I_{4\times 4}\right)^{-1}$$

then recursively $\sigma_{-2-n}(\Delta_{r+2}^{-1})$ given by

$$-\left(\sum_{\substack{0 \le j < n,\, 0 \le k \le 2 \\ \alpha \in \mathbb{Z}^4_{\ge 0} \\ -2-j-|\alpha|+k=-n}} \frac{(-i)^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \sigma_{-2-j}(\Delta_{r+2}^{-1})\right) \left(\partial_x^\alpha p_k\right) \right) \sigma_{-2}(\Delta_{r+2}^{-1}).$$

a_4-term as a period integral $C \cdot \int_{A_6} \Omega^{\alpha}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

- algebraic differential form

$$\Omega = f\, \widetilde{\sigma}_5,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{A}^7$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{A}^4$

- functions $f_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2)^\ell}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, u_5, u_6\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$
where r, k, m and ℓ non-negative integers

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- domain of integration $\mathbb{Q}$-semialgebraic set

$$A_6 = \left\{ (u_0, \ldots, u_6) \in \mathbb{A}^7(\mathbb{R}) \; : \; \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0) u_4^2 + u_5^2 + u_6^2 = 1 \\ 0 < u_i < 1, \quad i = 0,1,2,5,6 \end{array} \right\}$$

- the change of variables used here

$$u_0 = \sin^2(\eta), \qquad u_1 = \xi_1, \qquad u_2 = \xi_2$$

$$u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4, \qquad u_5 = \xi_5, \qquad u_6 = \xi_6$$

higher order terms a_{2n}

$$a_{2n} = \frac{1}{2^5\,\pi^{3+n}} \mathrm{Res}(\Delta_{2n}^{-1})$$

using auxiliary flat tori T^{2n-2}

$$\Delta_{2n} = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^{2n-2}}$$

need term σ_{-2n-2} homogeneous of order $-2n-2$ in expansion of pseudodifferential symbol of parametrix Δ_{2n}^{-1}

- recursive argument for structure of term σ_{-2n-2}

a_4-term as a period integral $C \cdot \int_{A_6} \Omega^{\alpha}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

- algebraic differential form

$$\Omega = f\, \tilde{\sigma}_5,$$

in affine coordinates $(u_0, u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{A}^7$, $\alpha \in \mathbb{G}_m$, and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{A}^4$

- functions $f_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha)$ $\mathbb{Q}$-linear combinations of rational functions

$$\frac{P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)}{\alpha^{2r} u_0^k (1-u_0)^m (u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2)^{\ell}}$$

where

$$P(u_0, u_1, u_2, u_3, u_4, u_5, u_6, \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

polynomials in $\mathbb{Q}[u_0, u_1, u_2, u_3, u_4, u_5, u_6\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4]$
where r, k, m and ℓ non-negative integers

- term $\mathrm{tr}(\sigma_{-2n-2})$ given by

$$\sum_{j=1}^{M_n} c_{j,2n}\, u_0^{\beta_{0,1,j}/2} (1-u_0)^{\beta_{0,2,j}/2} \frac{u_1^{\beta_{1,j}}\, u_2^{\beta_{2,j}} \cdots u_{2n+2}^{\beta_{2n+2,j}}}{Q_{\alpha,2n}^{\rho_{j,2n}}}\, \alpha^{k_{0,j}}\, \alpha_1^{k_{1,j}} \cdots \alpha_{2n}^{k_{2n,j}},$$

where

$$\alpha = a(t), \qquad \alpha_1 = a'(t), \qquad \alpha_2 = a''(t), \qquad \dots \qquad \alpha_{2n} = a^{2n}(t),$$

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2,$$

$$c_{j,2n} \in \mathbb{Q}, \quad \beta_{0,1,j}, \beta_{0,2,j}, k_{0,j} \in \mathbb{Z}, \quad \beta_{1,j}, \dots, \beta_{2n+2,j}, \rho_{j,2n}, k_{1,j}, \dots, k_{2n,j} \in \mathbb{Z}_{\geq 0}.$$

- using change of coordinates

$$u_0 = \sin^2(\eta), \qquad u_3 = \csc(\eta)\,\xi_3, \qquad u_4 = \sec(\eta)\,\xi_4$$

$$u_j = \xi_j, \qquad j = 1, 2, 5, 6, \dots, 2n+2$$

a_{2n}-terms as period integrals $C \cdot \int_{A_{2n}} \Omega^{\alpha}_{\alpha_1,\ldots,\alpha_{2n}}$

- algebraic differential form

$$\Omega^{\alpha}_{\alpha_1,\ldots,\alpha_{2n}}(u_0, u_1, \ldots, u_{2n+2})$$

- domain of definition complement

$$\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)$$

with hyperplanes $H_0 = \{u_0 = 0\}$ and $H_1 = \{u_0 = 1\}$ and $\widehat{CZ}_{\alpha,2n}$ the hypersurface defined by the vanishing of the quadric

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + \cdots + u_{2n+2}^2$$

- $\mathbb{Q}$-semialgebraic set A_{2n+2}

$$A_{2n+2} = \left\{ (u_0, \ldots, u_{2n+2}) \in \mathbb{A}^{2n+3}(\mathbb{R}) \,:\, \begin{array}{l} u_1^2 + u_2^2 + u_0 u_3^2 + (1-u_0)u_4^2 + \sum_{i=5}^{2n+2} u_i^2 = 1 \\ 0 < u_i < 1, \quad i = 0, 1, 2, 5, 6, \ldots, 2n+2 \end{array} \right\}$$

Periods and Motives

• Main Idea: can constrain the type of numbers that can occur as *periods* $\int_\sigma \omega$ on a given algebraic variety X on the basis of information about the *motive* $\mathfrak{m}(X)$ of X

• Motives (Grothendieck) are a universal cohomology theory for algebraic varieties (morphisms: equivalence classes of algebraic cycles in the product)

- pure motives: smooth projective varieties
- mixed motives: more general varieties (quasi-projective, singular...)

in applications to physics one typically deals with *mixed motives*

Mixed Motives and Mixed Tate Motives

- there is a triangulated $\otimes$-category $\mathcal{DM}$ of mixed motives (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(X) \to \mathfrak{m}(U \cap V)[1] \quad \text{Mayer-Vietoris}$$

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1] \quad \text{Gysin}$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2] \quad \text{homotopy}$$

$$\mathfrak{m}(X)^\vee = \mathfrak{m}^c(X)(-d)[-2d] \quad \text{duality}$$

- Mixed Tate motives: triangulated $\otimes$-subcateory $\mathcal{DMT} \subset \mathcal{DM}$ generated by the Tate objects $\mathbb{Q}(m)$
$\mathbb{Q}(1)$ formal inverse of Lefschetz motive $\mathbb{L} = h^2(\mathbb{P}^1)$
- Method: to show $\mathfrak{m}(X)$ mixed Tate realize it in terms of distinguished triangles where two out of three terms are mixed Tate $\Rightarrow$ third one also is (or one is and one is not, then third also not)
- Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Motives and the Grothendieck ring of varieties

• Usually difficult to determine explicitly the motive of $\mathfrak{m}(X)$ in the triangulated category of *mixed motives*

• Simpler invariant (universal Euler characteristic for motives): class $[X]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \smallsetminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$
(K_0 group of category of pure motives: virtual motives)

Matilde Marcolli | Arithmetic Spectral Action

Mixed Motives associated to these periods

$$\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

divisor Σ containing boundary of domain of integration A_{2n}

- motives of quadrics (Rost, Vishik)
  - hyperbolic form $\mathbb{H} := \langle 1, -1 \rangle$
  - $Q = d \cdot \mathbb{H}$ of dimension $2d$

$$\mathfrak{m}(Z_{d\mathbb{H}}) = \mathbb{Z}(d-1)[2d-2] \oplus \mathbb{Z}(d-1)[2d-2] \oplus \bigoplus_{i=0,\ldots,d-2,d,\ldots,2d-2} \mathbb{Z}(i)[2i]$$

  - $Q = d \cdot \mathbb{H} \perp \langle 1 \rangle$ in dimension $2d+1$

$$\mathfrak{m}(Z_{d\mathbb{H}\perp\langle 1\rangle}) = \bigoplus_{i=0,\ldots,2d-1} \mathbb{Z}(i)[2i]$$

  - if $\exists$ quadratic field extension $\mathbb{K}$ where Q hyperbolic

$$\mathfrak{m}(Z_Q) = \begin{cases} \mathfrak{m}_1 \oplus \mathfrak{m}_1(1)[2] & m = 2 \mod 4 \\ \mathfrak{m}_1 \oplus \mathcal{R}_{Q,\mathbb{K}} \oplus \mathfrak{m}_1(1)[2] & m = 0 \mod 4 \end{cases}$$

involving *forms of Tate motives*

- quadratic field extension $\mathbb{Q}(\sqrt{-1})$, assuming $\alpha \in \mathbb{Q}^*$

$$Q_{\alpha,2} = u_1^2 + \alpha^{-2}(u_2^2 + u_3^2 + u_4^2)$$

change of variables

$$X = u_1 + \frac{i}{\alpha} u_2, \quad Y = u_1 - \frac{i}{\alpha} u_2, \quad Z = \frac{i}{\alpha}(u_3 + i u_4), \quad W = \frac{i}{\alpha}(u_3 - i u_4)$$

identification of Z_α with the Segre quadric

$$\{XY - ZW = 0\} \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

- similar for a_{2n}-term case

$$Q_{\alpha,2n} = u_1^2 + \frac{1}{\alpha^2}(u_2^2 + u_3^2 + u_4^2) + u_5^2 + u_6^2 + \cdots + u_{2n+1}^2 + u_{2n+2}^2$$

inductively: change of coordinates

$$X = u_{2n+1} + i u_{2n+2}, \qquad Y = u_{2n+1} - i u_{2n+2}$$

puts $Q_{\alpha,2n}$ in the form

$$Q_{\alpha,2n} = Q_{\alpha,2n-2}(u_1, \ldots, u_{2n}) + XY.$$

classes in the Grothendieck ring

- $Z_{\alpha,2n}$ quadric in $\mathbb{P}^{2n+1}$ determined by $Q_{\alpha,2n}$

$$[\mathbb{P}^{2n+1} \smallsetminus Z_{\alpha,2n}] = \mathbb{L}^{2n+1} - \mathbb{L}^{n}$$

$$[\mathbb{A}^{2n+3} \smallsetminus \widehat{CZ}_{\alpha,2n}] = \mathbb{L}^{2n+3} - \mathbb{L}^{2n+2} - \mathbb{L}^{n+2} + \mathbb{L}^{n+1}$$

$$[\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1)] = \mathbb{L}^{2n+3} - 3\mathbb{L}^{2n+2} + 2\mathbb{L}^{2n+1} - \mathbb{L}^{n+2} + 3\mathbb{L}^{n+1} - 2\mathbb{L}^{n}$$

- based on an inductive argument using identities

1. $[\mathbb{A}^{N} \smallsetminus \hat{Z}] = (\mathbb{L} - 1)[\mathbb{P}^{N-1} \smallsetminus Z]$
2. $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = (\mathbb{L} - 1)[\mathbb{P}^{N} \smallsetminus CZ]$
3. $[CZ] = \mathbb{L}\,[Z] + 1$
4. $[\mathbb{A}^{N+1} \smallsetminus \widehat{CZ}] = \mathbb{L}^{N+1} - \mathbb{L}(\mathbb{L} - 1)[Z] - \mathbb{L}$
5. $[\mathbb{A}^{N+1} \smallsetminus (\widehat{CZ} \cup H \cup H')] = \mathbb{L}^{N+1} - 2\mathbb{L}^{N} - (\mathbb{L}-2)(\mathbb{L}-1)[Z] - (\mathbb{L}-2)$.

with $Z \subset \mathbb{P}^{N-1}$, $\hat{Z} \subset \mathbb{A}^{N}$ affine cone, CZ projective cone in $\mathbb{P}^{N}$, H and H' affine hyperplanes with $H \cap H' = \emptyset$, intersections $\widehat{CZ} \cap H$ and $\widehat{CZ} \cap H'$ sections $\hat{Z}$ of cone

Mixed Tate

- mixed motive (over field $\mathbb{Q}(\sqrt{-1})$)

$$\mathfrak{m}(\mathbb{A}^{2n+3} \smallsetminus (\widehat{CZ}_{\alpha,2n} \cup H_0 \cup H_1), \Sigma)$$

is mixed Tate

- over $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ quadratic form

$$Q_{\alpha,2n}|_{\mathbb{Q}(\sqrt{-1})} = (n+1) \cdot \mathbb{H},$$

so motive

$$\mathfrak{m}(Z_{\alpha,2n}|_{\mathbb{K}}) = \mathbb{Z}(n)[2n] \oplus \mathbb{Z}(n)[2n] \oplus \bigoplus_{i=0,\ldots,n-1,n+1,\ldots 2n} \mathbb{Z}(i)[2i]$$

- rest of the argument shown in example of a_2 for simplicity

• $\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)$ is mixed Tate

$$\mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha) \to \mathfrak{m}(\mathbb{P}^3) \to \mathfrak{m}(Z_\alpha)(1)[2] \to \mathfrak{m}(\mathbb{P}^3 \smallsetminus Z_\alpha)[1]$$

Gysin distinguished triangle of the closed codim one embedding $Z_\alpha \hookrightarrow \mathbb{P}^3$

• projective cone CZ_α in $\mathbb{P}^4$: homotopy invariance for $\mathbb{A}^1$-fibration $\mathbb{P}^4 \smallsetminus CZ_\alpha \to \mathbb{P}^3 \smallsetminus Z_\alpha$

$$\mathfrak{m}_c^j(\mathbb{P}^4 \smallsetminus CZ_\alpha) = \mathfrak{m}_c^{j-2}(\mathbb{P}^3 \smallsetminus Z_\alpha)(-1)$$

motive $\mathfrak{m}(\mathbb{P}^4 \smallsetminus CZ_\alpha)$ also mixed Tate

• motive $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ mixed Tate: $\mathbb{P}^1$-bundle $\mathcal{P}$ compactification of $\mathbb{G}_m$-bundle

$$\mathcal{T} = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha \to X = \mathbb{P}^4 \smallsetminus CZ_\alpha$$

and Gysin distinguished triangle

$$\mathfrak{m}(\mathcal{T}) \to \mathfrak{m}(\mathcal{P}) \to \mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})^*(1)[2] \to \mathfrak{m}(\mathcal{T})[1]$$

$\mathfrak{m}_c(\mathcal{P} \smallsetminus \mathcal{T})$ mixed Tate since $\mathcal{P} \smallsetminus \mathcal{T}$ two copies of X, so $\mathfrak{m}(\mathcal{T})$ mixed Tate

• union $\widehat{CZ}_\alpha \cup H_0 \cup H_1$ is mixed Tate: motives $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ and $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ and motive of intersection $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ are mixed Tate

$$\mathfrak{m}(U \cap V) \to \mathfrak{m}(U) \oplus \mathfrak{m}(V) \to \mathfrak{m}(U \cup V) \to \mathfrak{m}(U \cap V)[1]$$

Mayer-Vietoris distinguished triangle with $U = \mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha$ and $V = \mathbb{A}^5 \smallsetminus (H_0 \cup H_1)$

- $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ mixed Tate by previous
- $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ also mixed Tate since $\mathfrak{m}(H_0 \cup H_1)$ is
- $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ mixed Tate because intersection $\widehat{CZ}_\alpha \cap (H_0 \cup H_1)$ two sections of the cone and $\mathfrak{m}(\hat{Z}_\alpha)$ Tate
- then also $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ mixed Tate

• divisor Σ in $\mathbb{A}^5$ is a union of coordinate hyperplanes and their translates: mixed Tate

• $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1), \Sigma)$ also mixed Tate: distinguished triangle with $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ and $\mathfrak{m}(\Sigma)$

• union $\widehat{CZ}_\alpha \cup H_0 \cup H_1$ is mixed Tate: motives $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (H_0 \cup H_1))$ and $\mathfrak{m}(\mathbb{A}^5 \smallsetminus \widehat{CZ}_\alpha)$ and motive of intersection $\mathfrak{m}(\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ are mixed Tate

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• $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1), \Sigma)$ also mixed Tate: distinguished triangle with $\mathfrak{m}(\mathbb{A}^5 \smallsetminus (\widehat{CZ}_\alpha \cap (H_0 \cup H_1))$ and $\mathfrak{m}(\Sigma)$