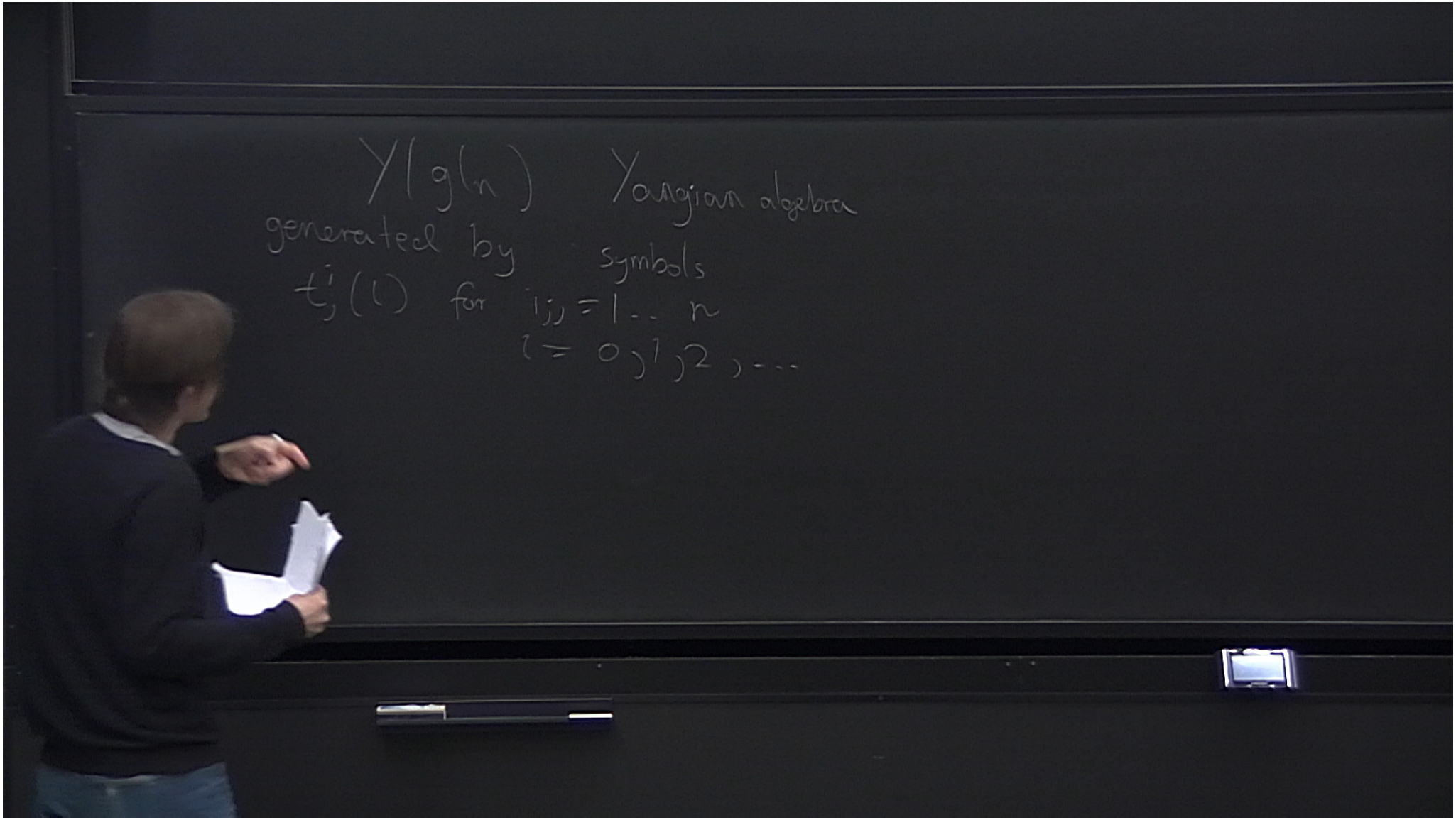
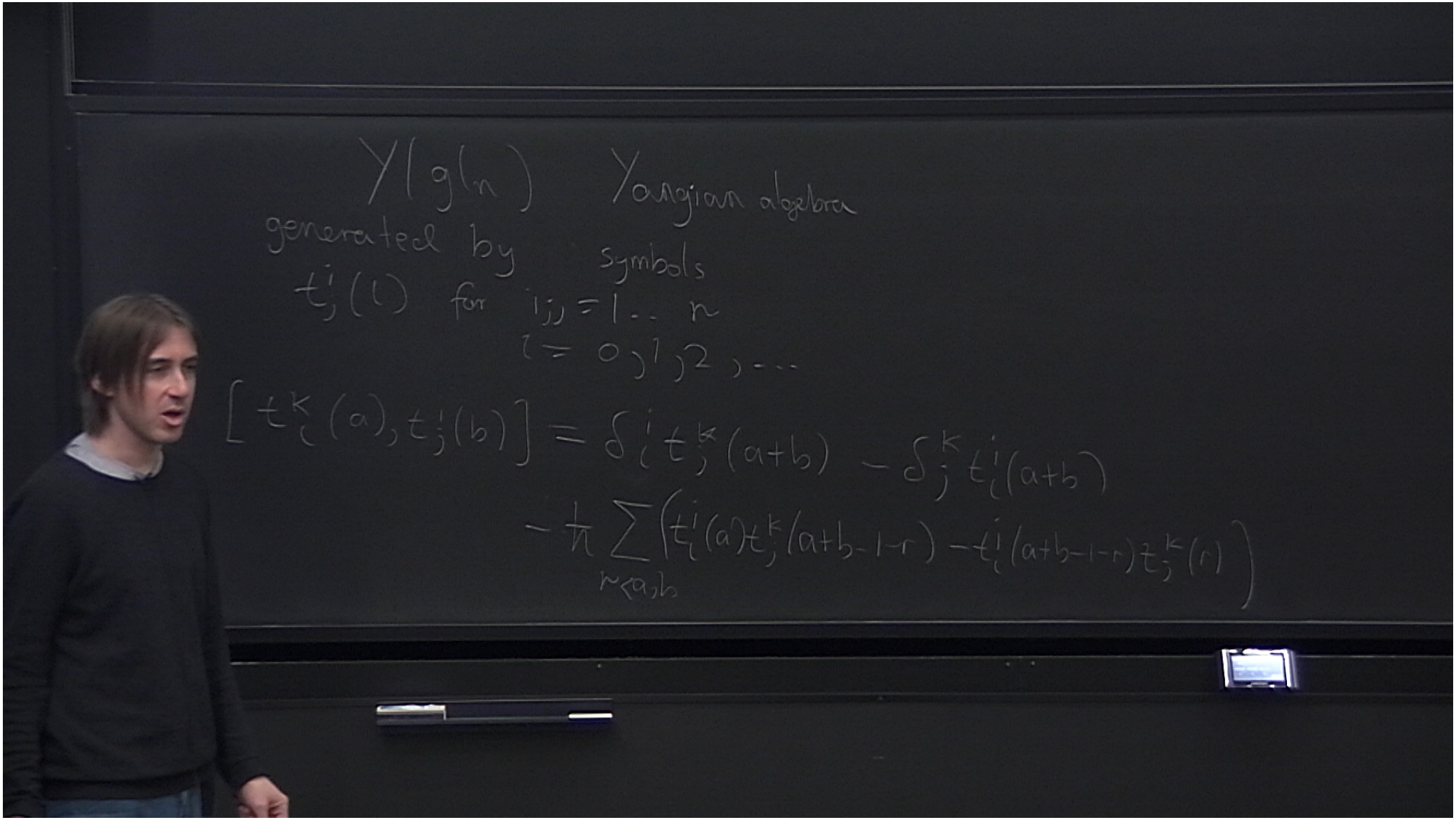
Title: PSI 2016/2017 Introduction to Quantum Integrable Systems - Lecture 11

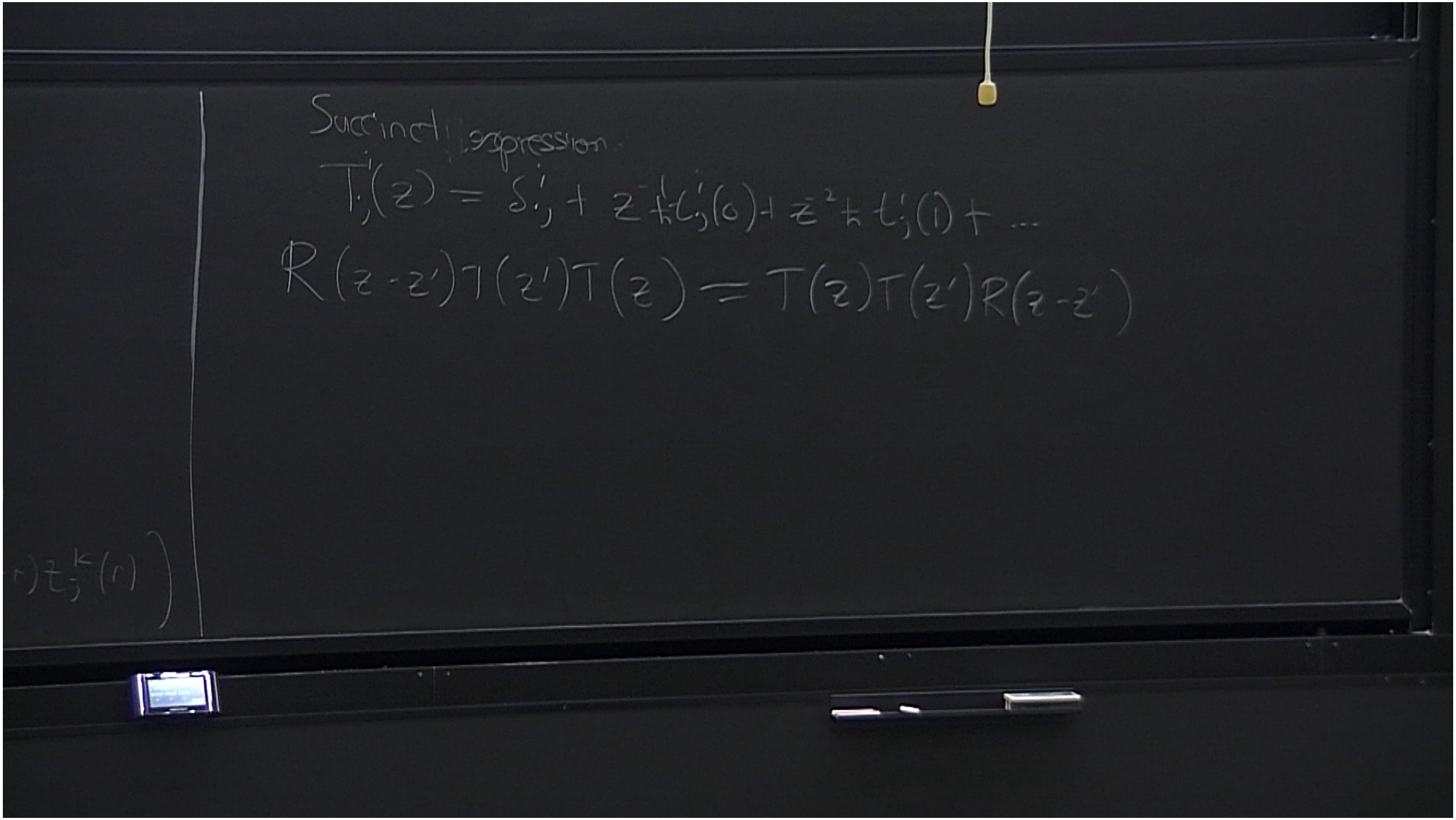
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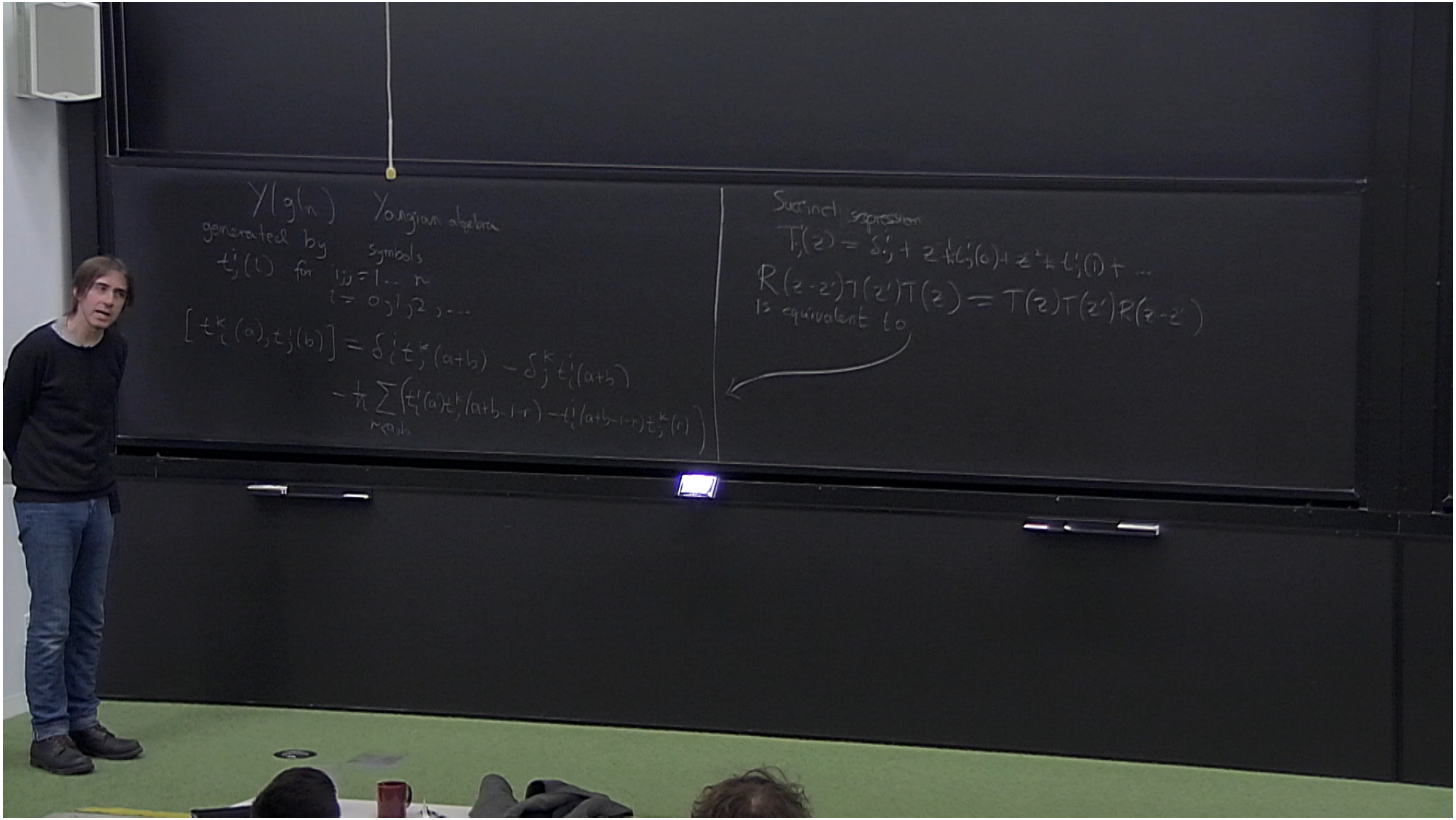
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Abstract:


Y(gl(n) Yangian algebra
generated by symbols
$t_{ij}^{(r)}$ for $i,j = 1 \ldots n$
$r = 0, 1, 2, \ldots$

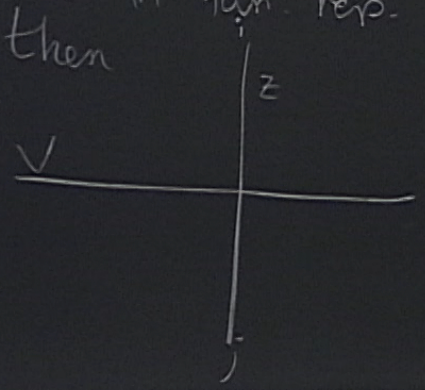

$Y(gl_n)$ Yangian algebra
generated by symbols
$t_{ij}^{(i)}$ for $i,j = 1 \dots n$
$i = 0, 1, 2, \dots$
$[t_{ij}^{k}(a), t_{ij}^{l}(b)] = \delta_{i}^{l} t_{j}^{k}(a+b) - \delta_{j}^{k} t_{i}^{l}(a+b)$
$- \hbar \sum_{r<a,b} \left(t_{i}^{l}(a) t_{j}^{k}(a+b-1-r) - t_{i}^{l}(a+b-1-r) t_{j}^{k}(r) \right)$


Succinct expression
$T^i_j(z) = \delta^i_j + z^{-1}\frac{1}{\hbar}t^i_j(0) + z^{-2}\hbar\, t^i_j(1) + \dots$
$R(z-z')T(z')T(z) = T(z)T(z')R(z-z')$


$Y(g(n))$ Yangian algebra
generated by symbols
$t_{ij}^{(r)}$ for $i,j = 1 \dots n$
$r = 0, 1, 2, \dots$
$[t_{ij}^{(a)}, t_{kl}^{(b)}] = \delta_{ik} t_{jl}^{(a+b)} - \delta_{jl} t_{ik}^{(a+b)}$
$- \frac{1}{\pi} \sum_{r=0}^{a} \left(t_{ik}^{(r)} t_{jl}^{(a+b-1-r)} - t_{ik}^{(a+b-1-r)} t_{jl}^{(r)} \right)$
Succinct expression
$T_{ij}(z) = \delta_{ij} + z^{-1} t_{ij}^{(0)} + z^{-2} t_{ij}^{(1)} + \dots$
$R(z-z') T(z) T(z') = T(z') T(z) R(z-z')$
is equivalent to

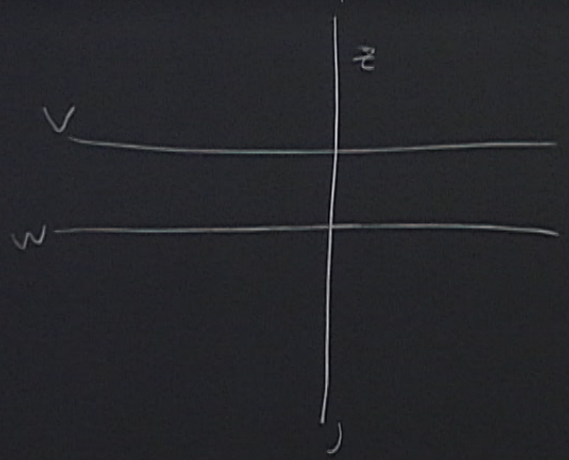
Coproduct Structure

If V is a horizontal Wilson line + vertical line in fun. rep. of $g(n)$ then



$\rightsquigarrow T_j^i(z)\ V \to V$

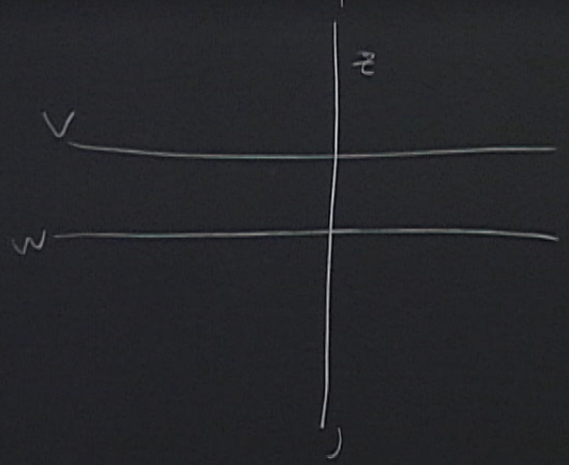
If we have two horizontal Wilson lines V, W



gives a linear map

$$T^i_j(z): V \otimes W \to V \otimes W$$

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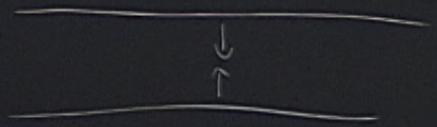


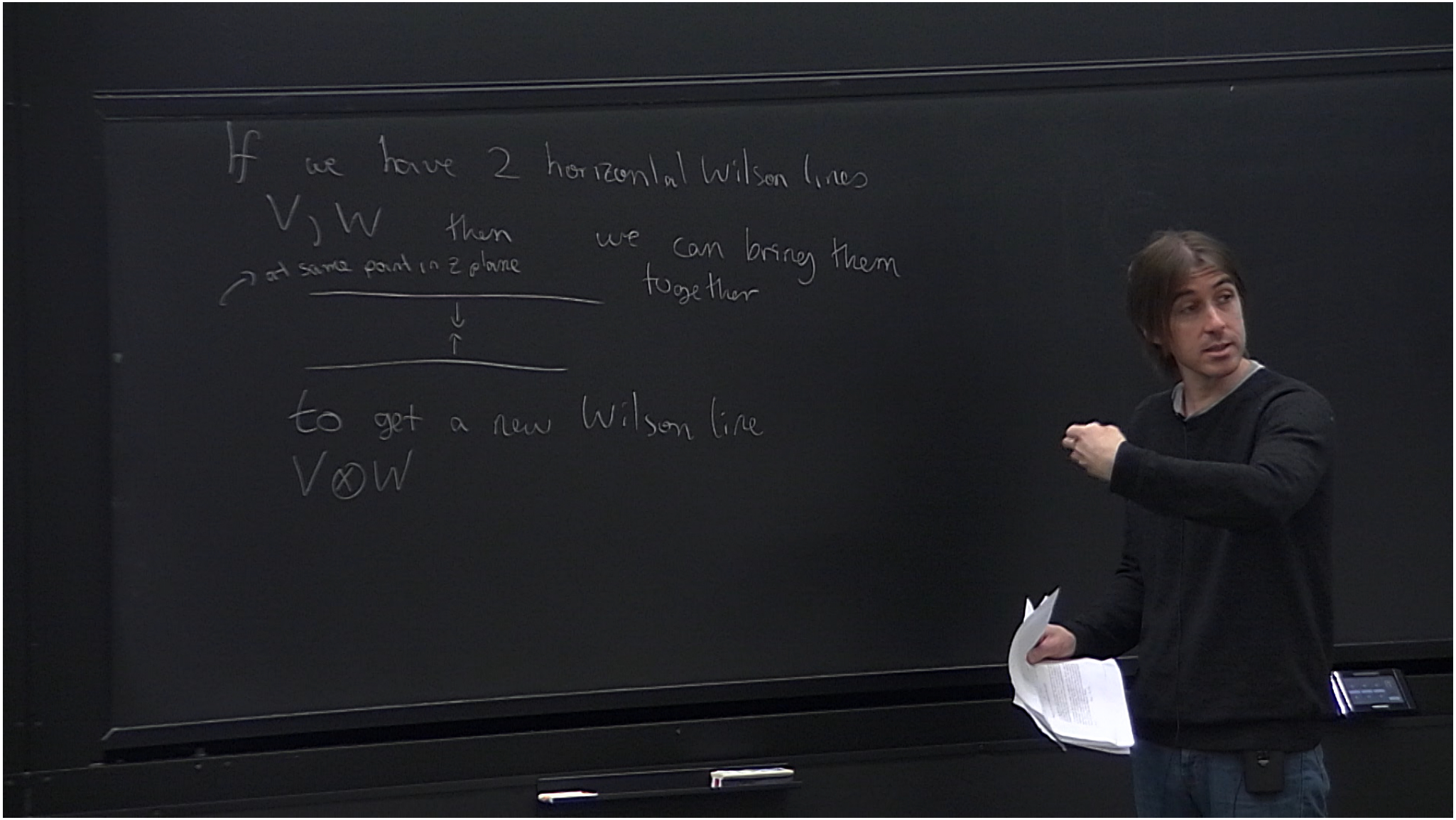
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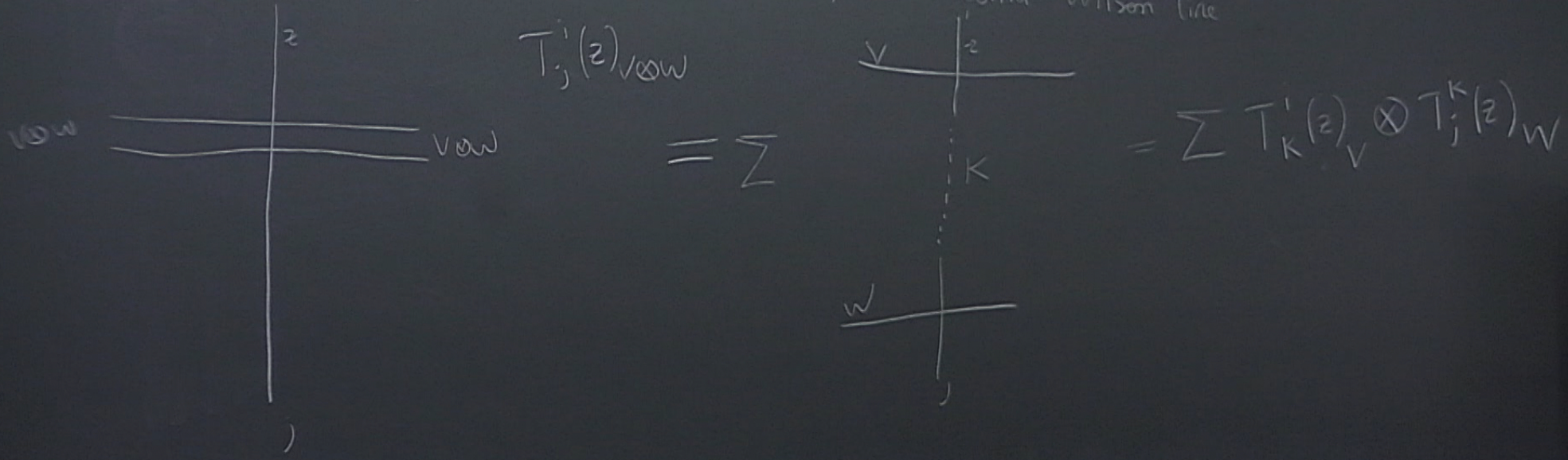
The theory is topological in x-y plane.
So distance between Wilson lines is irrelevant.

If we have 2 horizontal Wilson lines V, W then we can bring them together



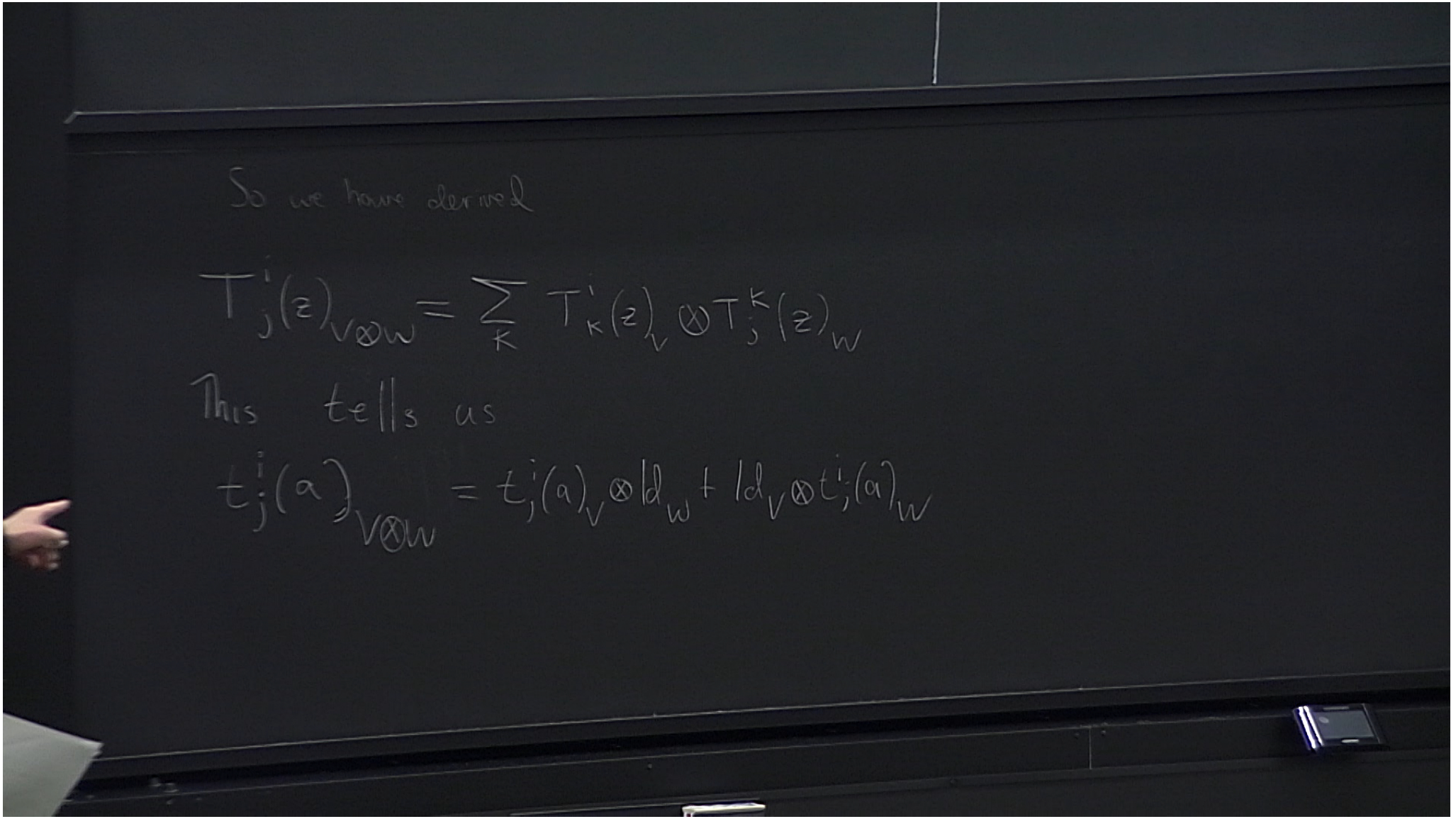

If we have 2 horizontal Wilson lines
V, W then we can bring them
at same point in z plane together
to get a new Wilson line
V⊗W

Let $T^i_j(z)_{V\otimes W}$, $T^i_j(z)_V$, etc. be operators for horizontal Wilson line



So we have derived

$$T^i_j(z)_{V\otimes W} = \sum_k T^i_k(z)_V \otimes T^k_j(z)_W$$

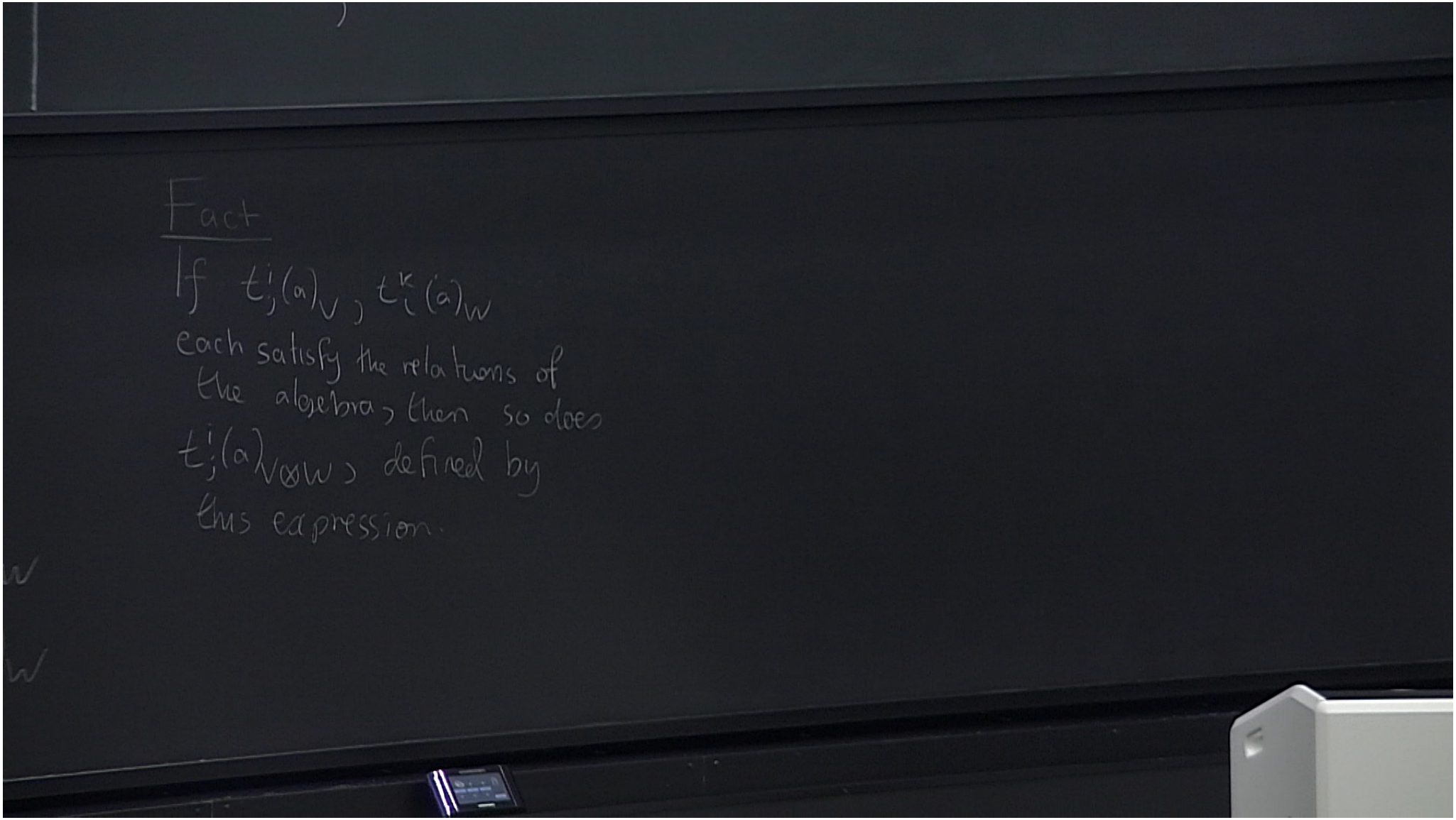

So we have derived
$T^i_j(z)_{V\otimes W} = \sum_k T^i_k(z)_V \otimes T^k_j(z)_W$
This tells us
$t^i_j(a)_{V\otimes W} = t^i_j(a)_V \otimes Id_W + Id_V \otimes t^i_j(a)_W$

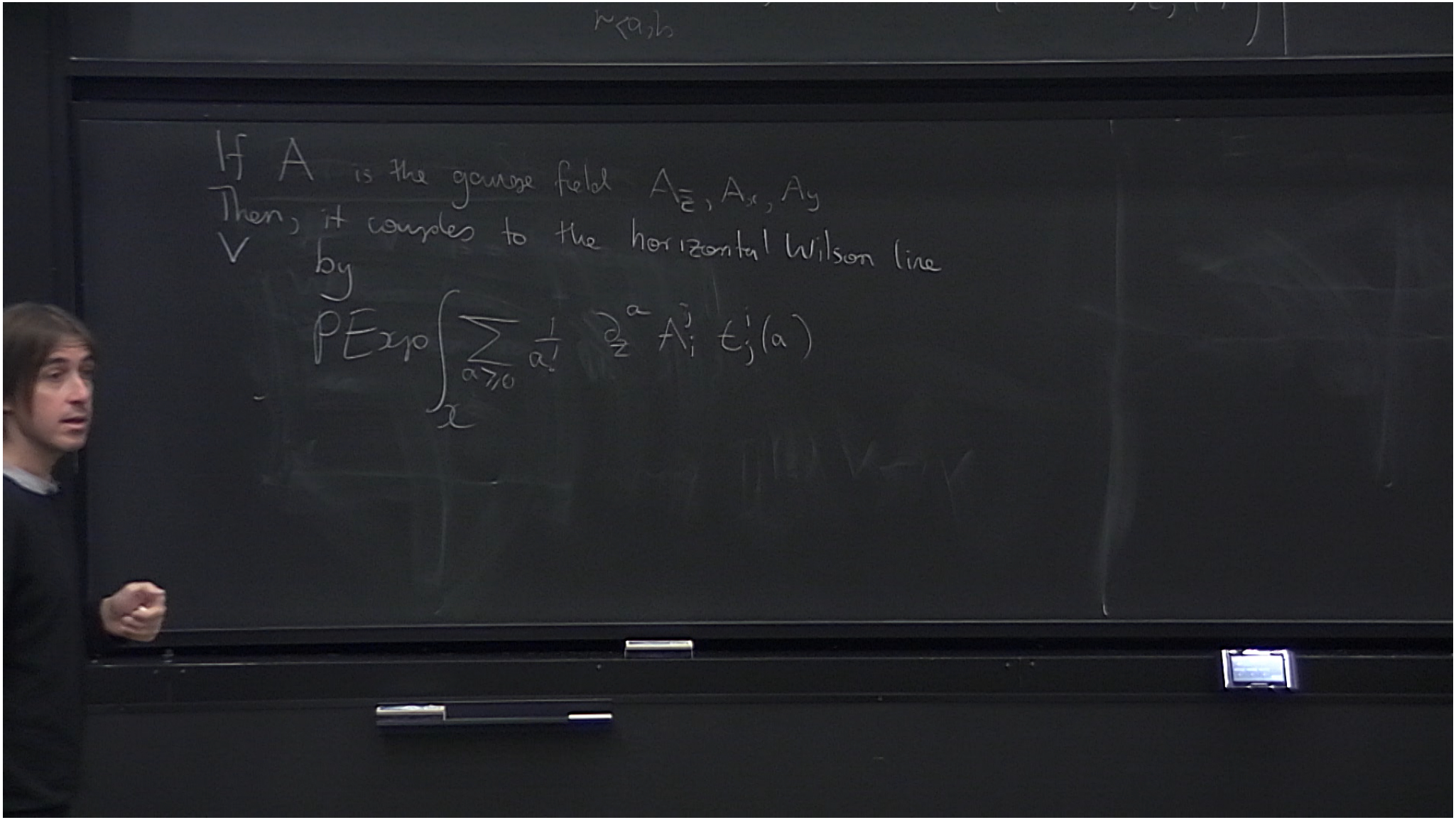
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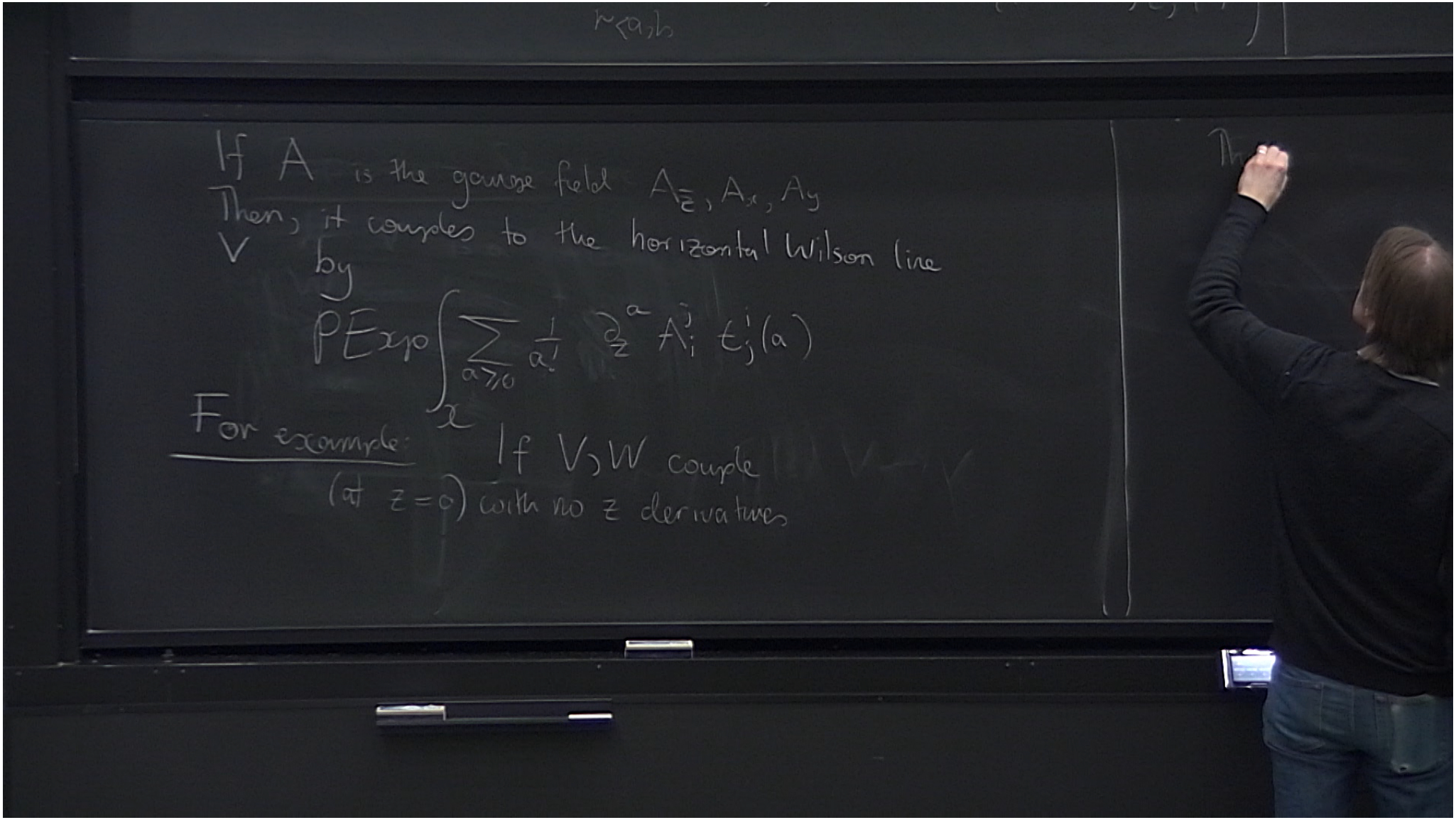
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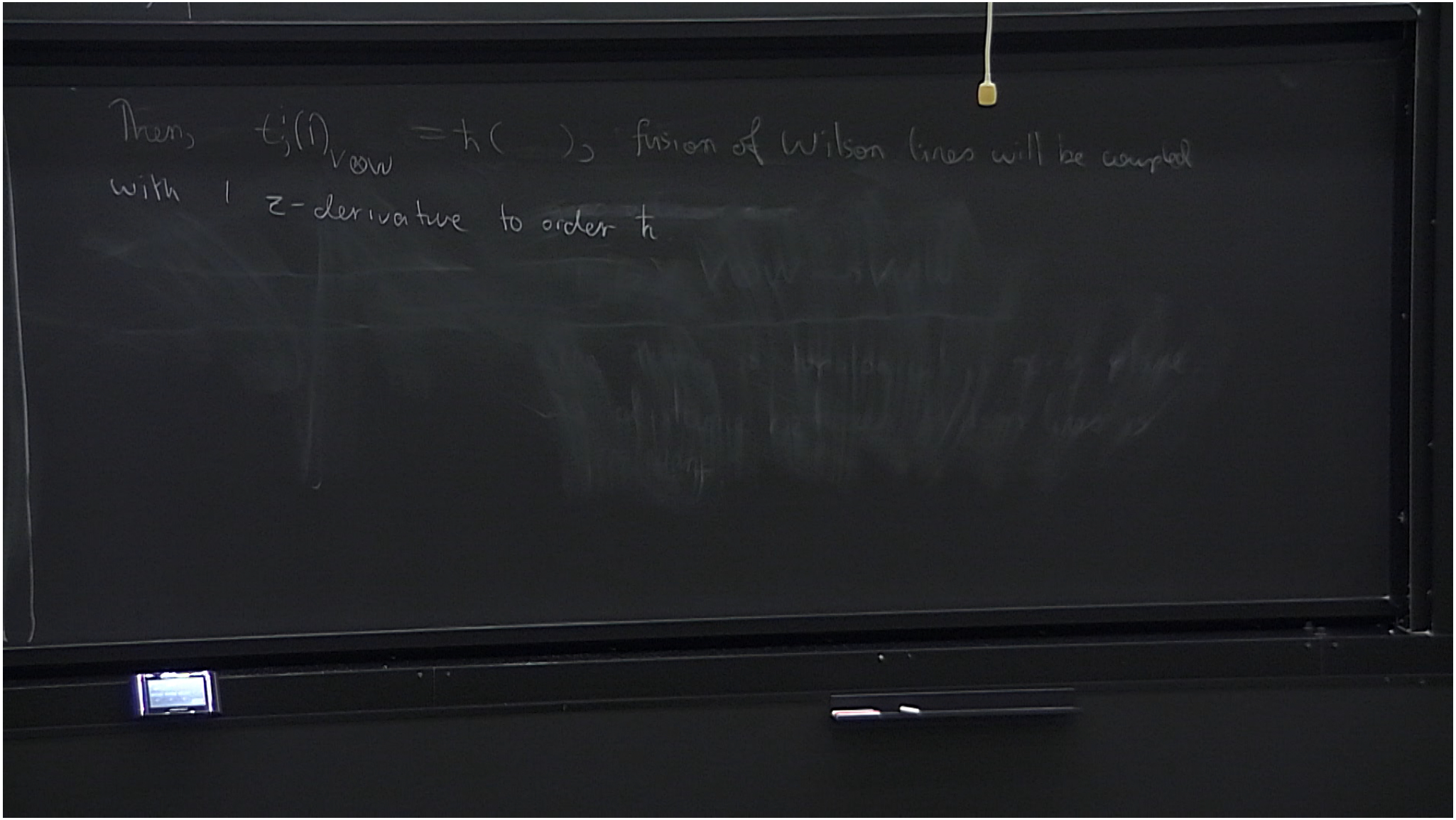
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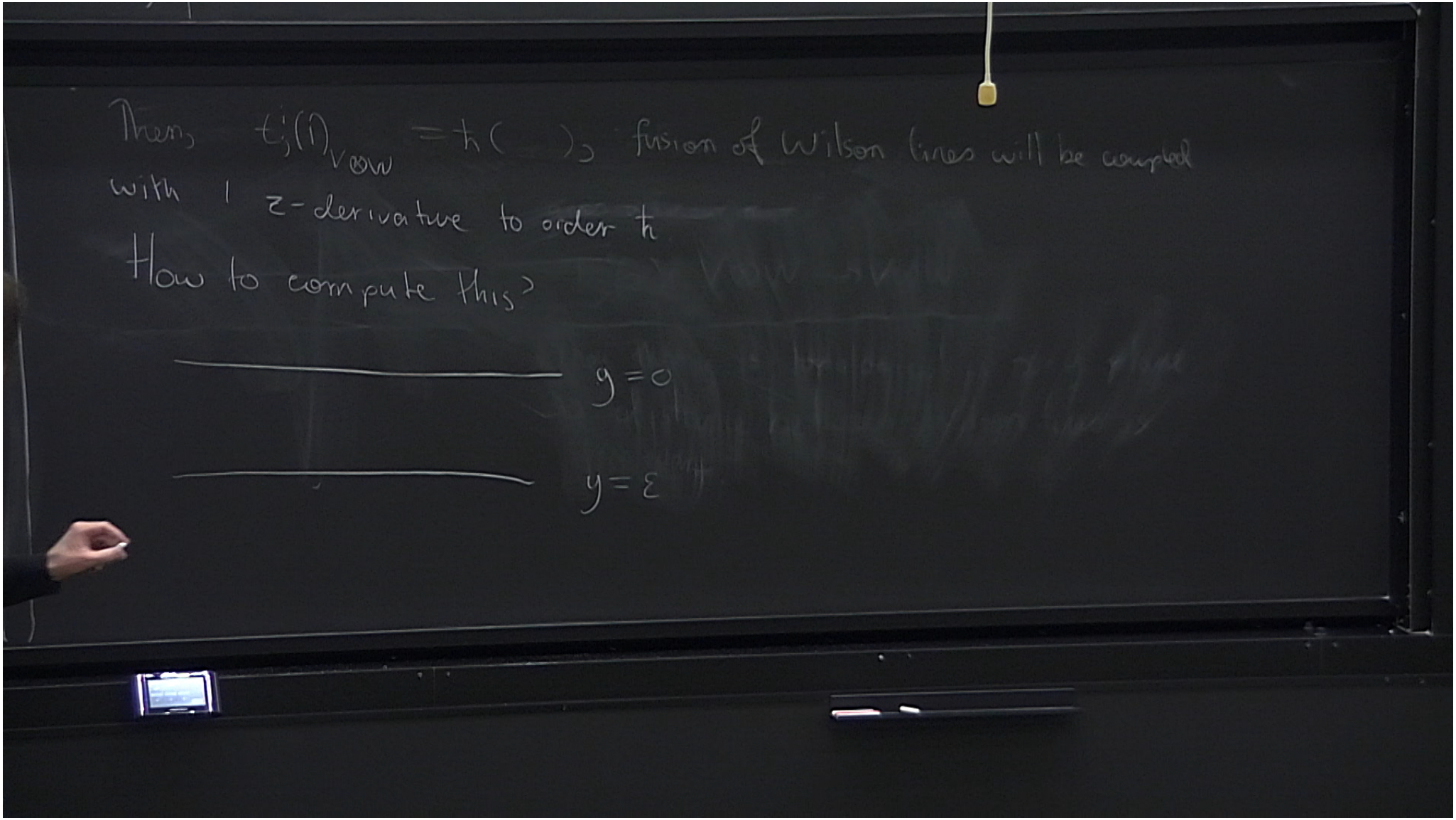
$$t^i_j(a)_{V\otimes W} = t^i_j(a)_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes t^i_j(a)_W + \hbar \sum_{r+s=a-1} t^i_k(r)_V \otimes t^k_j(s)_W$$


Fact
If $t^i_j(a)_V$, $t^k_l(a)_W$
each satisfy the relations of
the algebra, then so does
$t^i_j(a)_{V\otimes W}$, defined by
this expression.


If A is the gauge field $A_{\bar{z}}, A_x, A_y$
Then, it couples to the horizontal Wilson line V by
$$P\,Exp\int_x \sum_{a\geq 0} \frac{1}{a!}\,\partial_z^a A_i^j\, t_j^i(a)$$

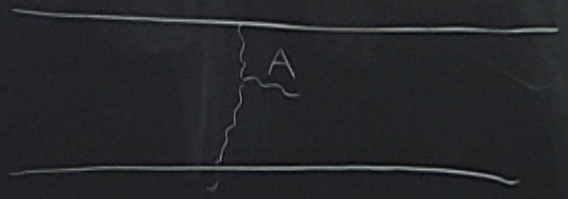

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Then, it couples to the horizontal Wilson line V by
$$P\,Exp\int_x \sum_{a\geq 0} \frac{1}{a!}\,\partial_z^a A_i^j\, t_j^i(a)$$
For example: If V, W couple (at $z=0$) with no z derivatives


Then, $t_i(1)_{vow} = h(\quad)$, fusion of Wilson lines will be counted
with 1 z-derivative to order h


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How to compute this?
$y = 0$
$y = \varepsilon$

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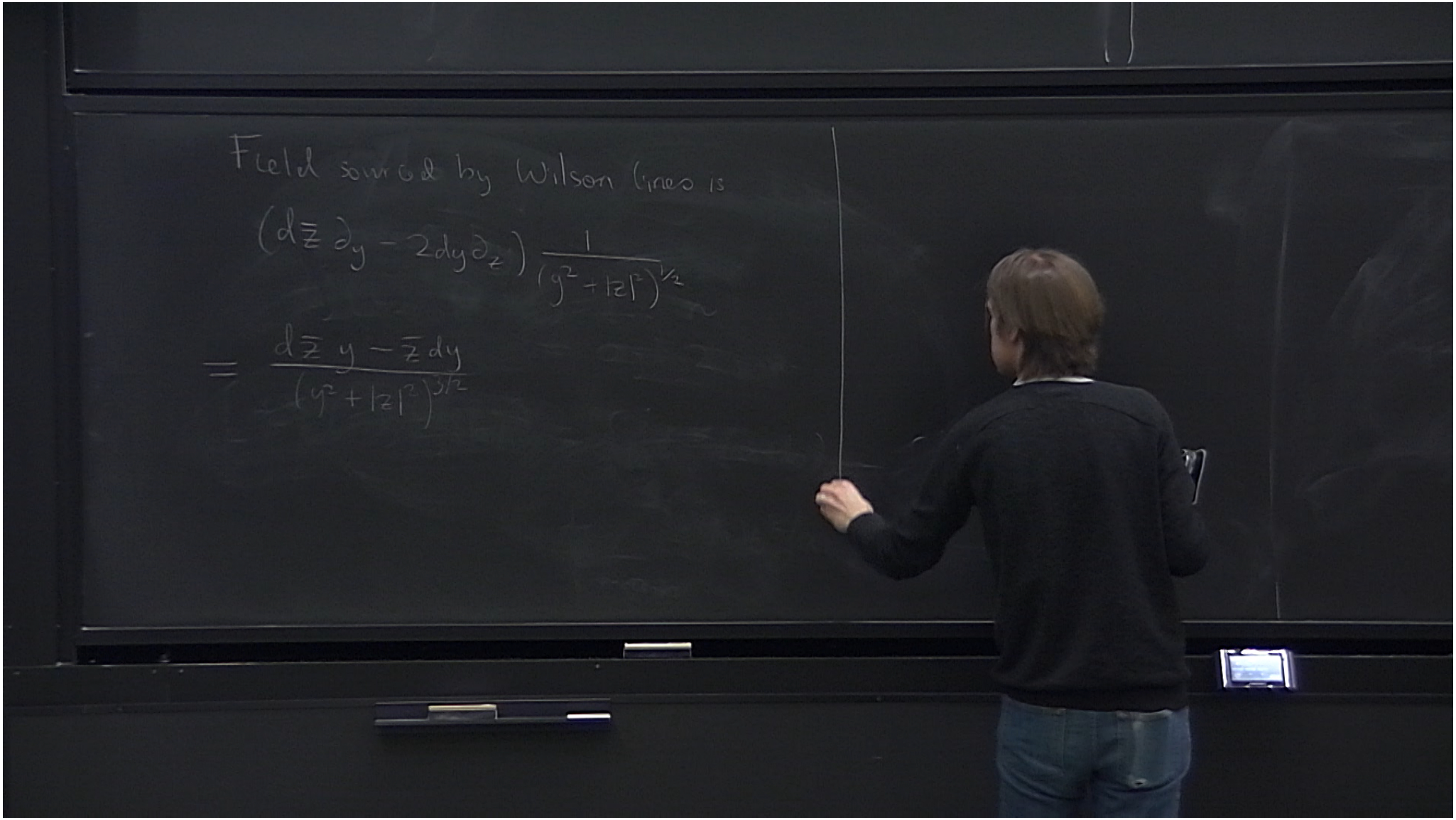
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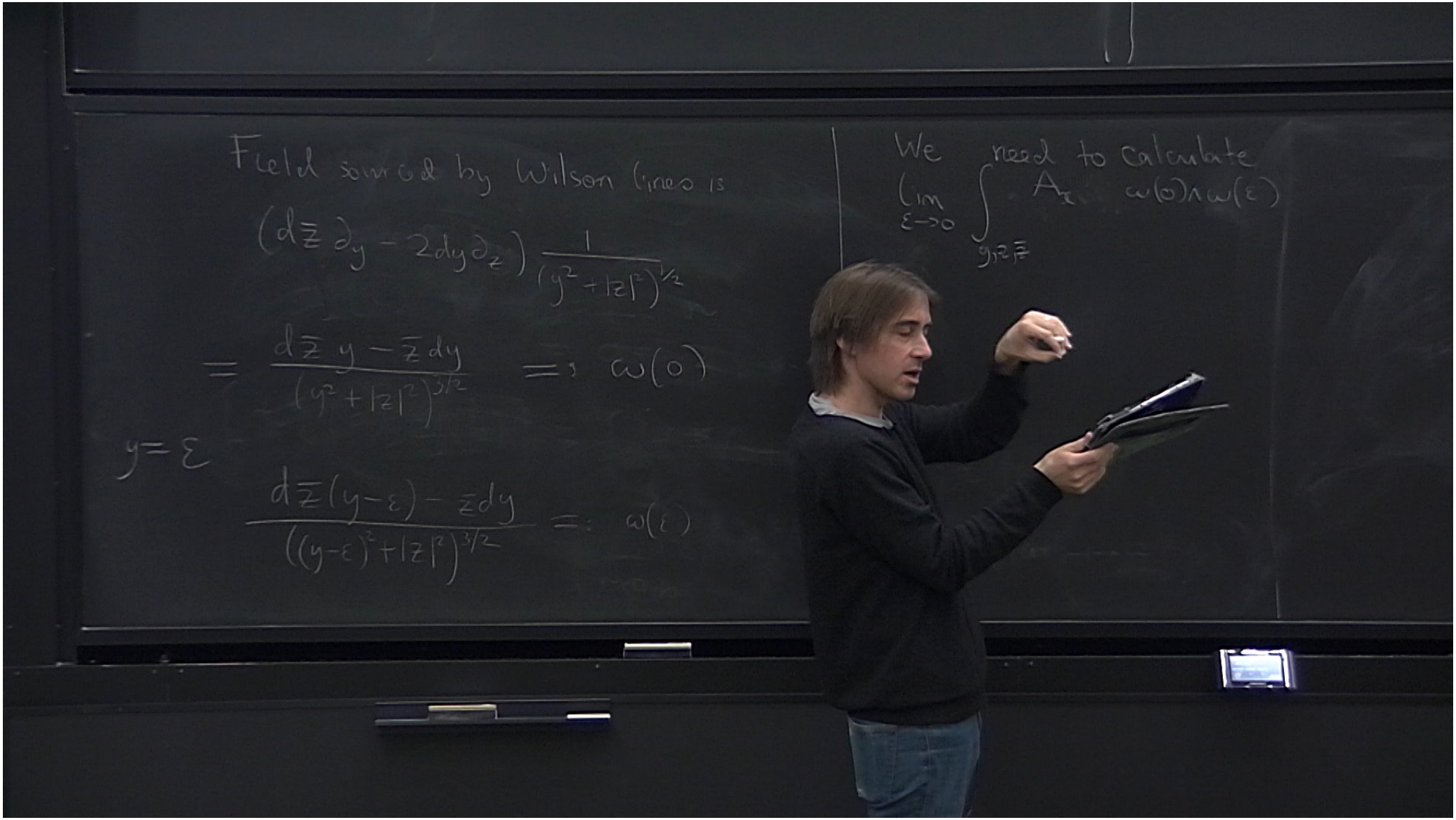


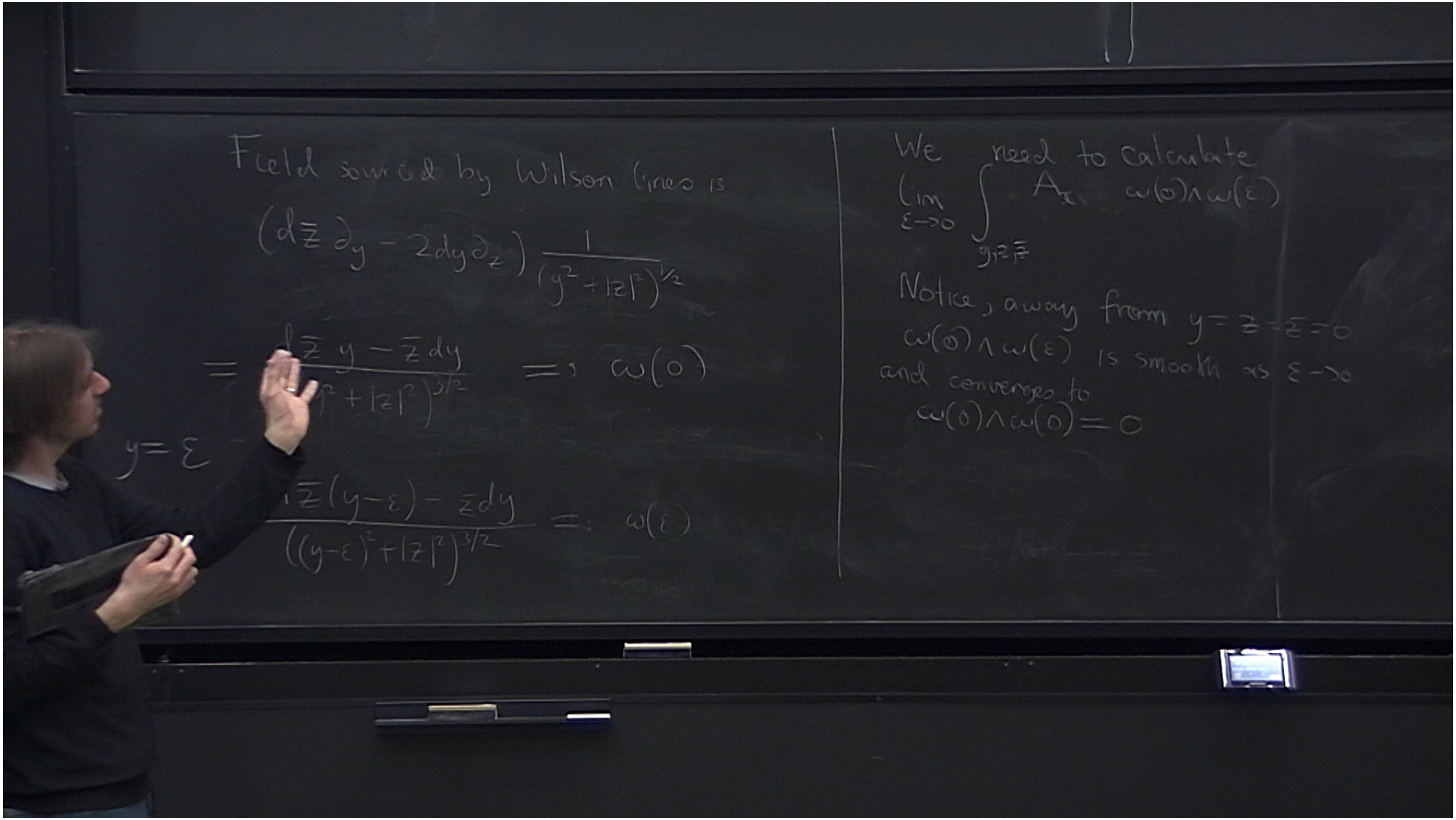
$y = 0$

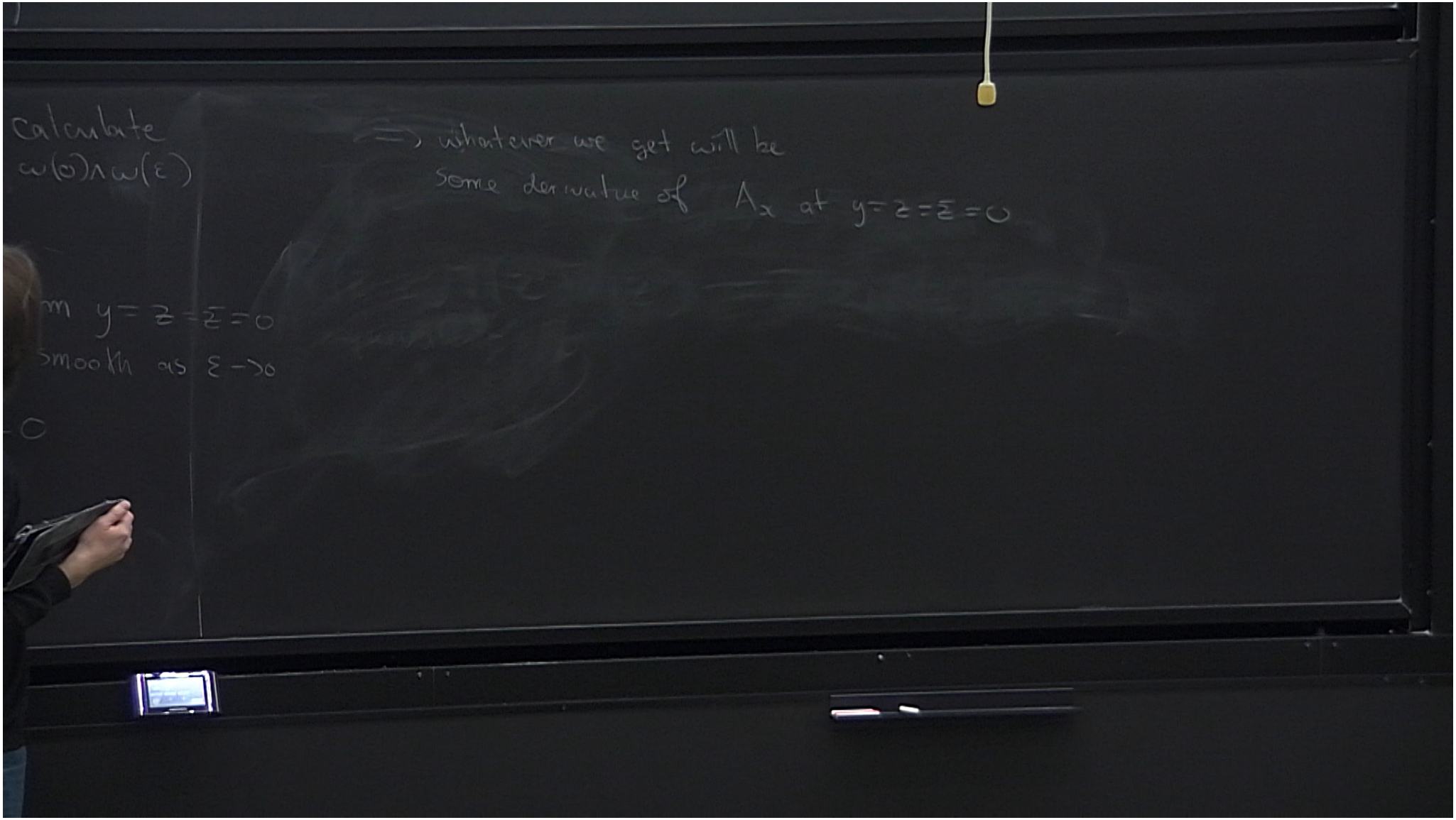
$y = \varepsilon$

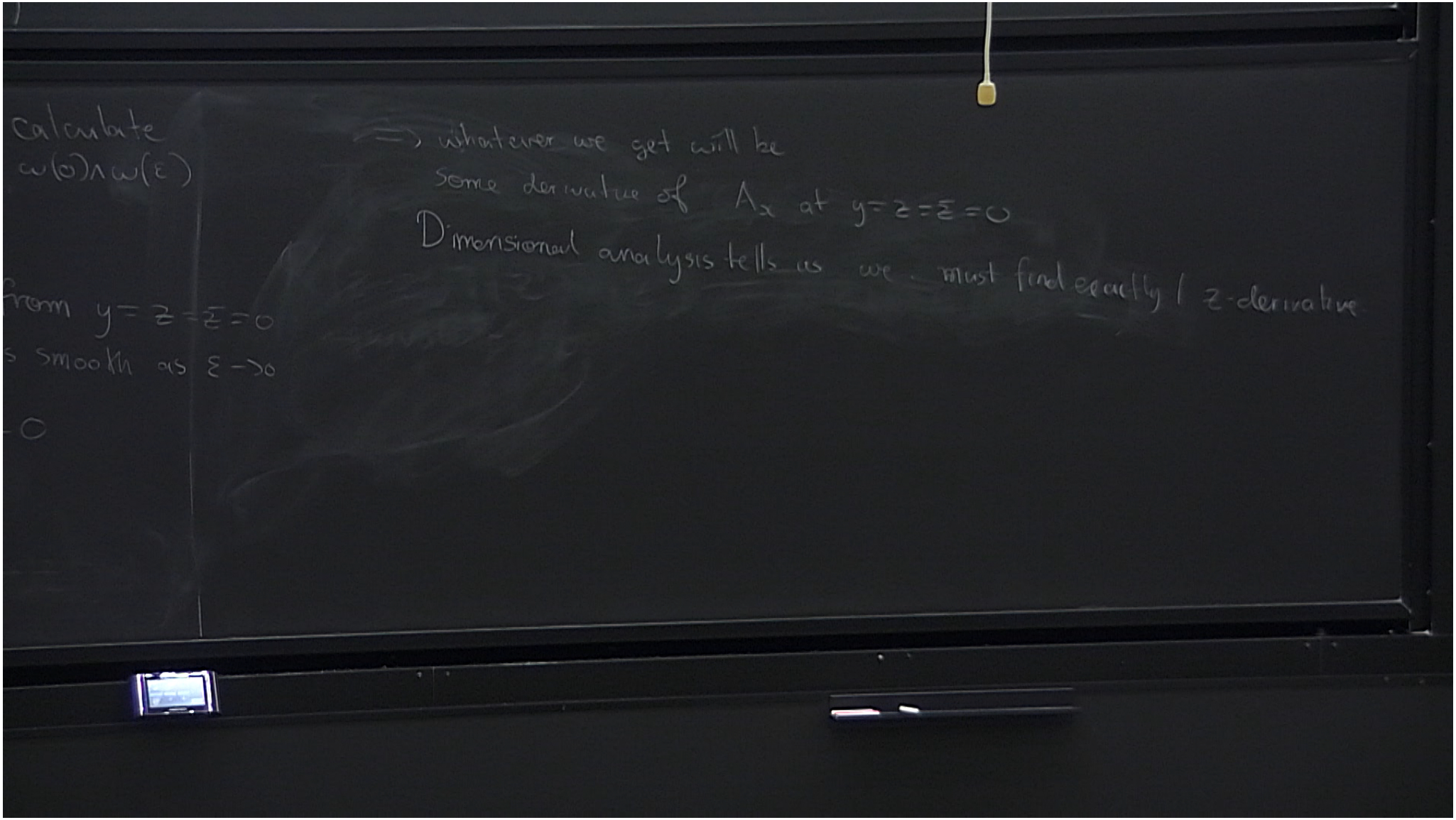
As $\varepsilon \to 0$ this will involve $\int_{y=0} \partial_z A$

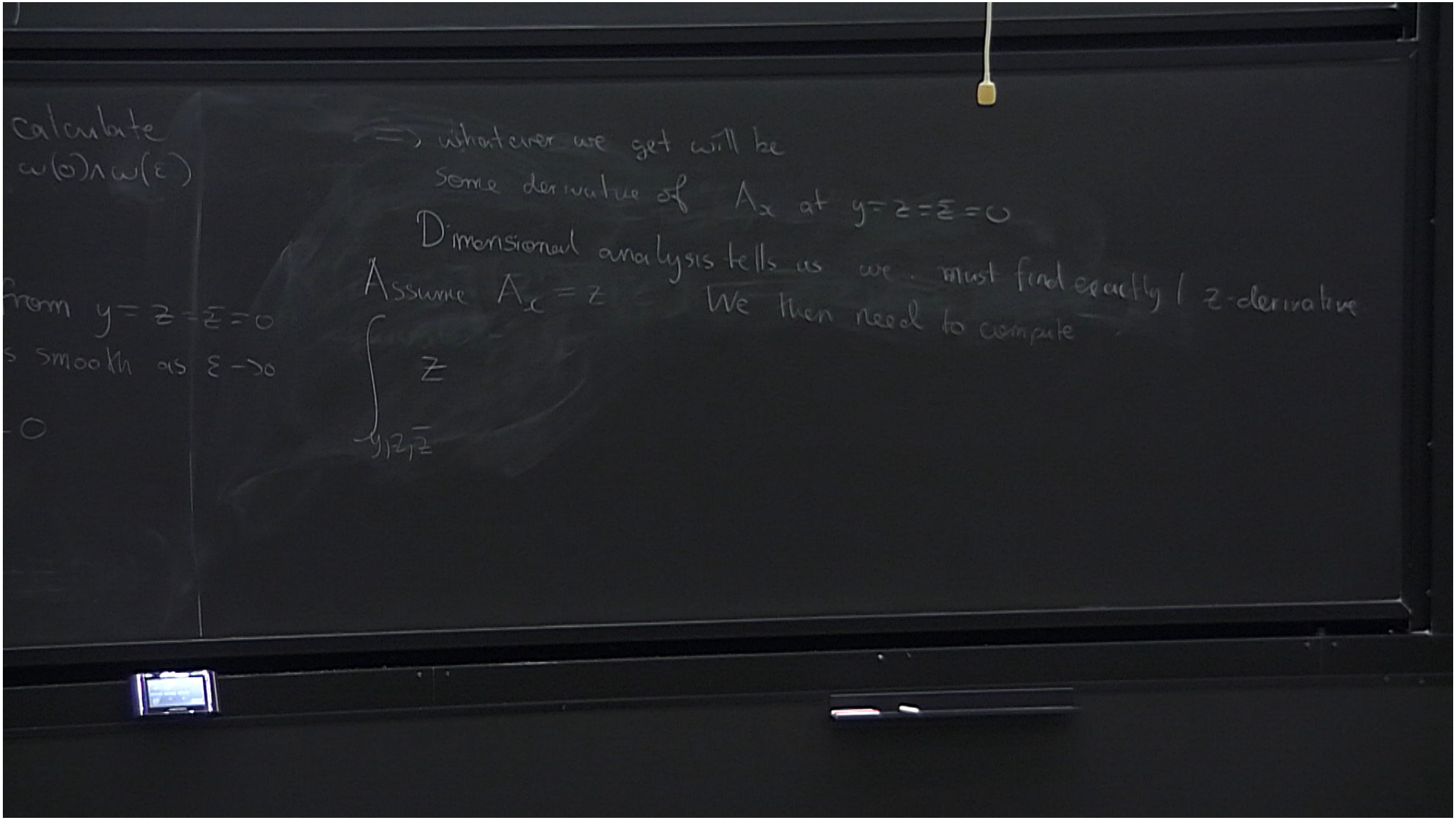

Field sourced by Wilson lines is
$(d\bar{z}\,\partial_y - 2dy\,\partial_z)\,\frac{1}{(y^2+|z|^2)^{1/2}}$
$= \frac{d\bar{z}\,y - \bar{z}\,dy}{(y^2+|z|^2)^{3/2}}$

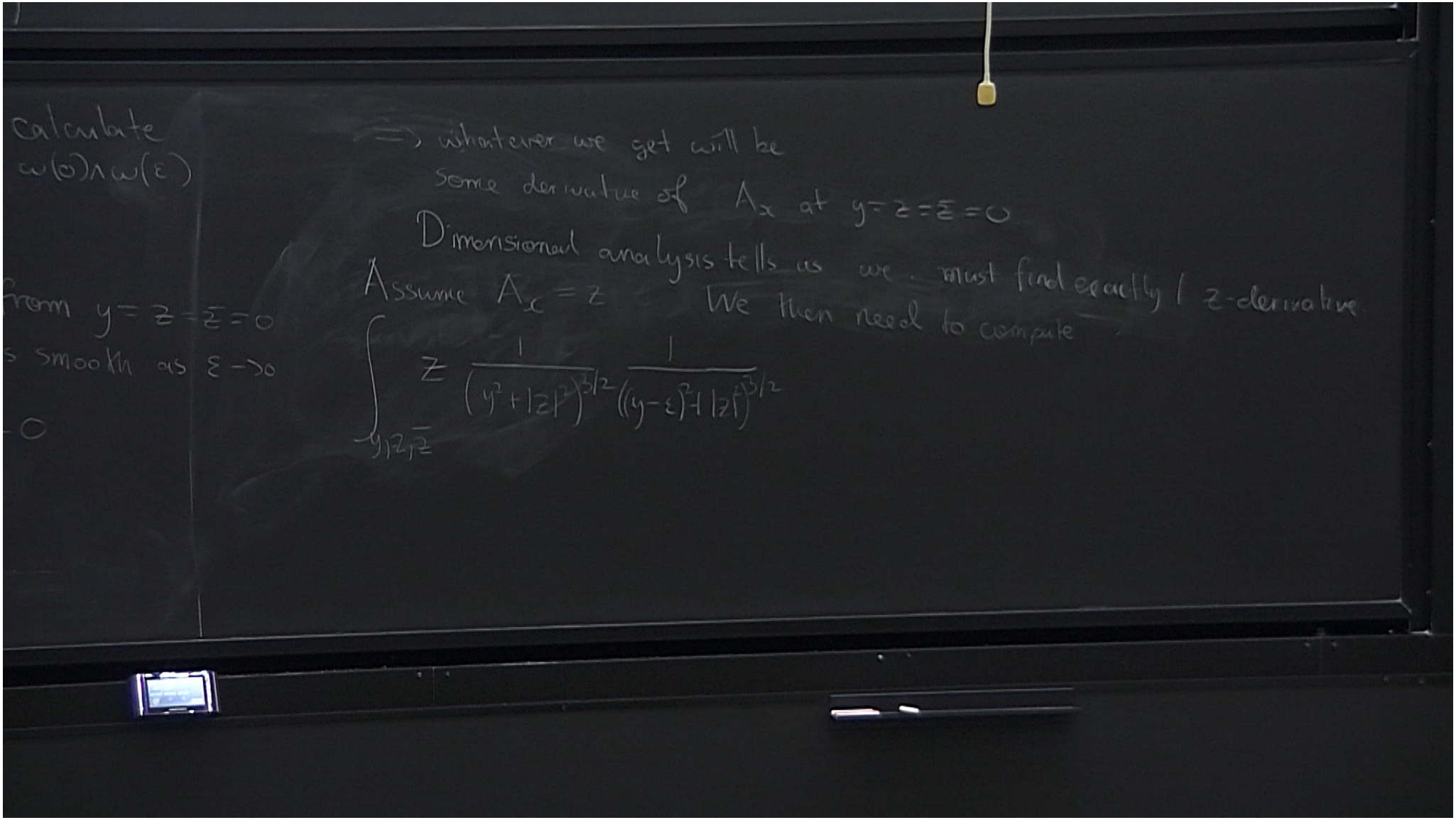

Field sourced by Wilson lines is
$(d\bar{z}\,\partial_y - 2dy\,\partial_z)\frac{1}{(y^2+|z|^2)^{1/2}}$
$= \frac{d\bar{z}\,y - \bar{z}\,dy}{(y^2+|z|^2)^{3/2}} =: \omega(0)$
$y=\varepsilon$
$\frac{d\bar{z}(y-\varepsilon) - \bar{z}dy}{((y-\varepsilon)^2+|z|^2)^{3/2}} =: \omega(\varepsilon)$
We need to calculate
$\lim_{\varepsilon\to 0}\int_{y,z,\bar{z}} A_z\, \omega(0)\wedge\omega(\varepsilon)$


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$\lim_{\varepsilon\to 0}\int_{y,z,\bar{z}} A_z\ \omega(0)\wedge\omega(\varepsilon)$
Notice, away from $y=z=\bar{z}=0$
$\omega(0)\wedge\omega(\varepsilon)$ is smooth as $\varepsilon\to 0$
and converges to
$\omega(0)\wedge\omega(0)=0$


Calculate
$\omega(\varepsilon)\wedge\omega(\varepsilon)$
$\Rightarrow$ whatever we get will be
some derivative of Λ_x at $y=z=\bar{z}=0$
$y=z=\bar{z}=0$
smooth as $\varepsilon\to 0$


Calculate
$\omega(\varepsilon)\wedge\omega(\varepsilon)$
$\Rightarrow$ whatever we get will be
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Dimensional analysis tells us we must find exactly 1 z-derivative
From $y=z=\bar{z}=0$
is smooth as $\varepsilon\to 0$
$=0$


calculate
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Dimensional analysis tells us we must find exactly 1 z-derivative
Assume $A_x = z$
We then need to compute
$\int_{y,z,\bar{z}} z$
from $y=z=\bar{z}=0$
s smooth as $\varepsilon \to 0$
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Calculate
$\omega(\varepsilon)\wedge\omega(\varepsilon)$
$\Rightarrow$ whatever we get will be some derivative of A_x at $y=z=\bar{z}=0$
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$$\int_{y,z,\bar{z}} z \frac{1}{(y^2+|z|^2)^{3/2}} \frac{1}{((y-\varepsilon)^2+|z|^2)^{3/2}}$$
From $y=z=\bar{z}=0$
is smooth as $\varepsilon \to 0$
$=0$

Calculate
$\omega(0)\wedge\omega(\varepsilon)$

$\Rightarrow$ whatever we get will be some derivative of A_x at $y=z=\bar{z}=0$

Dimensional analysis tells us we must find exactly 1 z-derivative

From $y=z=\bar{z}=0$

Assume $A_x = z$ We then need to compute

$\int_{y,z,\bar{z}} z \frac{1}{(y^2+|z|^2)^{3/2}} \frac{1}{((y-\varepsilon)^2+|z|^2)^{3/2}} (d\bar{z}\, y - \bar{z}\, dy)(d\bar{z}(y-\varepsilon) - \bar{z}\, dy)$

smooth as $\varepsilon \to 0$

$= 0$

Calculate
$\omega(\varepsilon)\wedge\omega(\varepsilon)$

$\Rightarrow$ whatever we get will be some derivative of A_x at $y=z=\bar{z}=0$

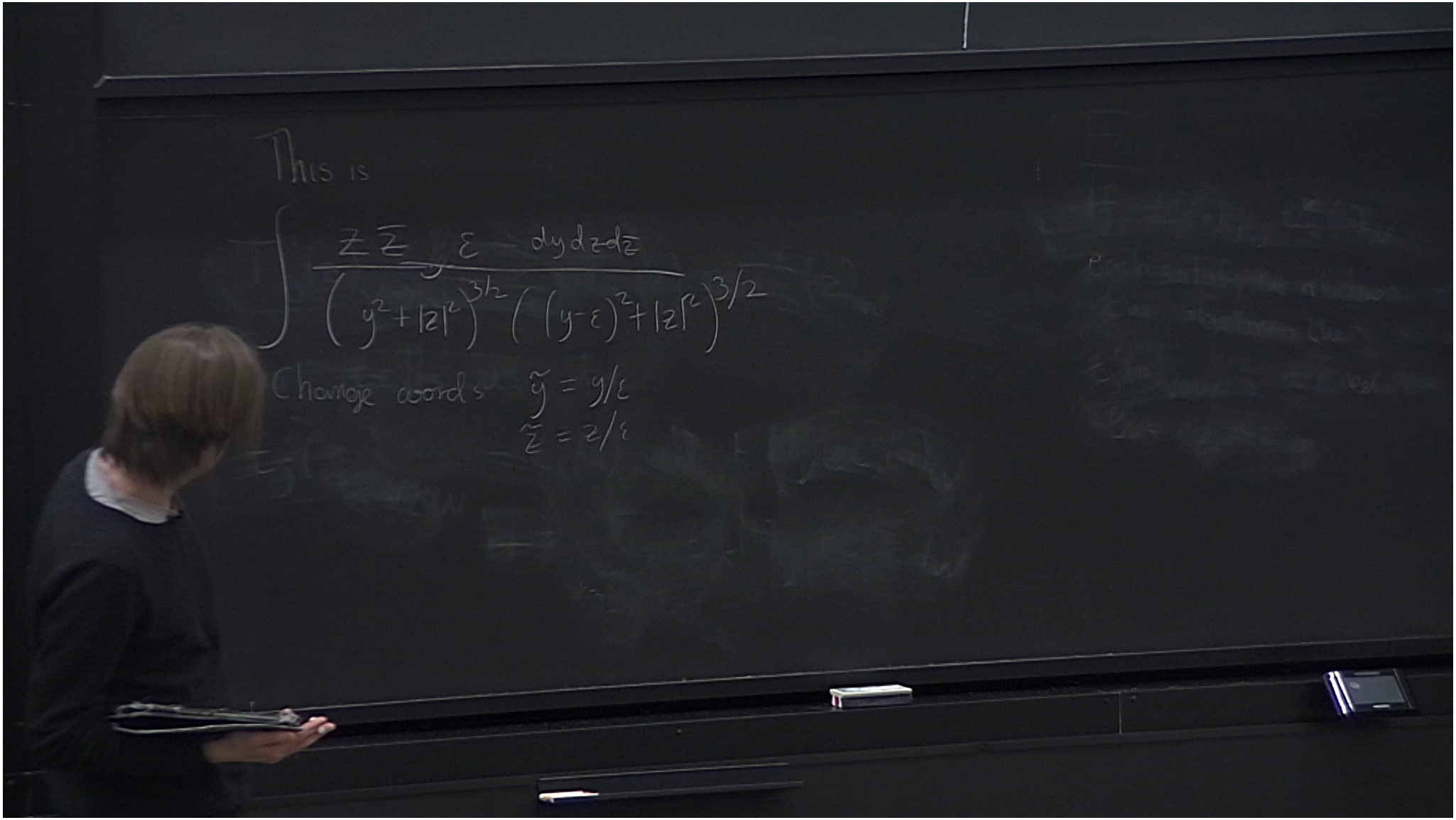
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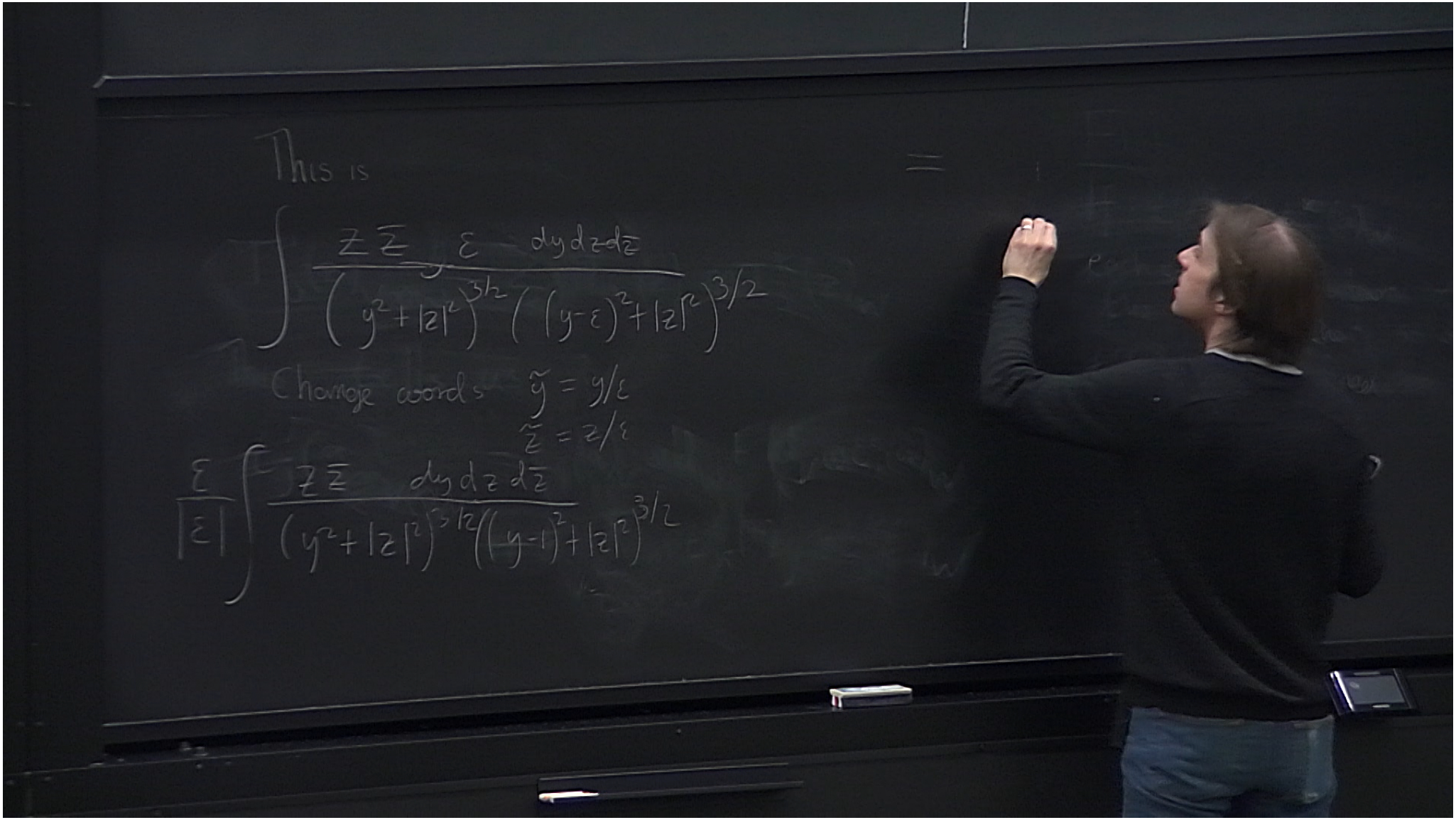
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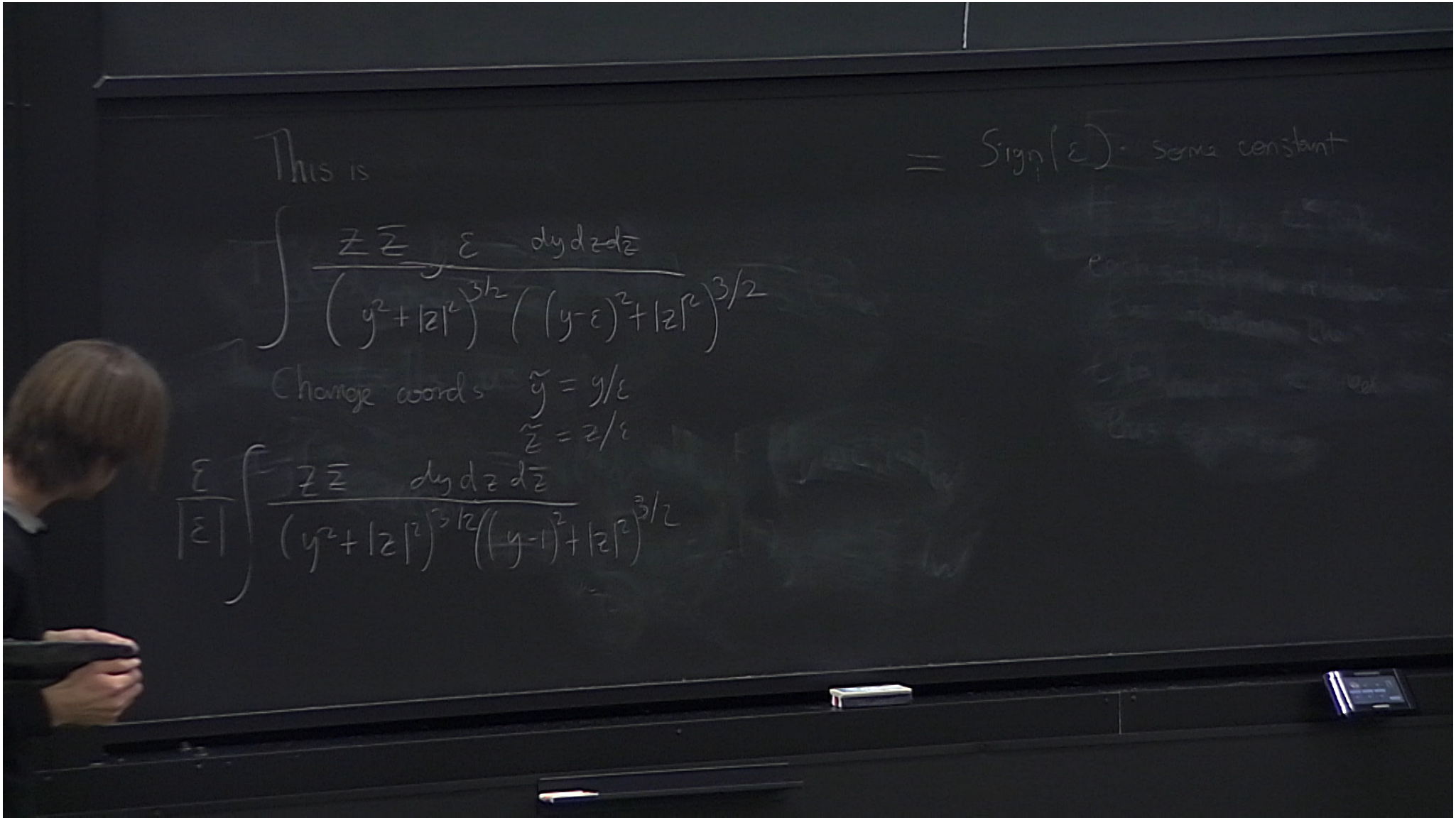
$$\int_{y,z,\bar{z}} z\,\frac{1}{(y^2+|z|^2)^{3/2}}\,\frac{1}{((y-\varepsilon)^2+|z|^2)^{3/2}}\left(d\bar{z}\,y-\bar{z}\,dy\right)\left(d\bar{z}\,(y-\varepsilon)-\bar{z}\,dy\right)$$

From $y=z=\bar{z}=0$
is smooth as $\varepsilon\to 0$

$=0$

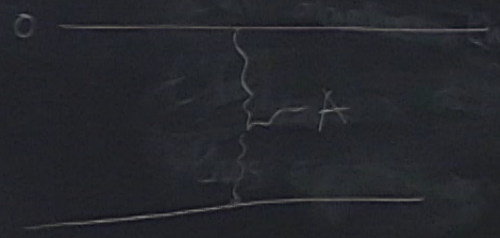

This is
$\int \frac{z\bar{z}\,\varepsilon\,dy\,dz\,d\bar{z}}{(y^2+|z|^2)^{3/2}((y-\varepsilon)^2+|z|^2)^{3/2}}$
Change coords $\tilde{y} = y/\varepsilon$
$\tilde{z} = z/\varepsilon$


This is
$\int \frac{z\bar{z}\ \varepsilon\ dy\,dz\,d\bar{z}}{(y^2+|z|^2)^{3/2}((y-\varepsilon)^2+|z|^2)^{3/2}}$
Change coords $\tilde{y} = y/\varepsilon$
$\tilde{z} = z/\varepsilon$
$\frac{\varepsilon}{|\varepsilon|}\int \frac{z\bar{z}\ dy\,dz\,d\bar{z}}{(y^2+|z|^2)^{3/2}((y-1)^2+|z|^2)^{3/2}}$


This is
$= \mathrm{Sign}(\varepsilon)$ · some constant
$\int \frac{z\bar{z}\ \varepsilon\ dy\,dz\,d\bar{z}}{(y^2+|z|^2)^{3/2}\left((y-\varepsilon)^2+|z|^2\right)^{3/2}}$
Change coords $\tilde{y} = y/\varepsilon$
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$= \mathrm{Sign}(\varepsilon) \cdot$ same constant

So we've calculated



$$\sim \int_\ell \mathrm{Sign}(\varepsilon)\, \partial_z A_x \, dx$$