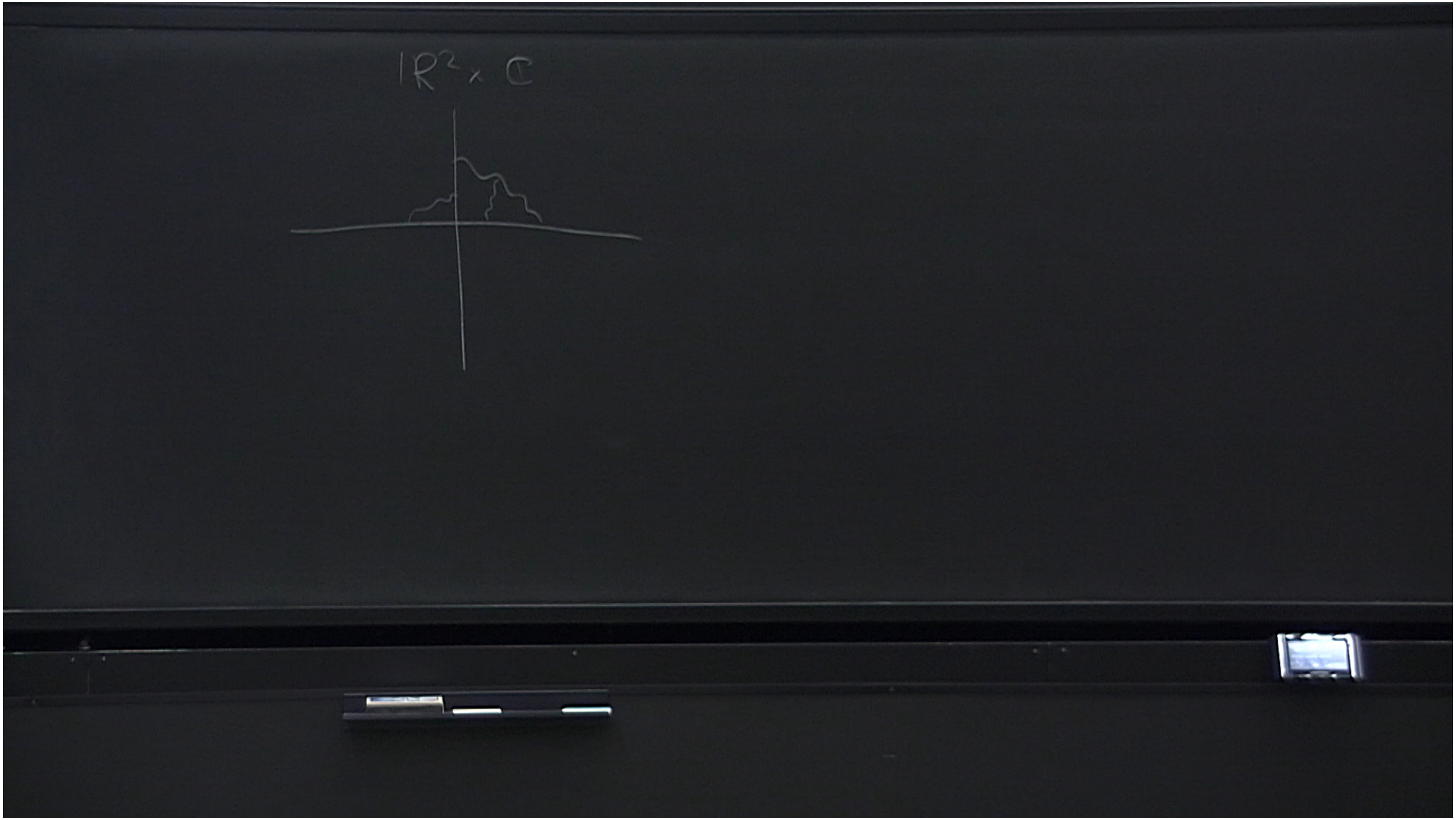
Title: PSI 2016/2017 Introduction to Quantum Integrable Systems - Lecture 10

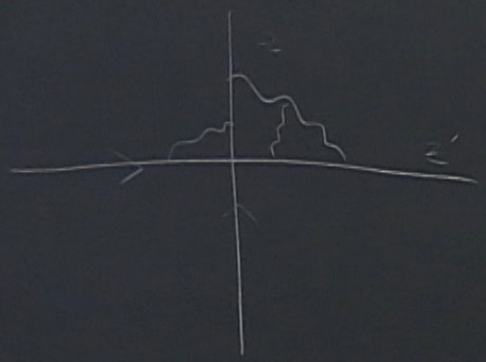
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Abstract:


$\mathbb{R}^2 \times \mathbb{C}$

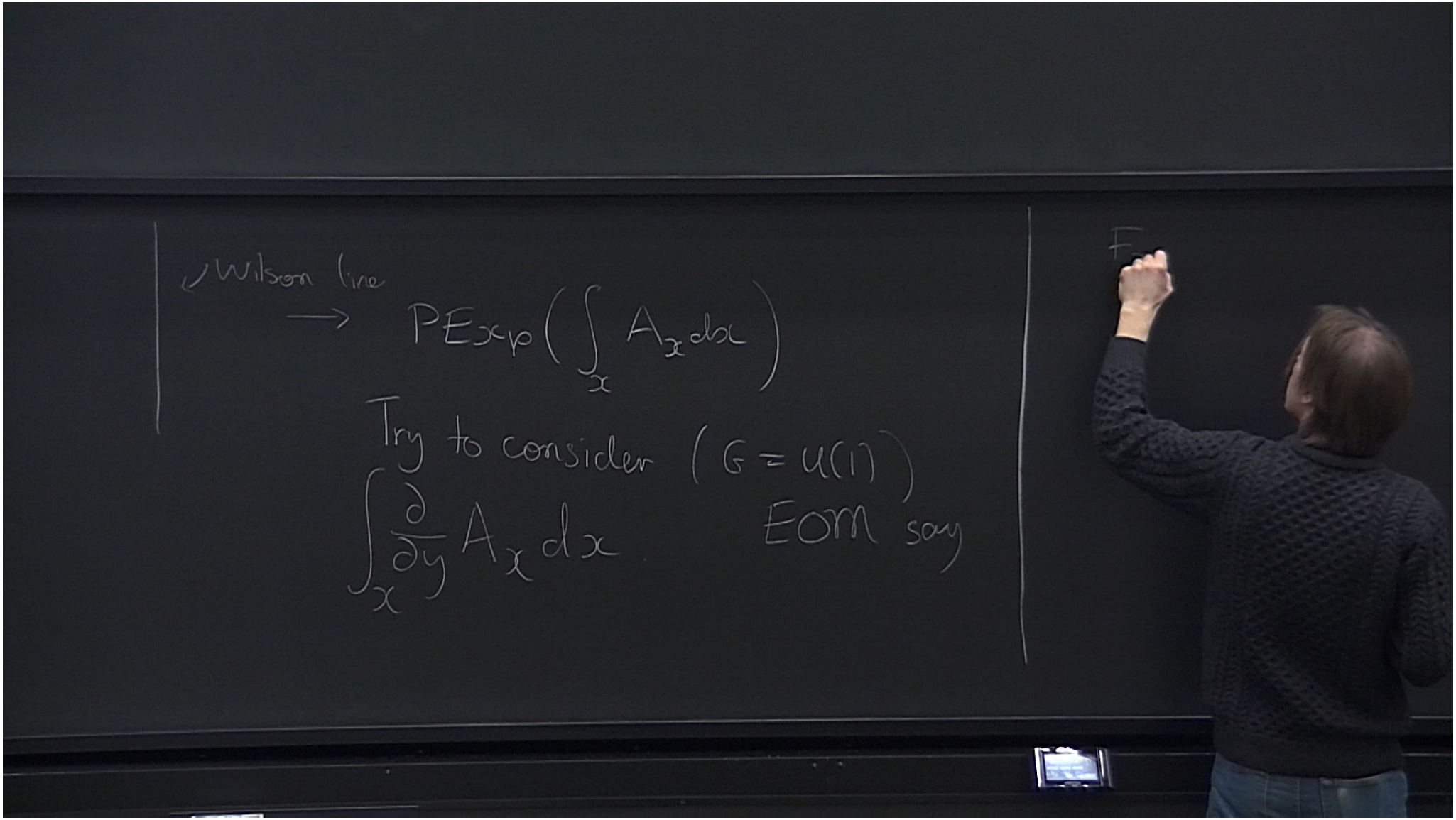
$\mathbb{R}^2 \times \mathbb{C}$

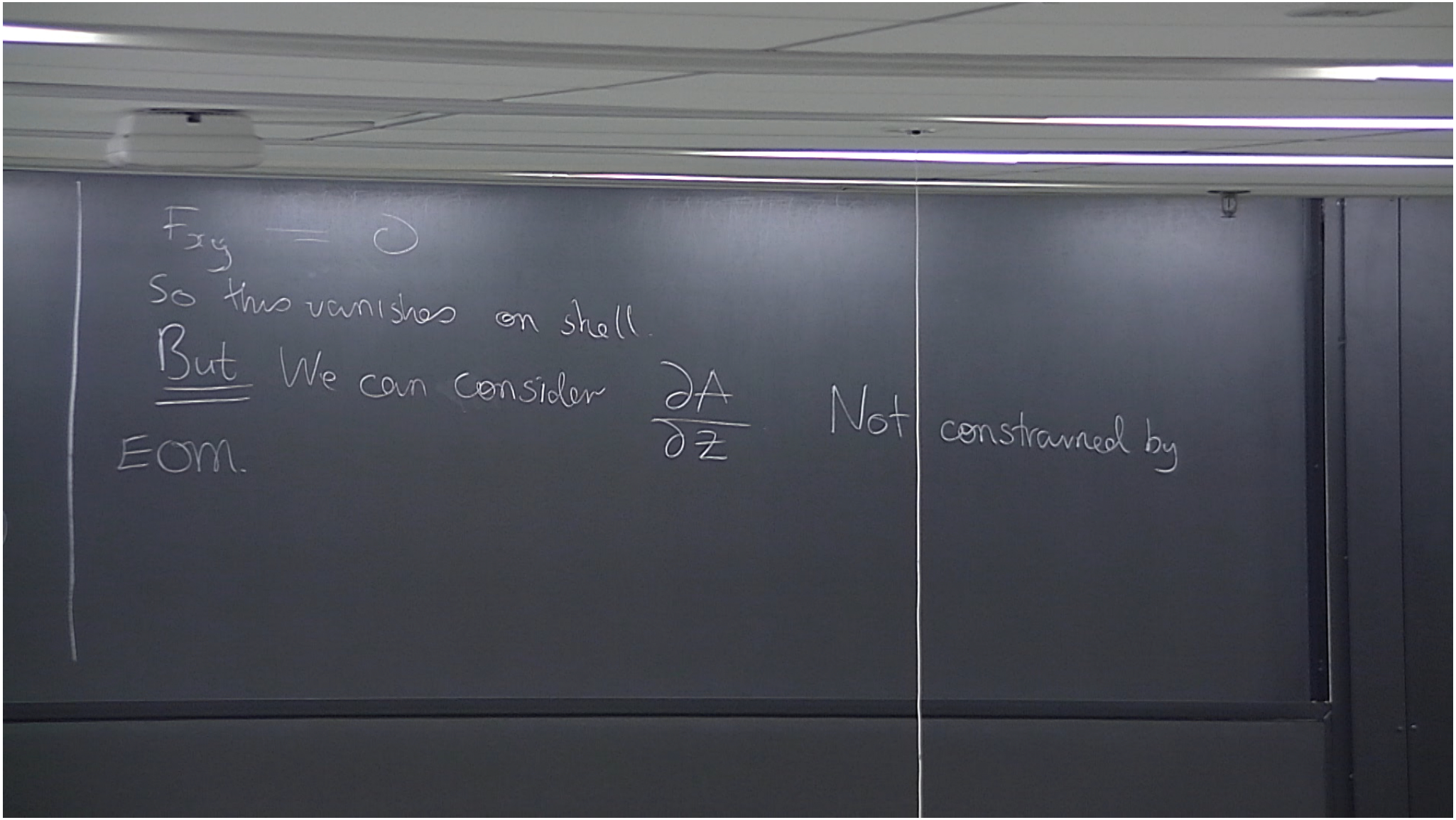


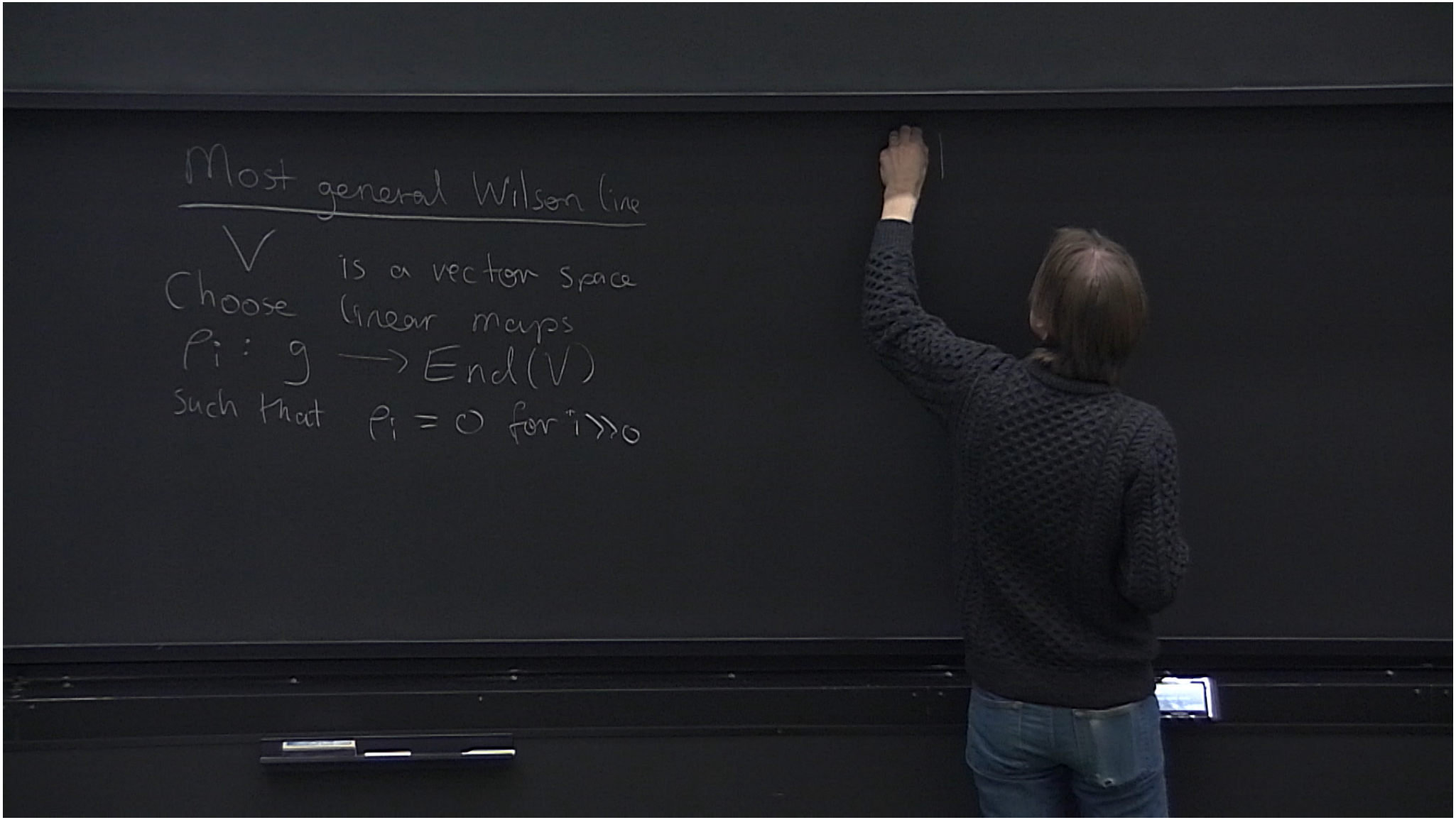
R-matrix

$R(z-z')$ comes from

evolution of states on Wilson lines.


Wilson line
$\longrightarrow$ $P\mathrm{Exp}\left(\int_x A_x dx\right)$
Try to consider $(G = U(1))$
$\int_x \frac{\partial}{\partial y} A_x dx$
EOM say


$F_{xy} = 0$
So this vanishes on shell.
But We can consider $\frac{\partial A}{\partial z}$ Not constrained by
EOM.


Most general Wilson line
V is a vector space
Choose linear maps
$\rho_i : g \longrightarrow End(V)$
such that $\rho_i = 0$ for $i \gg 0$

Then we can consider

$$P\mathrm{Exp}\left\{\int_x \sum_{i=0}^{\infty} \frac{a}{i!}\left(\left(\frac{\partial}{\partial z}\right)^i A_x\, da\right)\right\}$$

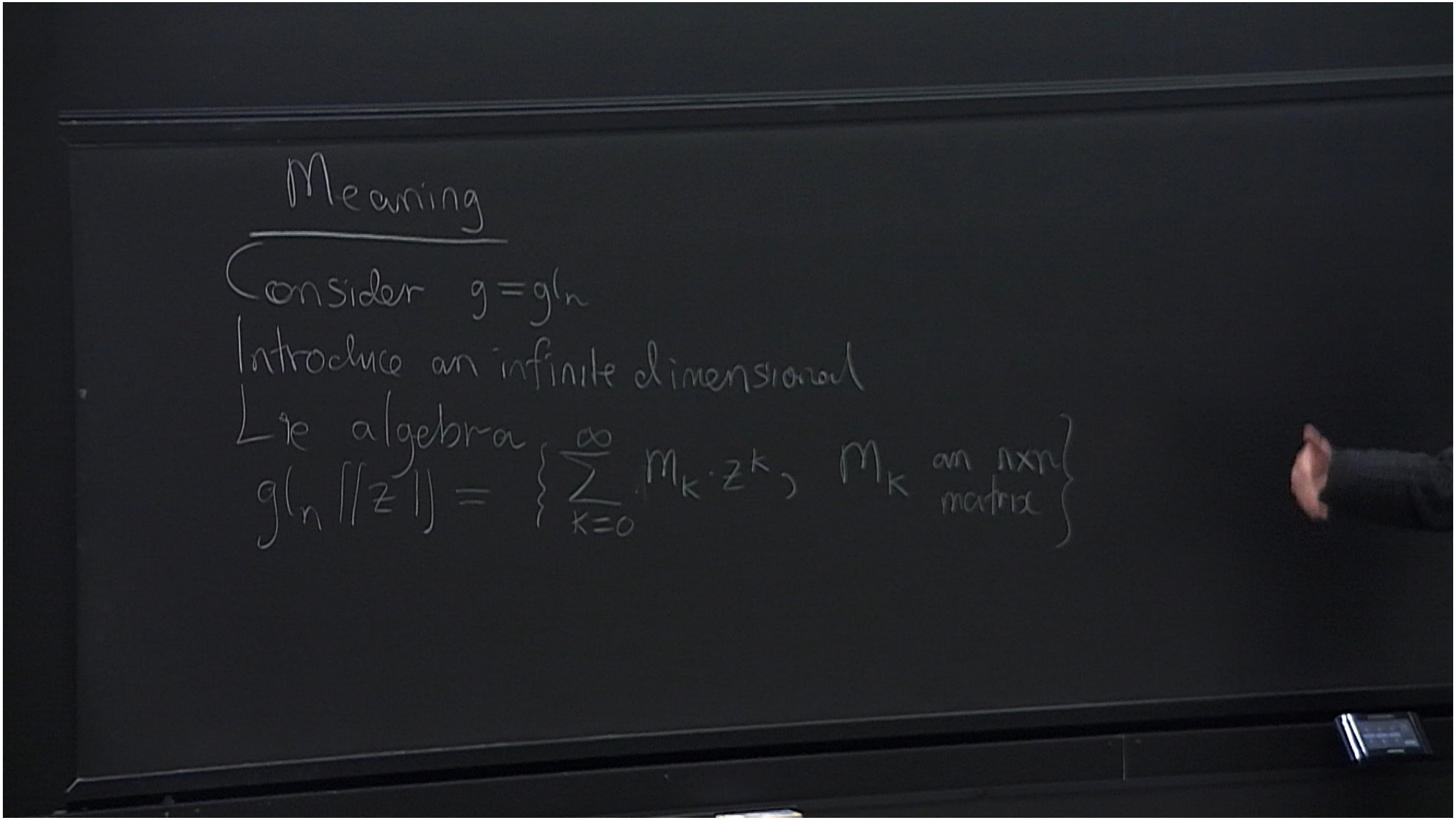
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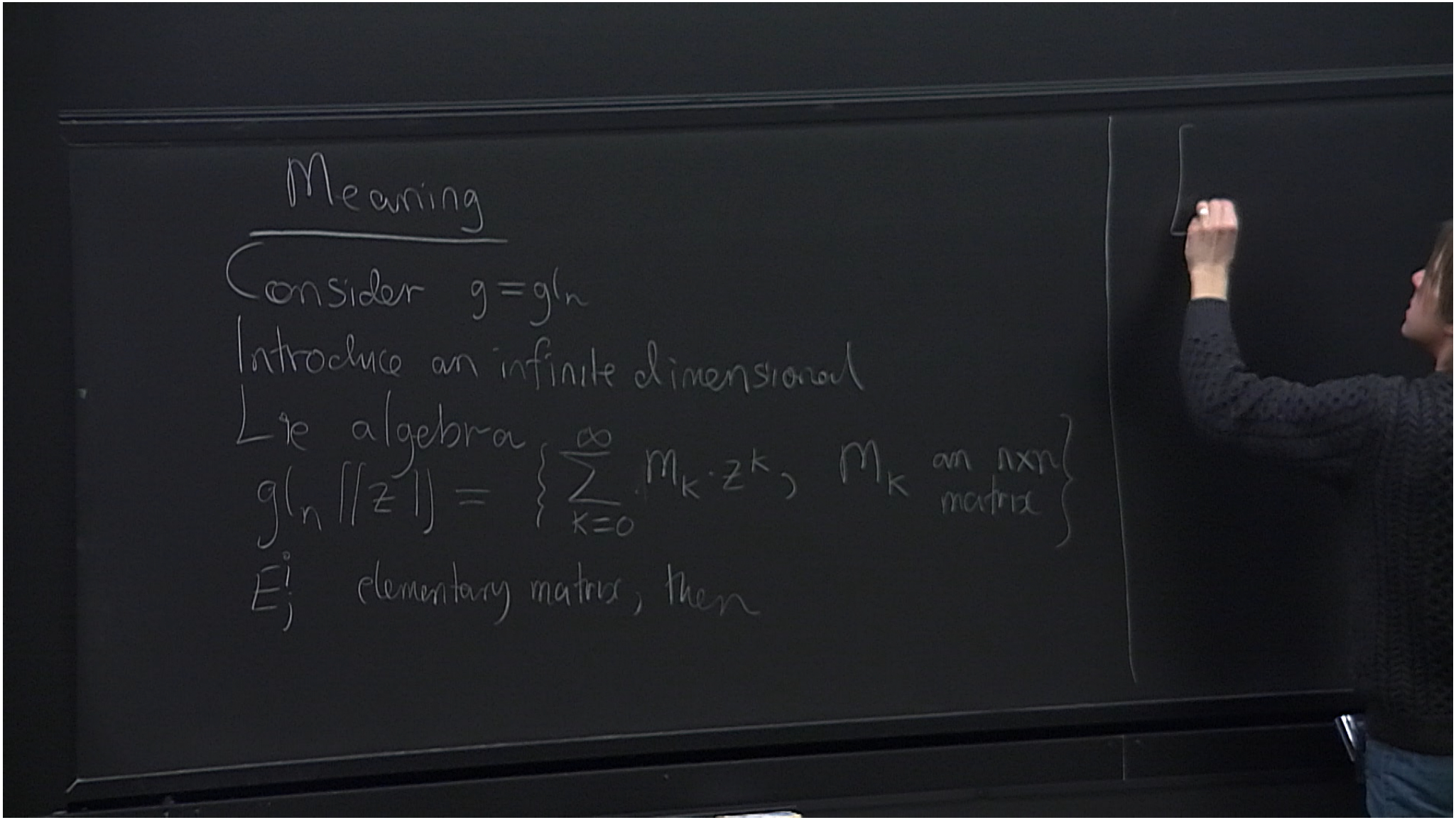
$$P\mathrm{Exp}\left\{\int_x \sum_{i=0}^{\infty} \frac{a}{i!}\left(\left(\frac{\partial}{\partial z}\right)^i A_x\, dz\right)\right\}$$

For this to be gauge invariant, we need $\forall\, X, Y \in g$

$$[\rho_i(X), \rho_j(Y)] = \rho_{i+j}([X,Y])$$

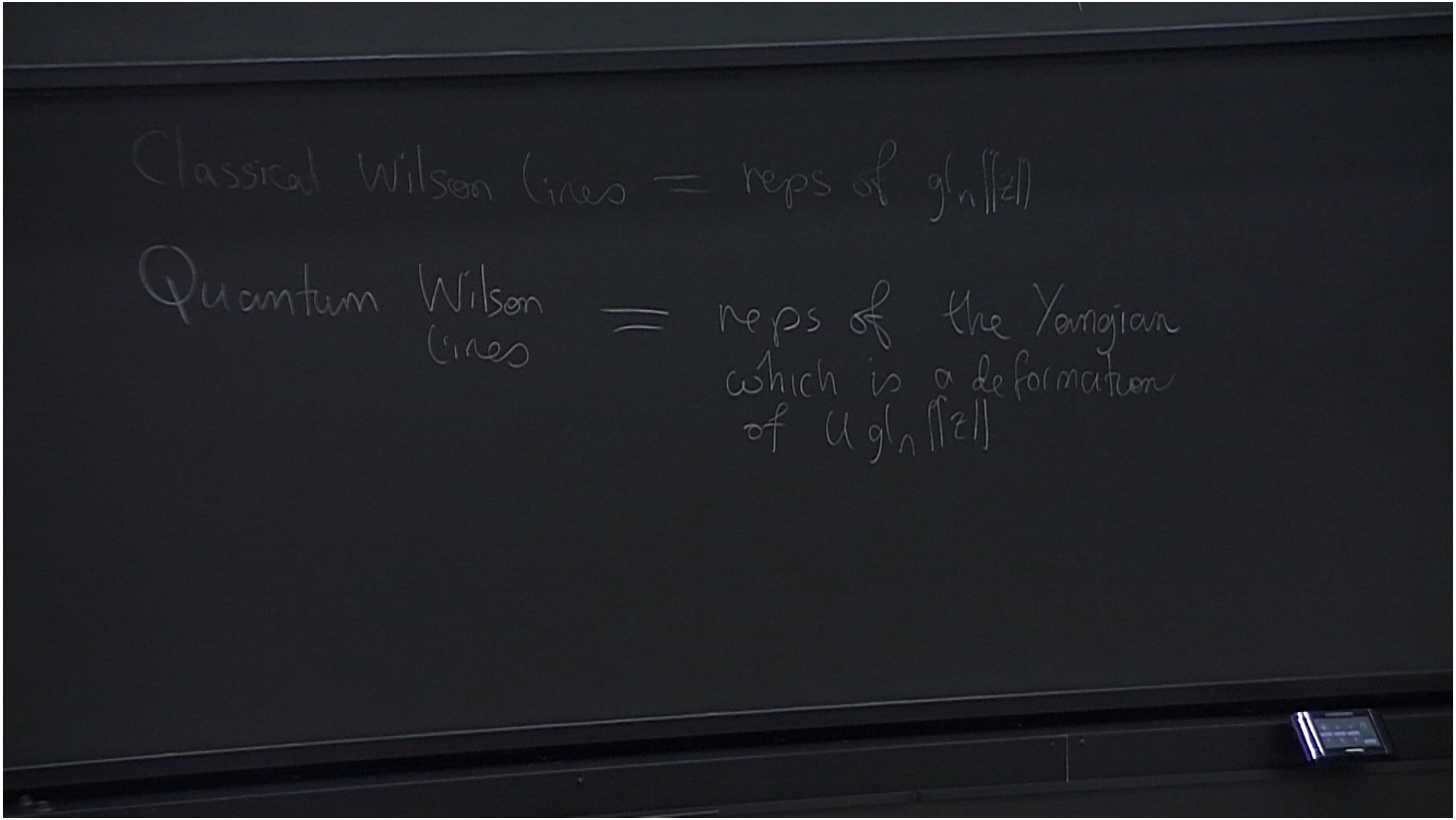



Meaning
Consider $g = gl_n$
Introduce an infinite dimensional
Lie algebra
$gl_n[[z]] = \left\{ \sum_{k=0}^{\infty} M_k \cdot z^k, \; M_k \text{ an } n \times n \text{ matrix} \right\}$


Meaning
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Introduce an infinite dimensional
Lie algebra
$gl_n[[z]] = \left\{ \sum_{k=0}^{\infty} M_k \cdot z^k, \; M_k \text{ an } n \times n \text{ matrix} \right\}$
E_i^j elementary matrix, then

$$\left[E^i_j z^a, E^k_l z^b\right] = \delta_{kj} E^i_l z^{a+b} - \delta_{il} E^k_j z^{a+b}$$

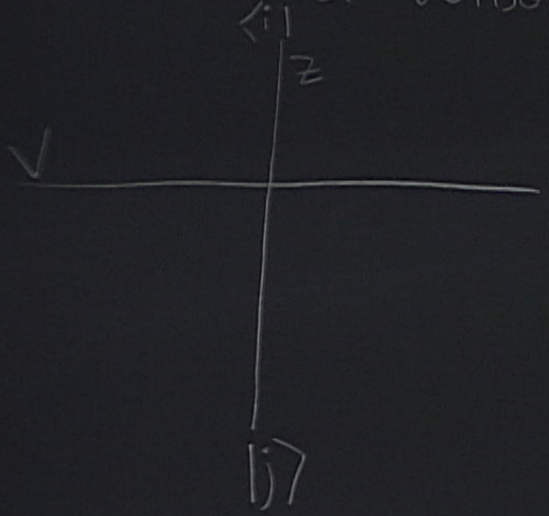
The ρ_i are precisely a representation of $gl_n(\mathbb{C}[z])$.


Classical Wilson lines = reps of $gl_n[[z]]$
Quantum Wilson lines = reps of the Yangian which is a deformation of $U gl_n[[z]]$

If we have some horizontal Wilson line

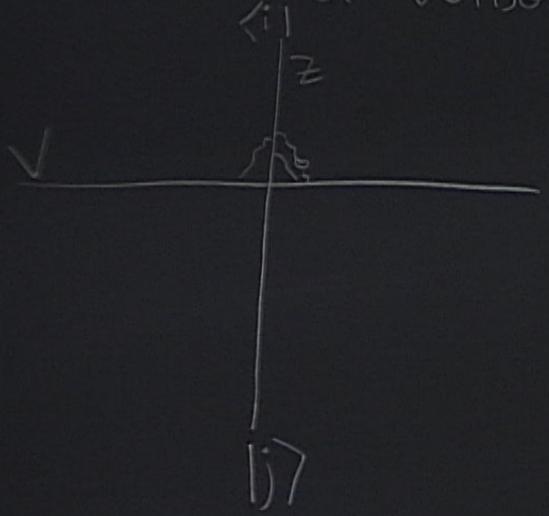
A vertical Wilson line near $z = \infty$

$|i\rangle$ in fundamental rep of gl_r

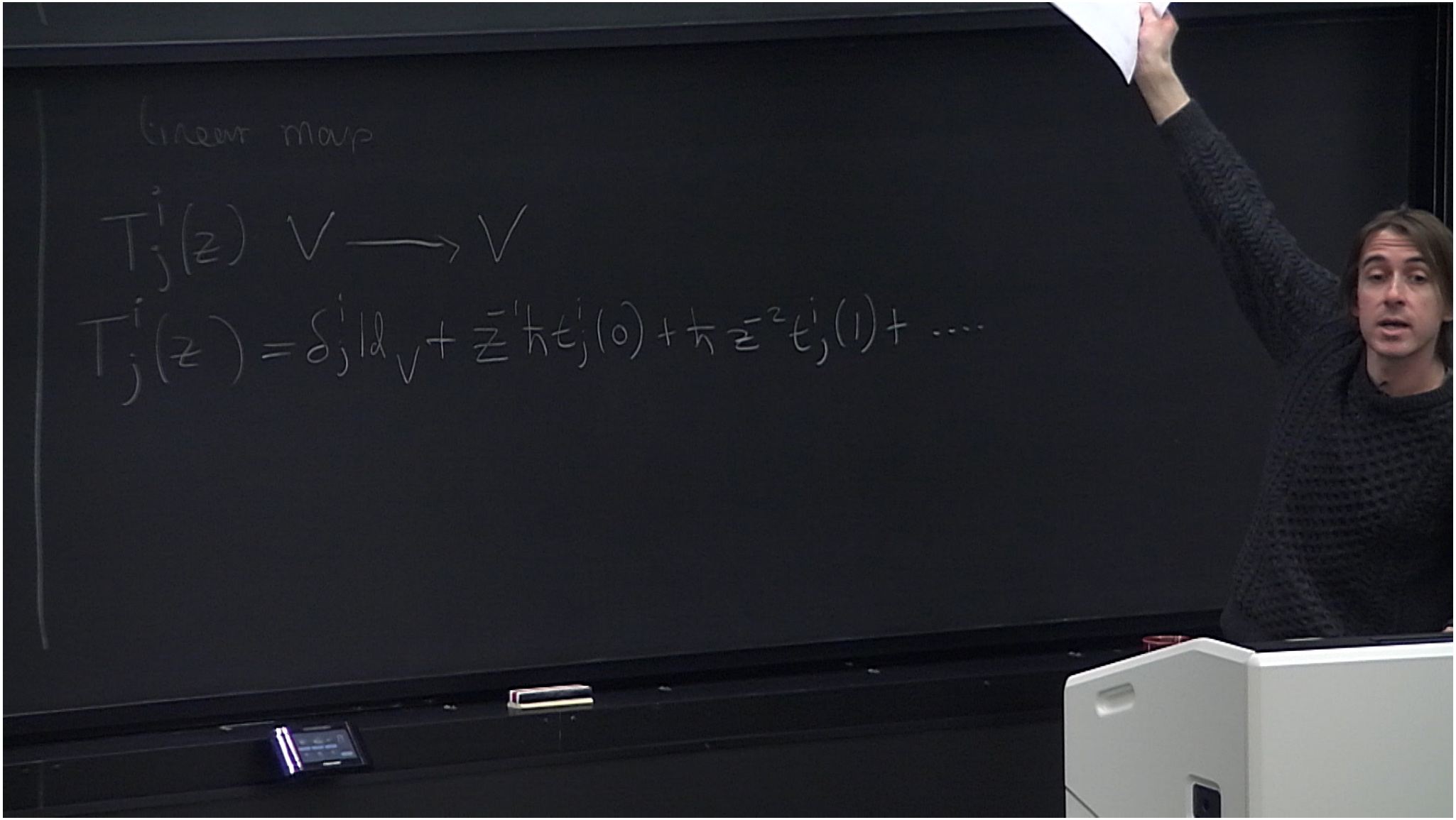


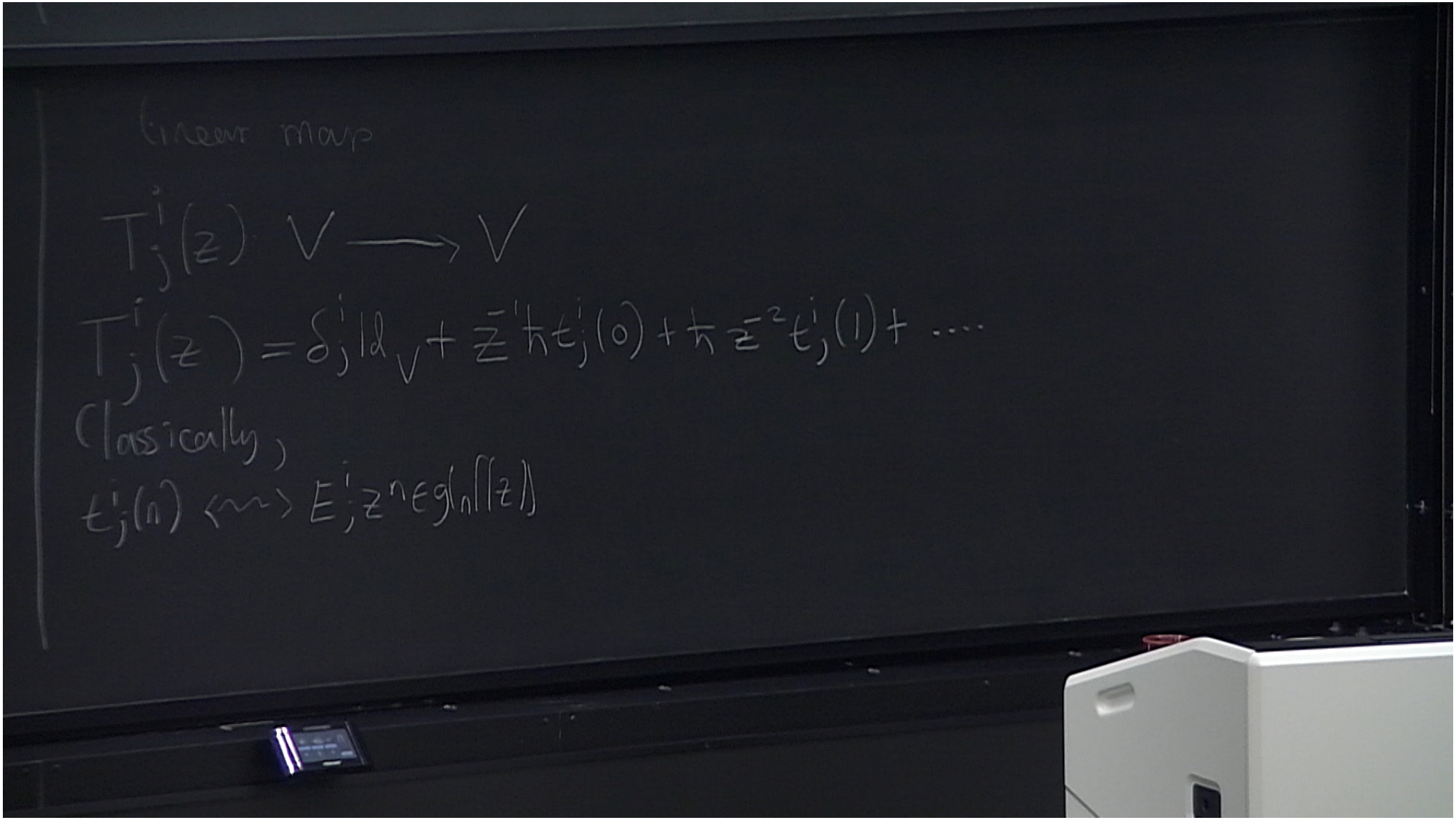
If we have some horizontal Wilson line

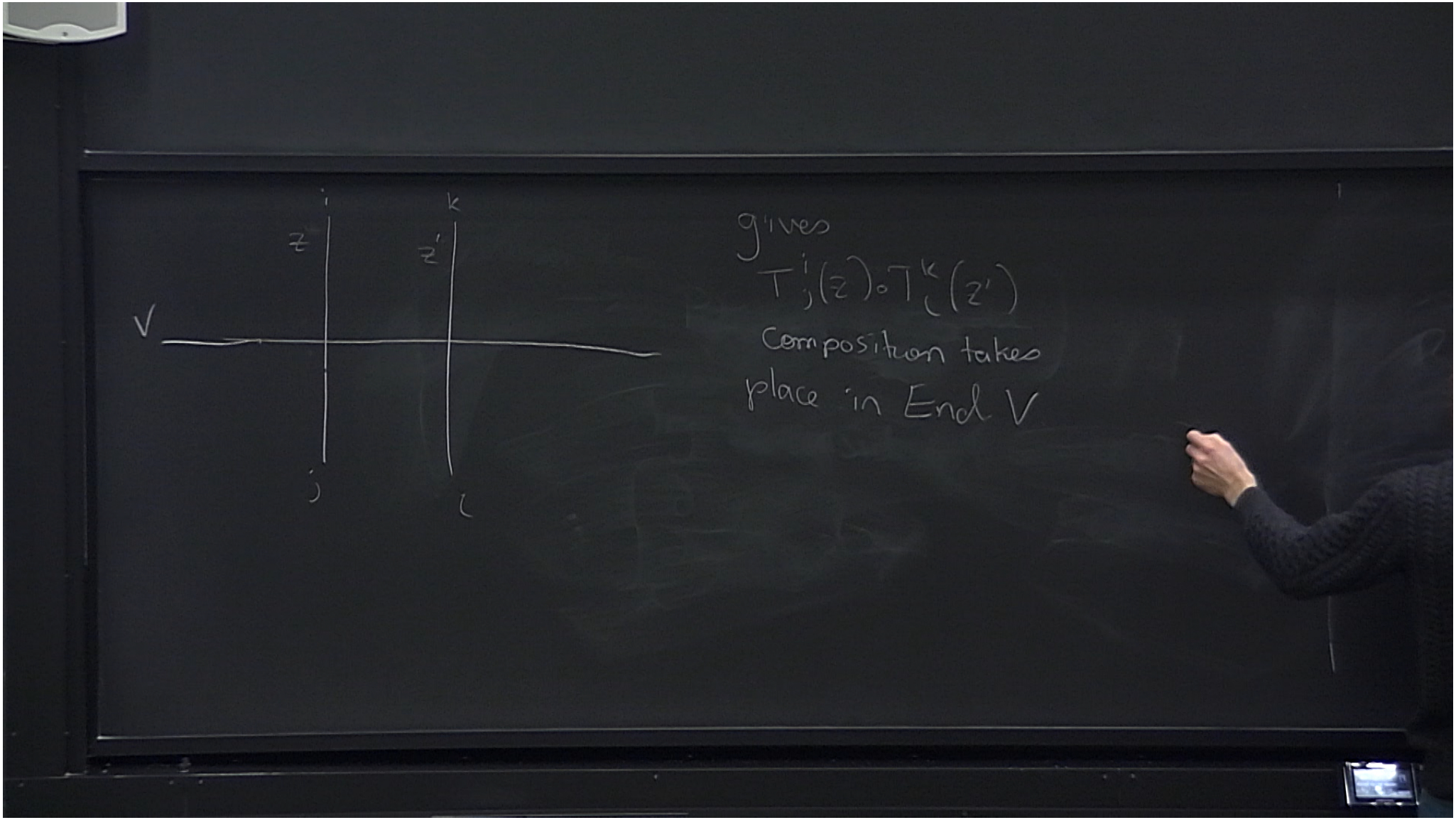
A vertical Wilson line near $z = \infty$ } in fundamental rep of gl_r

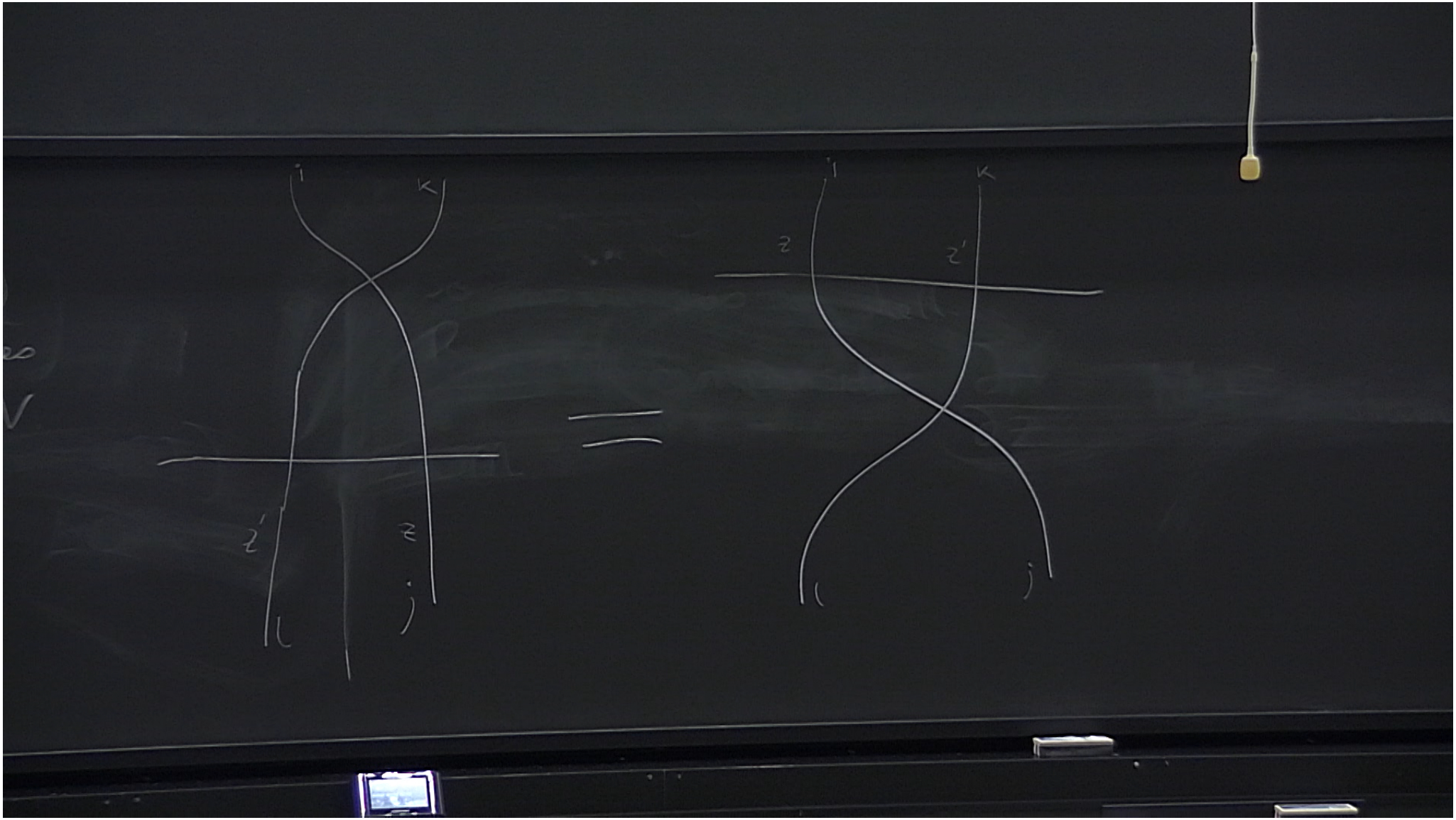


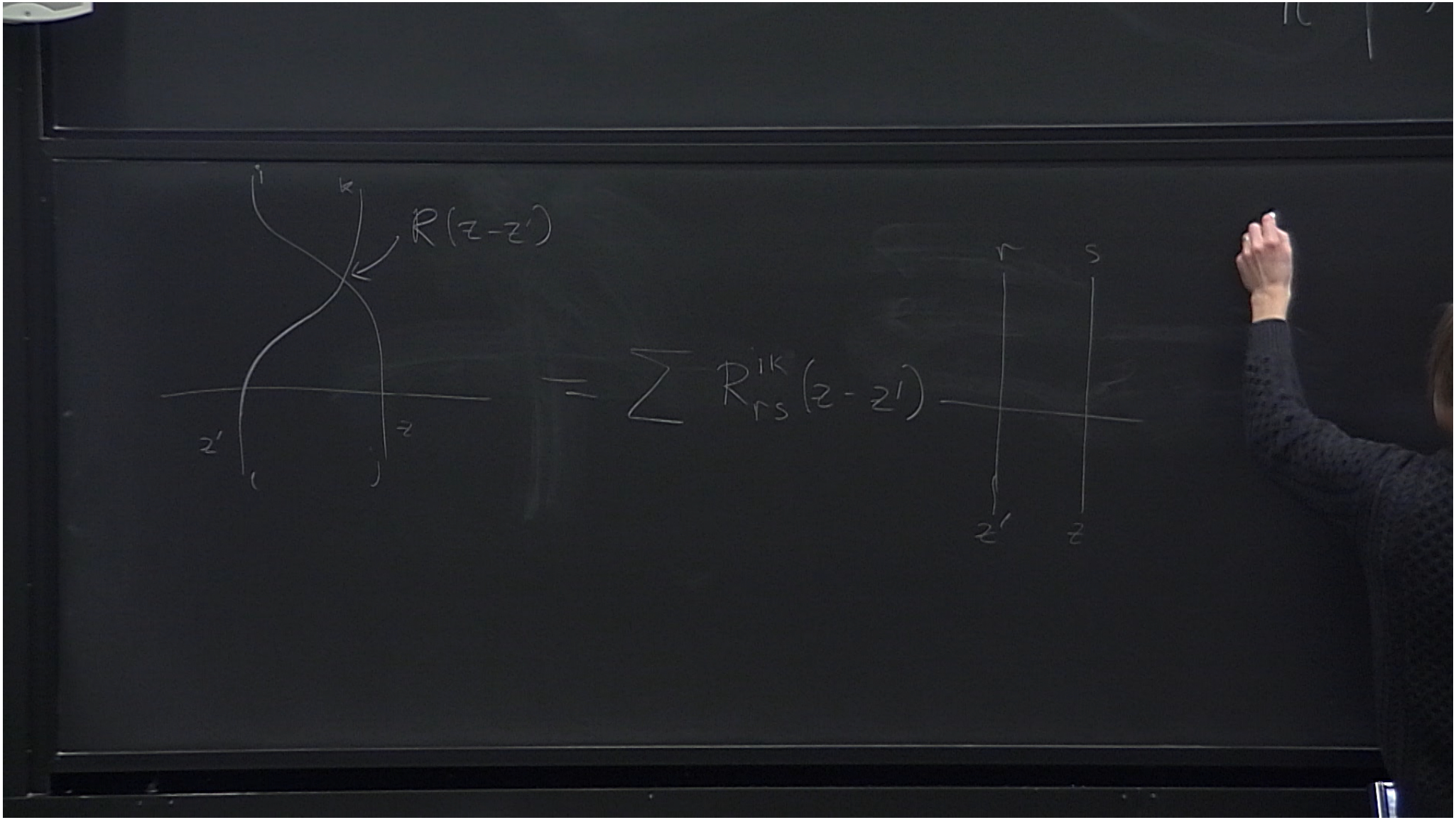
This gives a

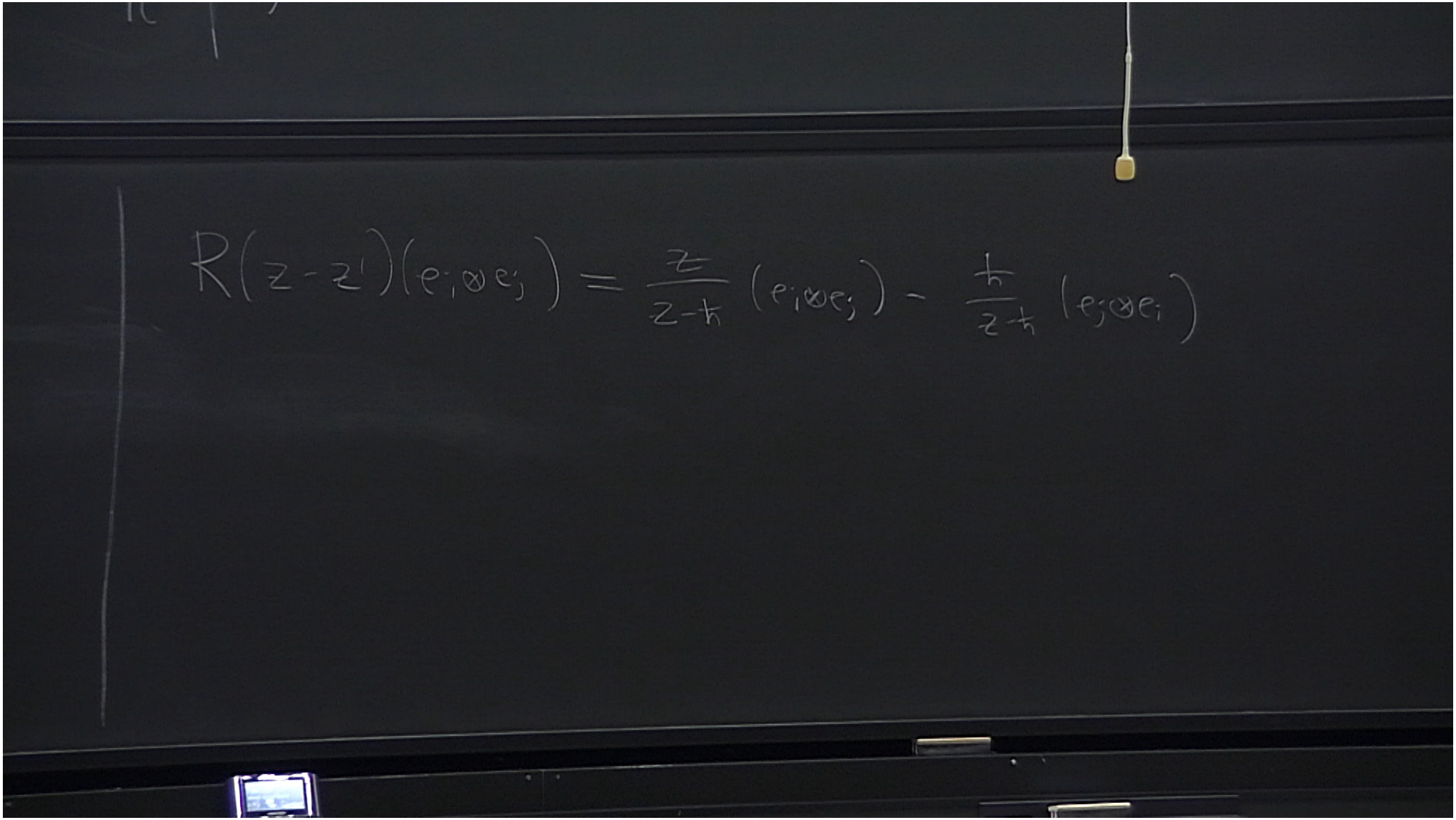

linear map
$T_j^i(z)$ $V \longrightarrow V$
$T_j^i(z) = \delta_j^i Id_V + z^{-1}\hbar t_j^i(0) + \hbar z^{-2} t_j^i(1) + \cdots$


linear map
$T^i_j(z)\ V \longrightarrow V$
$T^i_j(z) = \delta^i_j Id_V + z^{-1}\hbar t^i_j(0) + \hbar z^{-2} t^i_j(1) + \cdots$
Classically,
$t^i_j(n) \leftrightsquigarrow E^i_j z^n \in gl_n(\mathbb{C}[z])$

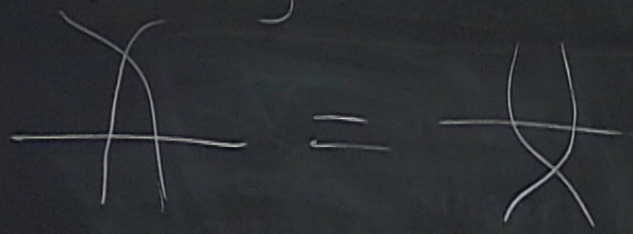

i
k
z
z'
V
j
l
gives
$T^i_j(z) \cdot T^k_l(z')$
Composition takes
place in End V


i
k
z'
z
j
i
k
z
z'
j


i
k
R(z-z')
z'
z
j
= Σ R^{ik}_{rs}(z-z')
r
s
z'
z


$R(z-z')(e_i \otimes e_j) = \frac{z}{z-\hbar}(e_i \otimes e_j) - \frac{\hbar}{z-\hbar}(e_j \otimes e_i)$

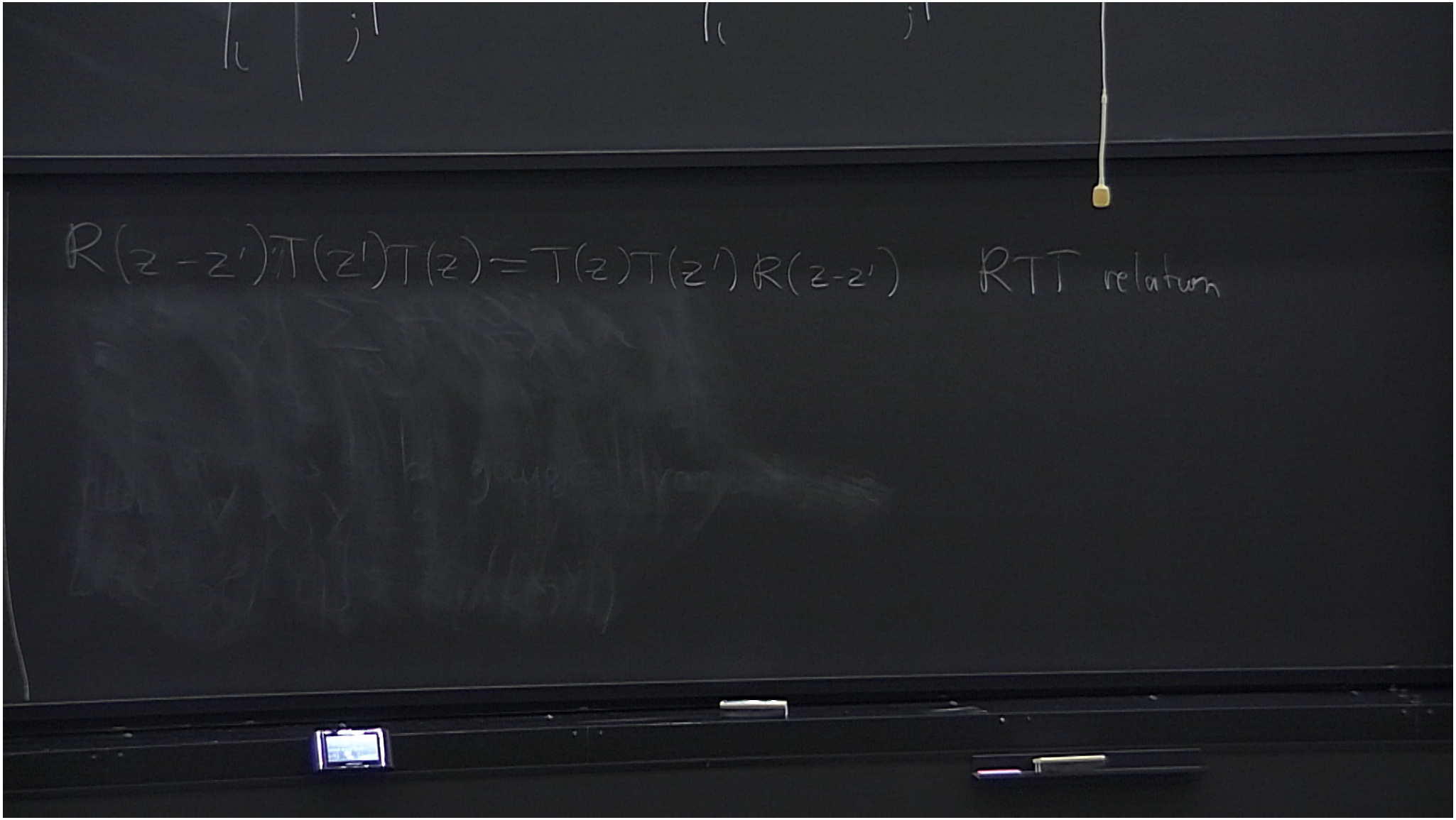
Equating



gives us

$$\sum_{r,s} R^{ik}_{rs}(z-z')T^r_j(z')T^s_\ell(z) = \sum_{r,s} T^i_r(z)T^k_s(z')R^{rs}_{j\ell}(z-z')$$

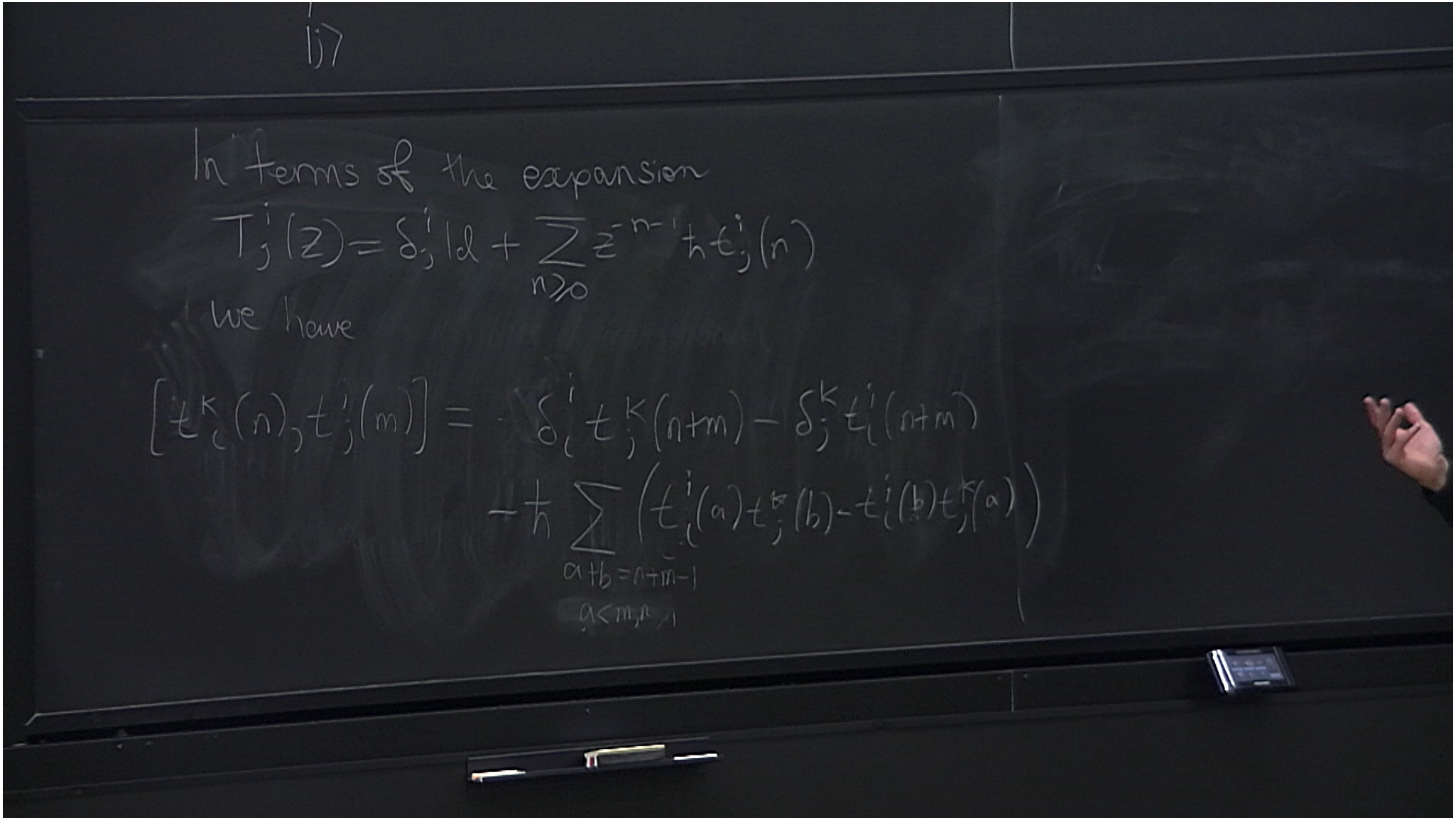
$R(z-z')T(z$


$R(z-z')T(z')T(z) = T(z)T(z')R(z-z')$ RTT relation

$$R(z-z')T(z')T(z) = T(z)T(z')R(z-z') \quad \text{RTT relation}$$

Use expression for $R(z-z')$

$$\left[T_i^k(z'), T_j^l(z)\right] = \frac{\hbar}{z-z'}\left(T_i^l(z')T_j^k(z) - T_i^l(z)T_j^k(z')\right)$$


ij
In terms of the expansion
$T^i_j(z) = \delta^i_j \mathrm{Id} + \sum_{n \geq 0} z^{-n-1} h t^i_j(n)$
we have
$[t^k_i(n), t^l_j(m)] = \delta^l_i t^k_j(n+m) - \delta^k_j t^l_i(n+m)$
$- h \sum_{\substack{a+b=n+m-1 \\ a<m,n}} \left(t^l_i(a) t^k_j(b) - t^l_i(b) t^k_j(a) \right)$

$t_j^i(n) \leftrightarrow E_j^i z^n \in gl_n[[z]]$

This is the presentation of the Yangian algebra.

Example

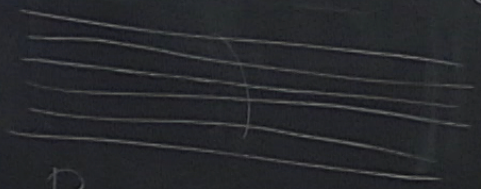
$$[t_i^k(1), t_j^i(1)] = \delta_i^i t_j^k(2) - \delta_j^k t_i^i(2) + \hbar\left(t_i^i(0) t_j^k(1) - t_i^i(1) t_j^k(0)\right)$$

Spin chains

The steeles at end of w

$G = SL_2$

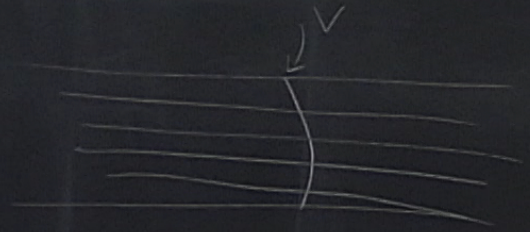
Replace $\mathbb{R}^2$ by $\mathbb{R} \times S^1$



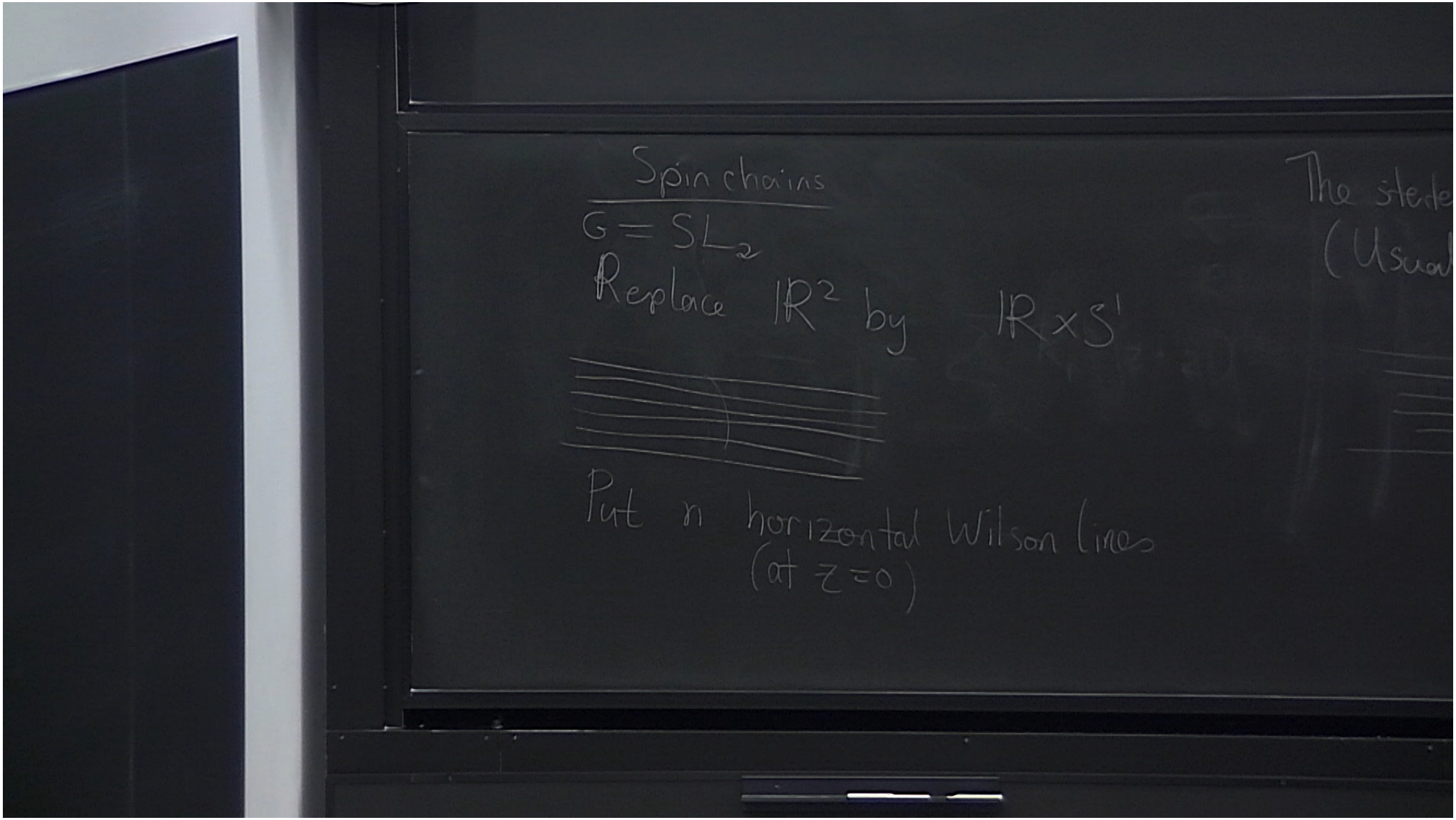
Put n horizontal Wilson lines

The states at end of Wilson lines is $(\mathbb{C}^2)^{\otimes n}$

(Usual space of states of spin chain)

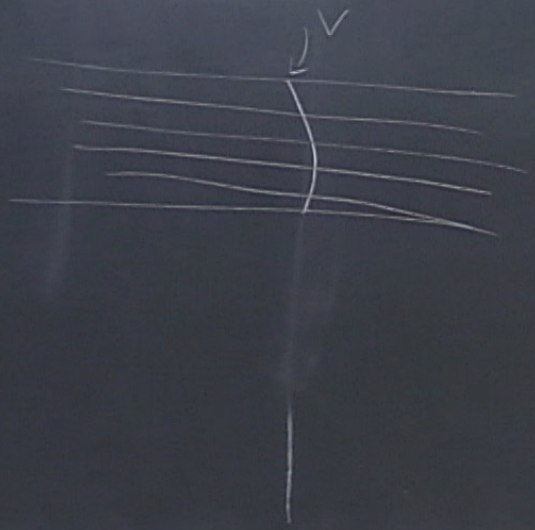


A wilson line wrapping S^1 at $-z$


Spin chains
G = SL2
Replace IR2 by IR x S1
Put n horizontal Wilson lines
(at z = 0)

The states at end of Wilson lines is $(\mathbb{C}^2)^{\otimes n}$

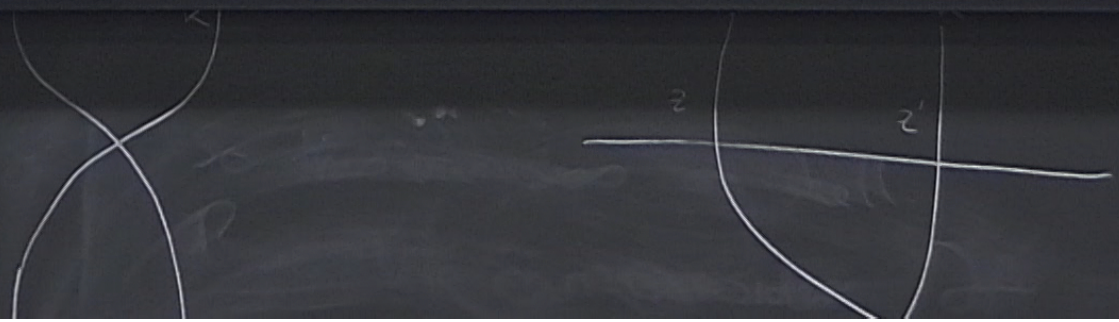
(Usual space of states of spin chain)

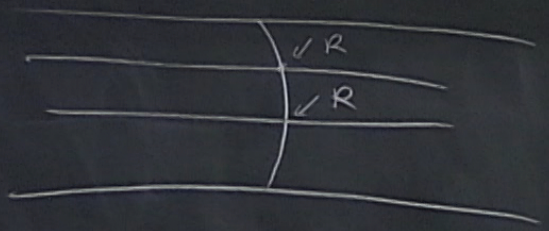


A wilson line wrapping S^1 at z gives an operator

$$T(z): (\mathbb{C}^2)^{\otimes n} \longrightarrow (\mathbb{C}^2)^{\otimes n}$$

$$[T(z), T(z')] = 0$$



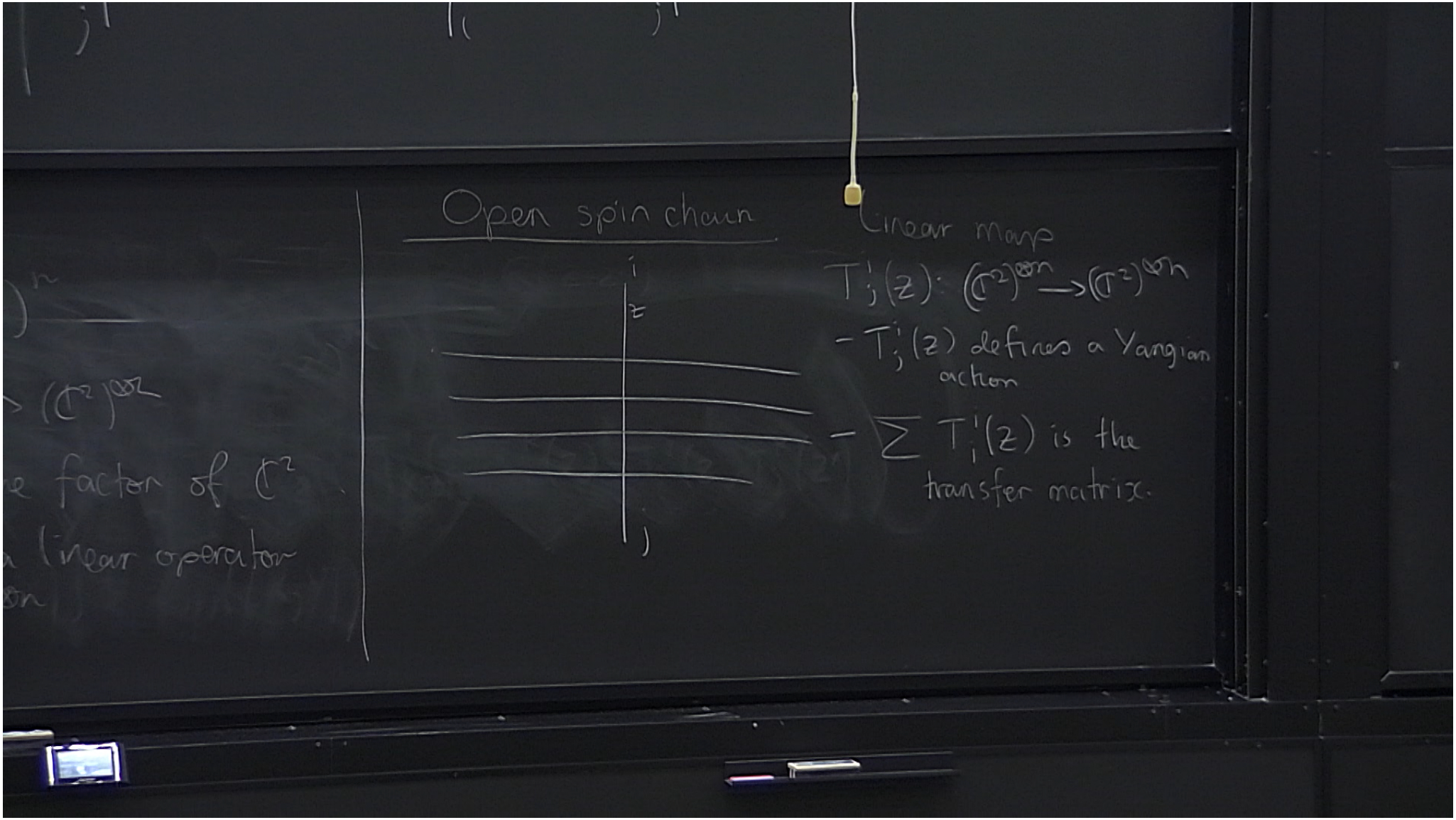


$$T(z) = \mathrm{Tr}\ R(z)^n$$

$R : (\mathbb{C}^2)^{\otimes 2} \longrightarrow (\mathbb{C}^2)^{\otimes 2}$

R^n compose in one factor of $\mathbb{C}^2$

$\mathrm{Tr}\, R(z)^n$ is a linear operator on $(\mathbb{C}^2)^{\otimes n}$


Open spin chain
i
z
j
Linear maps
$T^i_j(z)$: $(\Gamma^2)^{\otimes n} \to (\Gamma^2)^{\otimes n}$
- $T^i_j(z)$ defines a Yangian action
- $\sum T^i_i(z)$ is the transfer matrix.