Title: F-fields

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Abstract: <p>An F-field on a manifold M is a local system of algebraically closed fields of characteristic p. You can study local systems of vector spaces over this local system of fields. On a 3-manifold, they are are rigid, and the rank one local systems are counted by the Alexander polynomial. On a surface, they come in positive-dimensional moduli (perfect of characteristic p), but they are more "stable" than ordinary local systems in the GIT sense. When M is symplectic, maybe an F-field should remind you of a B-field, it can be used to change the Fukaya category in about the same way. On S^1 x R^3, this version of the Fukaya category is related to Deligne-Lusztig theory, and I found something like a cluster structure on the Deligne-Lusztig pairing varieties by studying it.</p>

Standard analogy:

$\mathbb{F}_p$ is like a circle.

Frob is like a generator for $\pi_1(\text{circle})$.

This lecture: What if we take this very literally?

Def: let

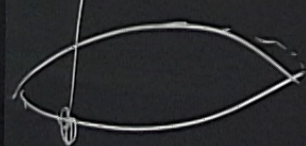
An

Def: let M be a manifold.

An F-field on M is a locally const. sheaf of fields with fiber R.

Best example:

$S^1_{\mathbb{R}}$ carries an F. field, $x \mapsto \mathrm{Frob}(x) = x^p$;



$S^1_{\mathbb{F}_p}$

circumference is $\log(p)$

$S^1_{\mathbb{F}_{p^q}}$

circumf. $q\log(p)$.

$$\Gamma(S^1_{\mathbb{F}_p}; \underline{k}) = \mathbb{F}_p.$$

eld,

$S^1_{\mathbb{F}_{p^q}}$ circumf. $q\log(p)$.

$S^1_{\mathbb{F}_p}$;

circumference is $\log(p)$

Every other good example is pulled back from this one.

<u>Def</u>: Let M be a manifold.

An f-field on M is a map

$$f : M \longrightarrow S^1_{\mathbb{F}_p}.$$

<u>Standard analogy</u>:

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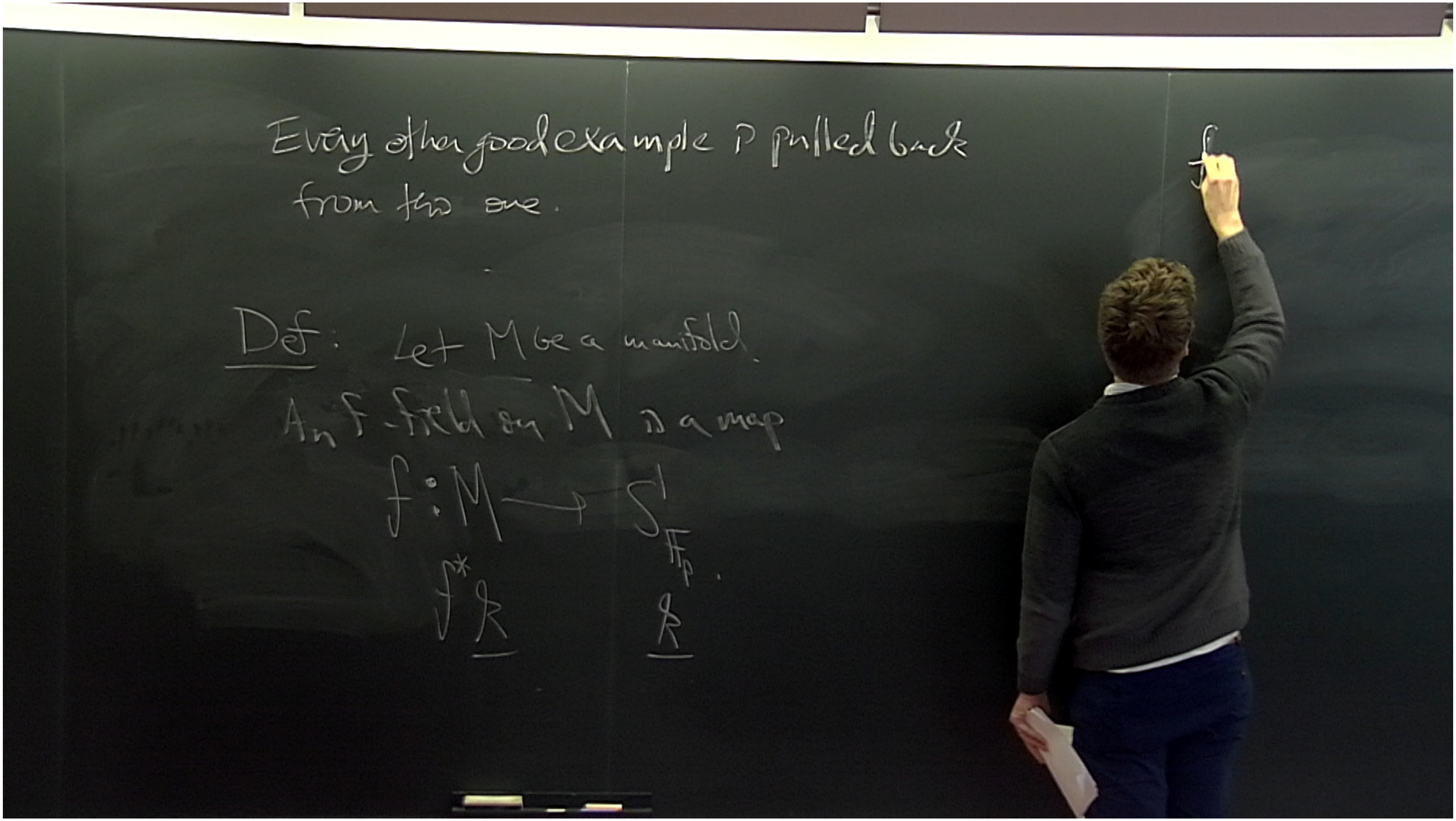
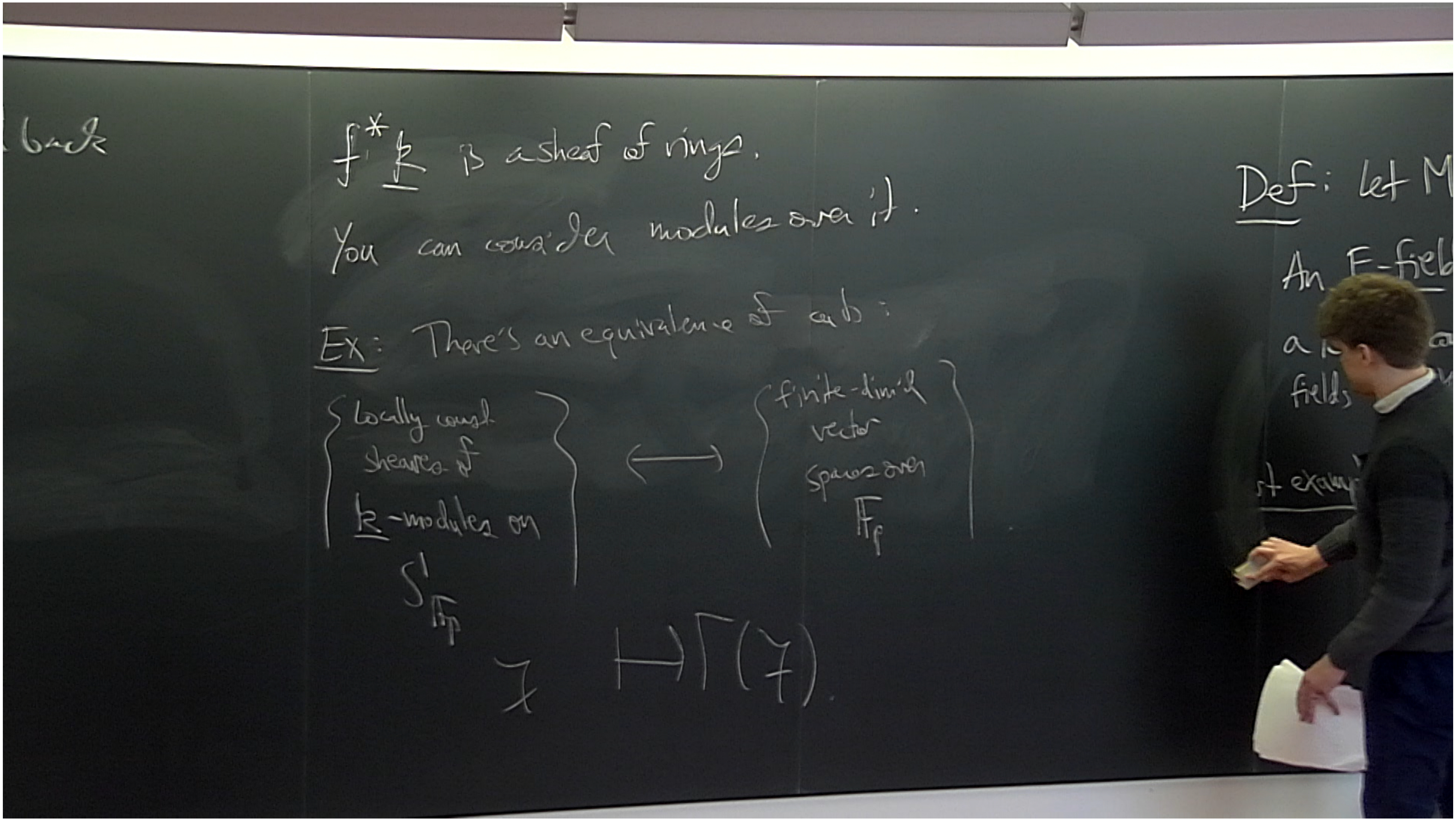
Frob is like a


Every other good example is pulled back
from this one.
Def: Let M be a manifold.
An F-field on M is a map
$f : M \to S_{\mathbb{F}_p}$.
$f^* \underline{R}$
$\underline{R}$


back
$f^*\underline{k}$ is a sheaf of rings.
You can consider modules over it.
Ex: There's an equivalence of cats:
{locally const sheaves of $\underline{k}$-modules on $S^1_{\mathbb{F}_p}$} $\longleftrightarrow$ {finite-dim'l vector spaces over $\mathbb{F}_p$}
$\mathcal{F} \longmapsto \Gamma(\mathcal{F})$.
Def: let M
An F-fiel
fields
st examp

Ex: $K \subset S^3$ a knot.

$f : S^3 - K \longrightarrow S^1_{\mathbb{F}_p}$.

of degree n on H_1.

Then there are finitely many isomorphism classes of $f^* \underline{k}$. of rank r;

When $r = 1$ and $p \gg 0$, the number is $\Delta_K(p^n)$

$\Gamma(S^1_{\mathbb{F}_p}; \underline{k}) = \mathbb{F}_p$.

$\Delta_K(x) = c_0 + c_1 x + c_2 x^2 + \cdots$

$c_0 \in \mathbb{Z}_{\geq 0}$

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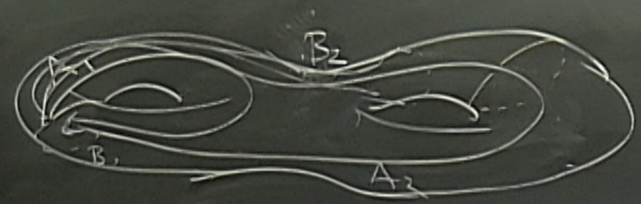
When $r=1$ and $p \gg 0$, the number is $\Delta_K(p^n)$ $\left(/(p^n - 1) \text{ if we do an orbifold count.} \right)$

$\Gamma(S^1_{\mathbb{F}_p}; \underline{k}) = \mathbb{F}_p$.

$\Delta_K(x) = c_0 + c_1 x + c_2 x^2 + \cdots$

$c_0 \in \mathbb{Z}_{>0}$

What about a surface?



$\xrightarrow{f} S^1_{\mathbb{F}_p}$

orange circle $= f^{-1}(\text{point})$.

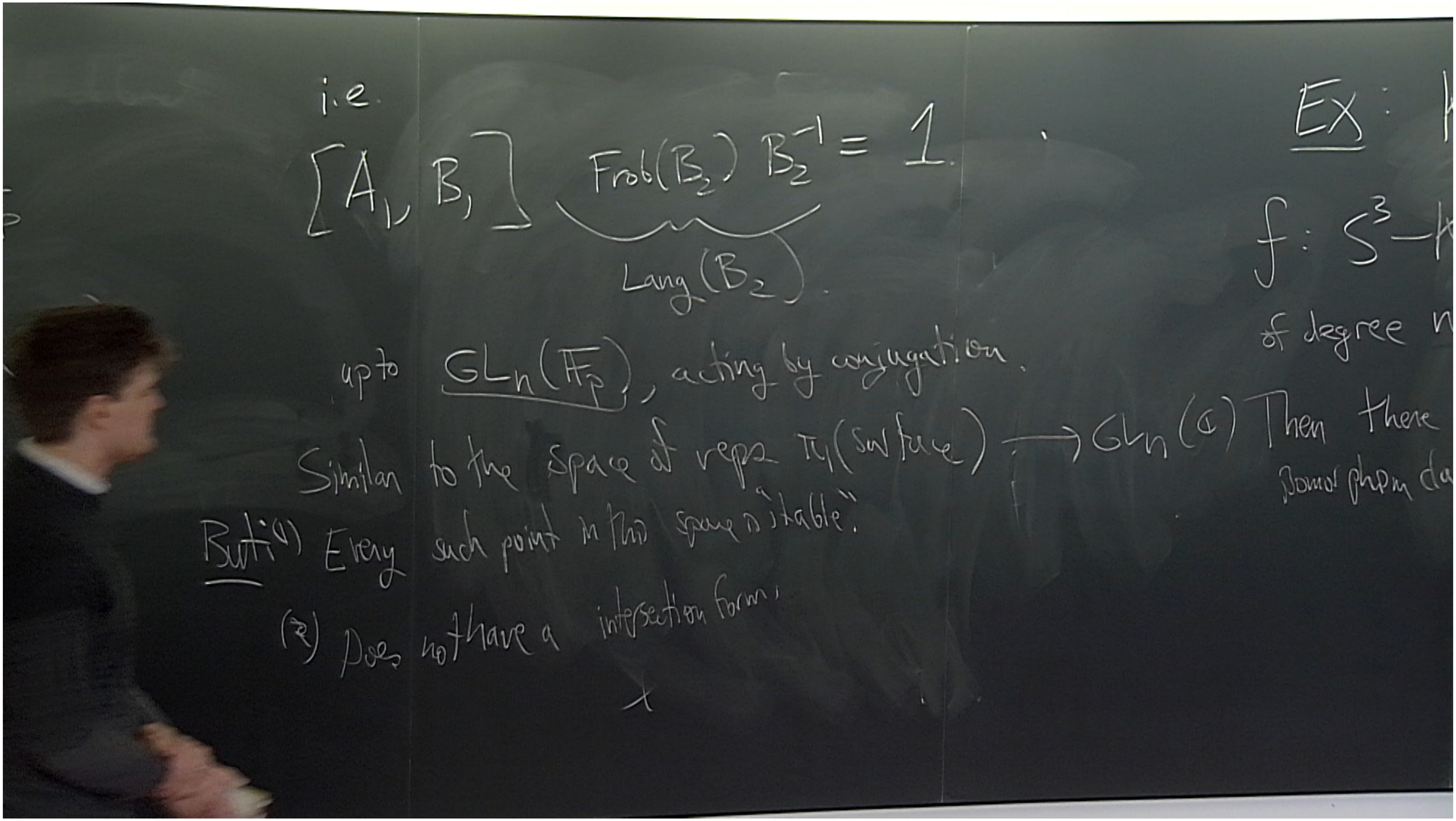
$$[A_1, B_1][A_2, B_2] = 1$$

So an $f^*\underline{k}$-module is given by.

3 $n \times n$ matrices w/ entries in k.

subject to

$$A_1 B_1 A_1^{-1} B_1^{-1} \boxed{\mathrm{Frob}\, B_2\, \mathrm{Frob}^{-1}} B_2^{-1} = 1$$


i.e.
$[A_1, B_1] \underbrace{\mathrm{Frob}(B_2)\, B_2^{-1}}_{\mathrm{Lang}(B_2)} = 1.$
up to $GL_n(\mathbb{F}_p)$, acting by conjugation.
Similar to the space of reps $\pi_1(\text{surface}) \longrightarrow GL_n(\mathbb{C})$
But (1) Every such point in this space is "stable".
(2) Does not have a intersection form.
Ex:
$f: S^3 -$
of degree
Then there

Variants of $\Bbbk$:

Any automorphism at all can be repurposed as a local system on a circle ...

(1) R a $\Bbbk$-algebra,

$$R \longrightarrow R \qquad r \longmapsto r^p$$

is usually not invertible. When it's invertible R is called a perfect ring.

$\underline{R}$ on $S^1_{\mathbb{F}_p}$

$f^*\underline{R}$

3: The Frob of k lifts to char zero.

$$\mathbb{F}_p \rightsquigarrow \mathbb{Q}_p$$

$$\mathbb{F}_{p^2} \rightsquigarrow \mathbb{Q}_{p^2}$$

$$\vdots$$

$$\mathrm{Frob} \circlearrowright k \hookrightarrow \mathbb{Q}_{p^\infty} \circlearrowleft \sigma$$

max unramified ext.

it induces a sheaf of rings on $S^1_{\mathbb{F}_p}$ $\quad \underline{\mathbb{Q}_p^{\infty}}$.

Now local systems of $\underline{\mathbb{Q}_p^{\infty}}$-modules over $S^1_{\mathbb{F}_p}$ are "Dieudonné isocrystals".

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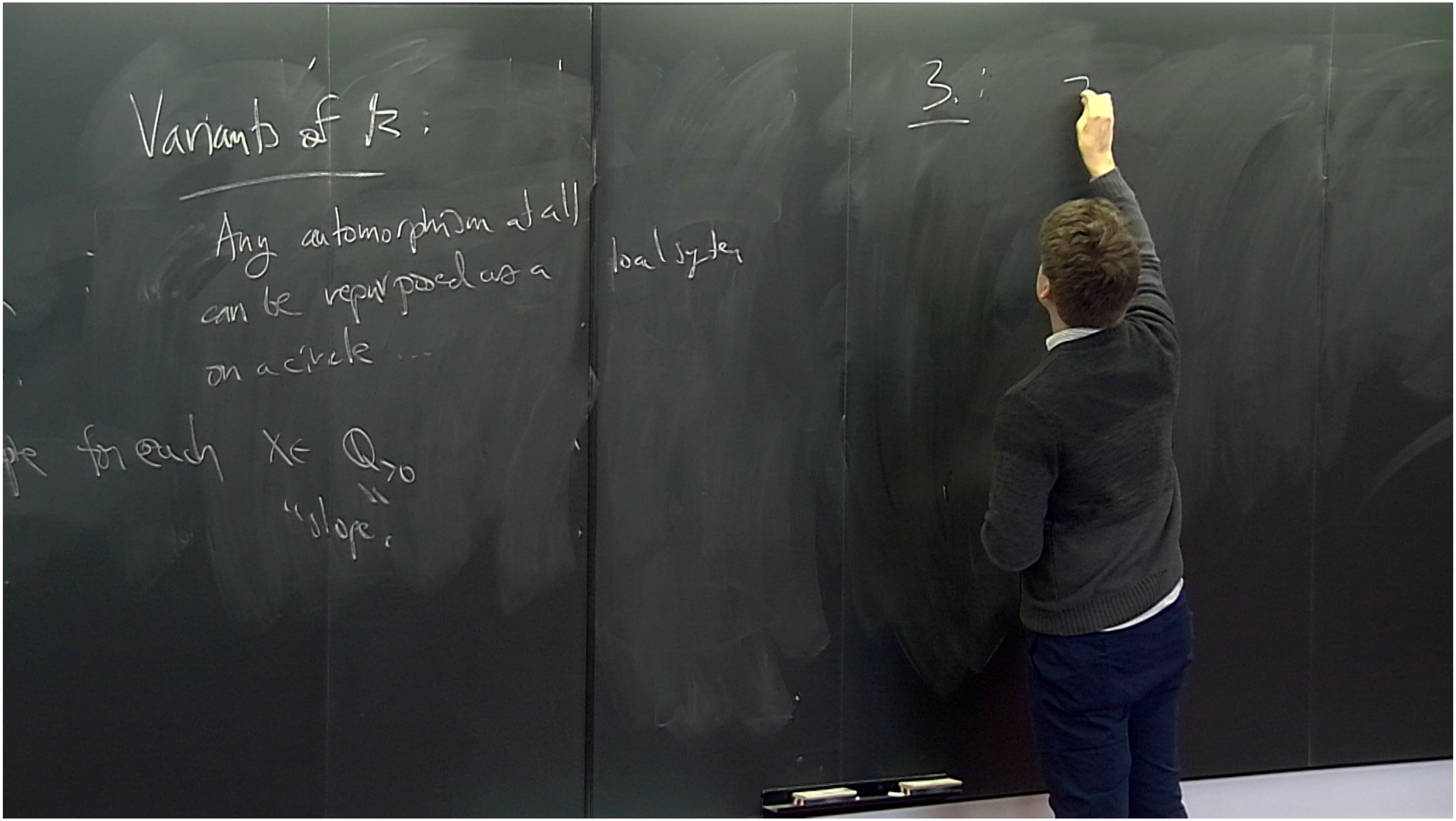
max unramified ext.

Variants of k
Any automorphism of all
can be represented as a local system
on a circle
"slope"
3.


Variants of
Any automorphism at all can be repurposed as a local system on a circle ...
for each $x \in \mathbb{Q}_{>0}$
"slope".
3. :

3. : If G is a reductive alg. group over $\mathbb{F}_p$

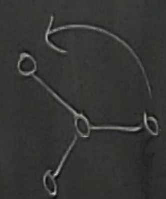
over $\mathbb{F}_p$ it is classified by

its "root datum." i.e. Dynkin diagram + ε.

\+ an automorphism.

corresponds to an interesting group / $\mathbb{F}_p$.

$^3D_4$



$S^1_{\mathbb{F}_p}$ carries sheaves of groups

called $\underline{G}$, one per reductive group over $\mathbb{F}_p$.

There's an equivalence of gpoids

a finite group

$$\left\{ \begin{array}{c} \underline{G}\text{-torsors} \\ \text{on } S^1_{\mathbb{F}_p} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} G(\mathbb{F}_p)\text{-torsors} \\ \text{over a point} \end{array} \right\}$$

Let (M,ω) be a symplectic mfld.

$Fuk(M,\omega)$ is an A_∞-category.

Typical object is a $\underline{L} \subset M$ Lagrangian subm.

+ a $\underline{\nabla}$ a $U(1)$-flat connection on L.

What's ∇ for? It's for getting a $\mathbb{C}$-moduli space of objects in Fuk.

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What if you want to find interesting $\overline{\mathbb{F}}_p$-moduli in SG?

How about putting an $\mathbb{F}_1$-field on $(M, \omega = d\alpha)$

$$f : M \longrightarrow S^1_{\mathbb{F}_p}$$

Study Lagrangians $L \subset M$

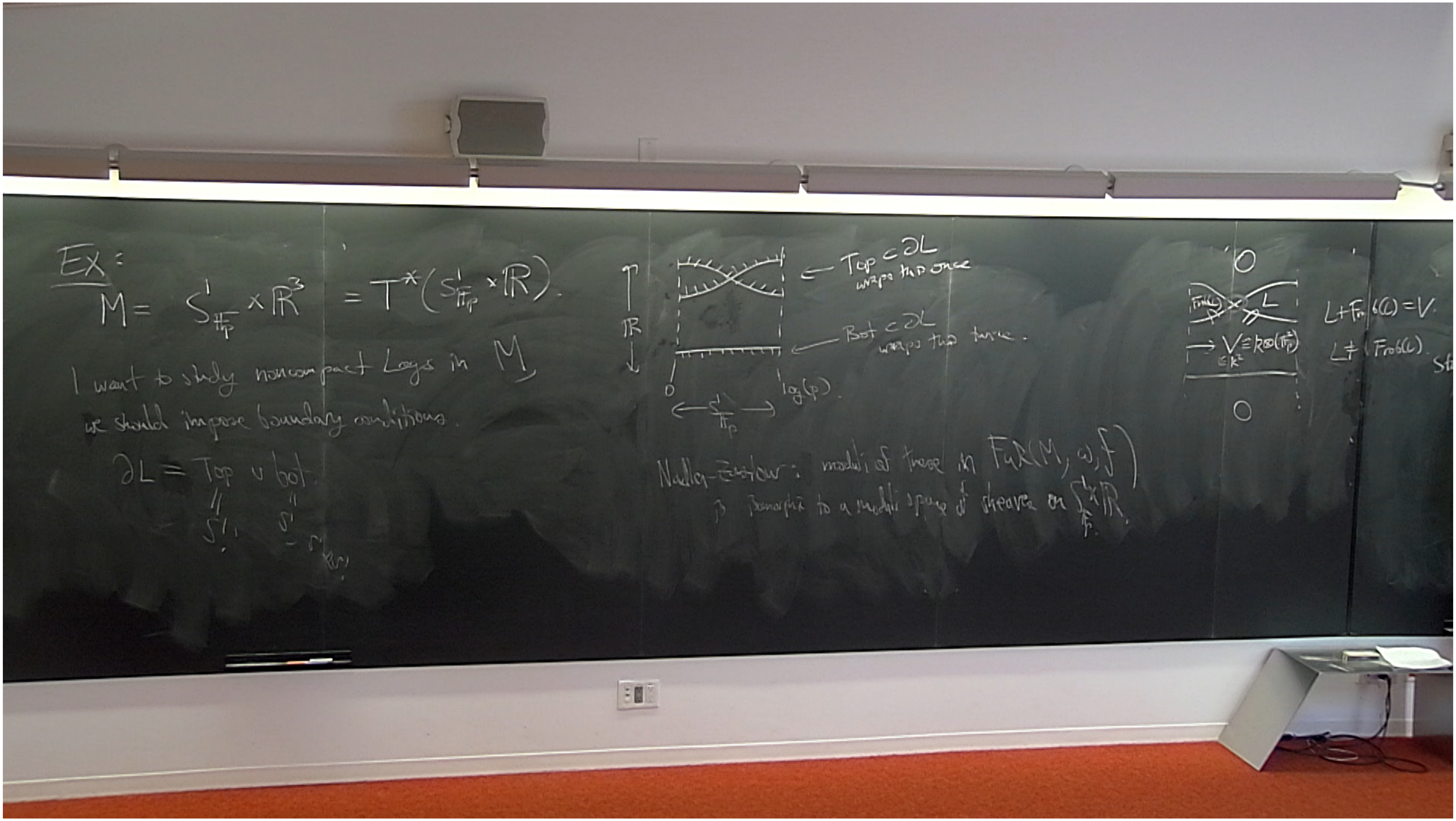
+ local systems of $(f^* \underline{E})|_L$-modules?

Ex:

$$M = S^1_{\mathbb{F}_p} \times \mathbb{R}^3 = T^*(S^1_{\mathbb{F}_p} \times \mathbb{R}).$$

I want to study noncompact Lags in M,

we should impose boundary conditions.


Ex:
M = S^1_{F_p} × R^3 = T^*(S^1_{F_p} × R).
I want to study noncompact Lags in M
we should impose boundary conditions.
∂L = Top ∪ bot
Top ⊂ ∂L wraps the once
Bot ⊂ ∂L wraps the torus.
log(p)
Nadler-Zaslow: moduli of these in Fuk(M, ω, f)
is isomorphic to a moduli space of sheaves on S^1_{F_p} × R.
L + Frob(L) = V.
L ≠ Frob(L).

Ex:

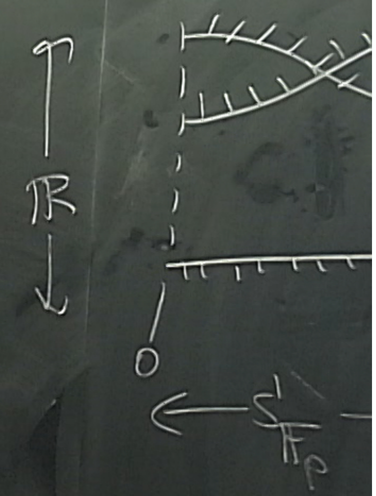
$$M = S^1_{\mathbb{F}_p} \times \mathbb{R}^3 = T^*(S^1_{\mathbb{F}_p} \times \mathbb{R}).$$

I want to study noncompact Lags in M.

we should impose boundary conditions.

$$\partial L = \mathrm{Top} \cup \mathrm{bot}.$$

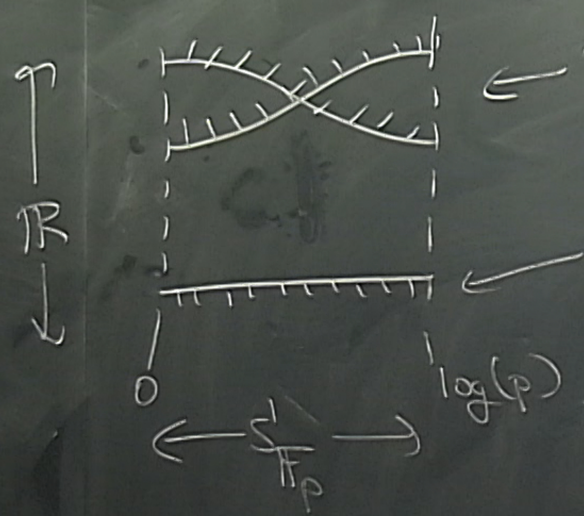
$$\mathrm{Top} = S^1, \quad \mathrm{bot} = S^1 - S^1(\varepsilon)$$



Nadler-Z

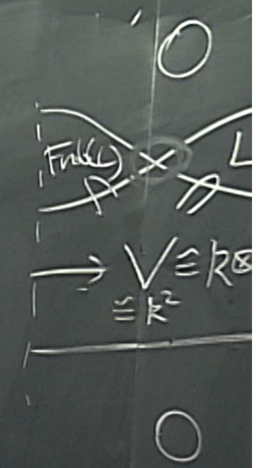
$\times \mathbb{R})$.

M

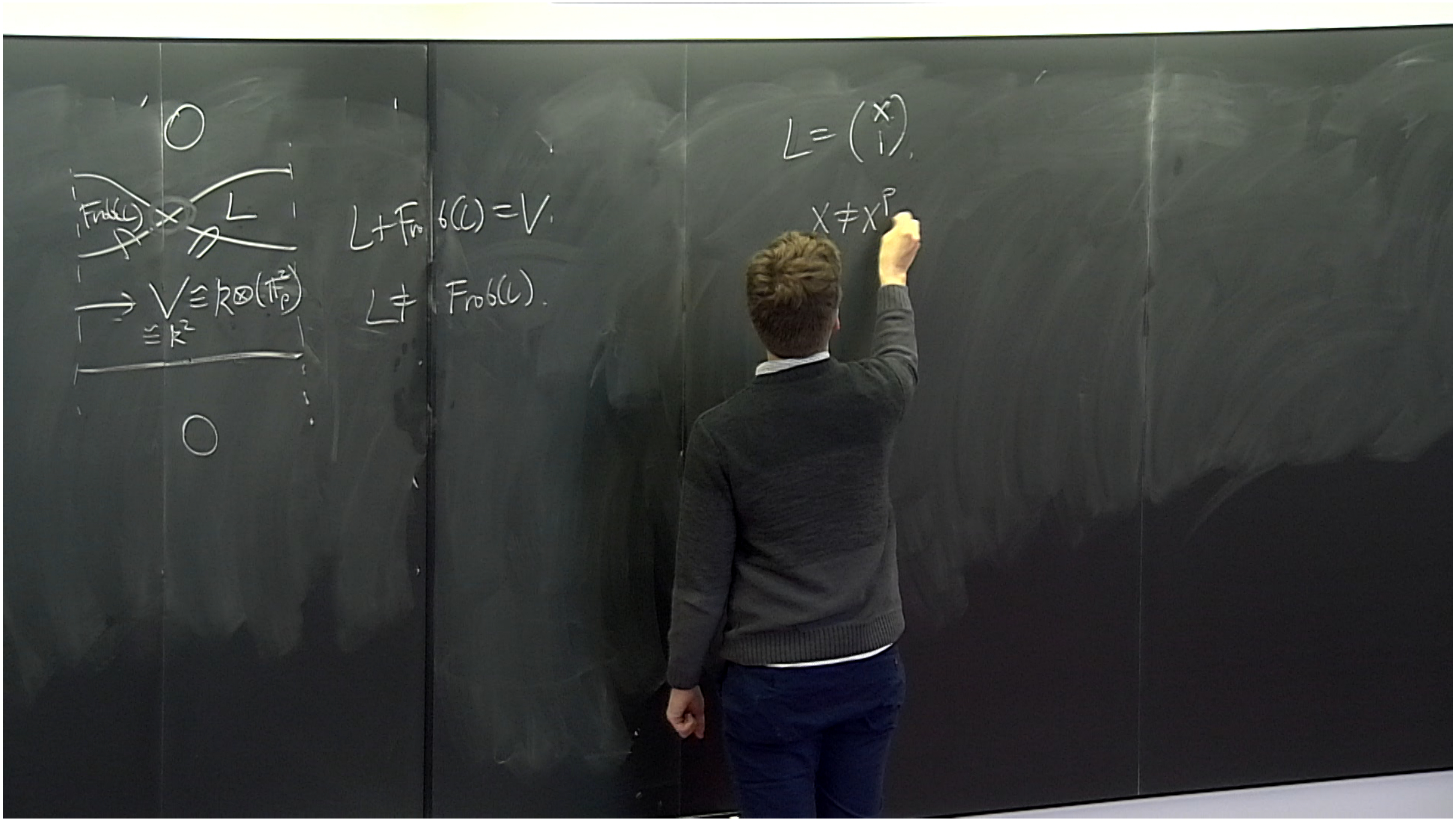


Top $\subset \partial L$
wraps this once.

Bot $\subset \partial L$
wraps this twice.



Nadler–Zaslow: moduli of these in $\mathrm{Fuk}(M, \omega, f)$
is isomorphic to a moduli space of sheaves on $S^1_{\mathbb{F}_p} \times \mathbb{R}$.


Frob(L)
L
V ≅ k ⊗ (F_p^2)
≅ k^2
L + Frob(L) = V
L ≠ Frob(L)
L = (x 1)
x ≠ x^p

$$L = \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

$$x \neq x^p.$$

Drinfeld curve

$$\frac{\mathbb{P}^1(\mathbb{R}) - \mathbb{P}^1(\mathbb{F}_p)}{GL_2(\mathbb{F}_p)} \cong \frac{\{(x,y) \in \mathbb{R}^2 \mid xy^p - x^p y = 1\}}{GL_2(\mathbb{F}_p) \times GL_1(\mathbb{F}_{p^2})}$$

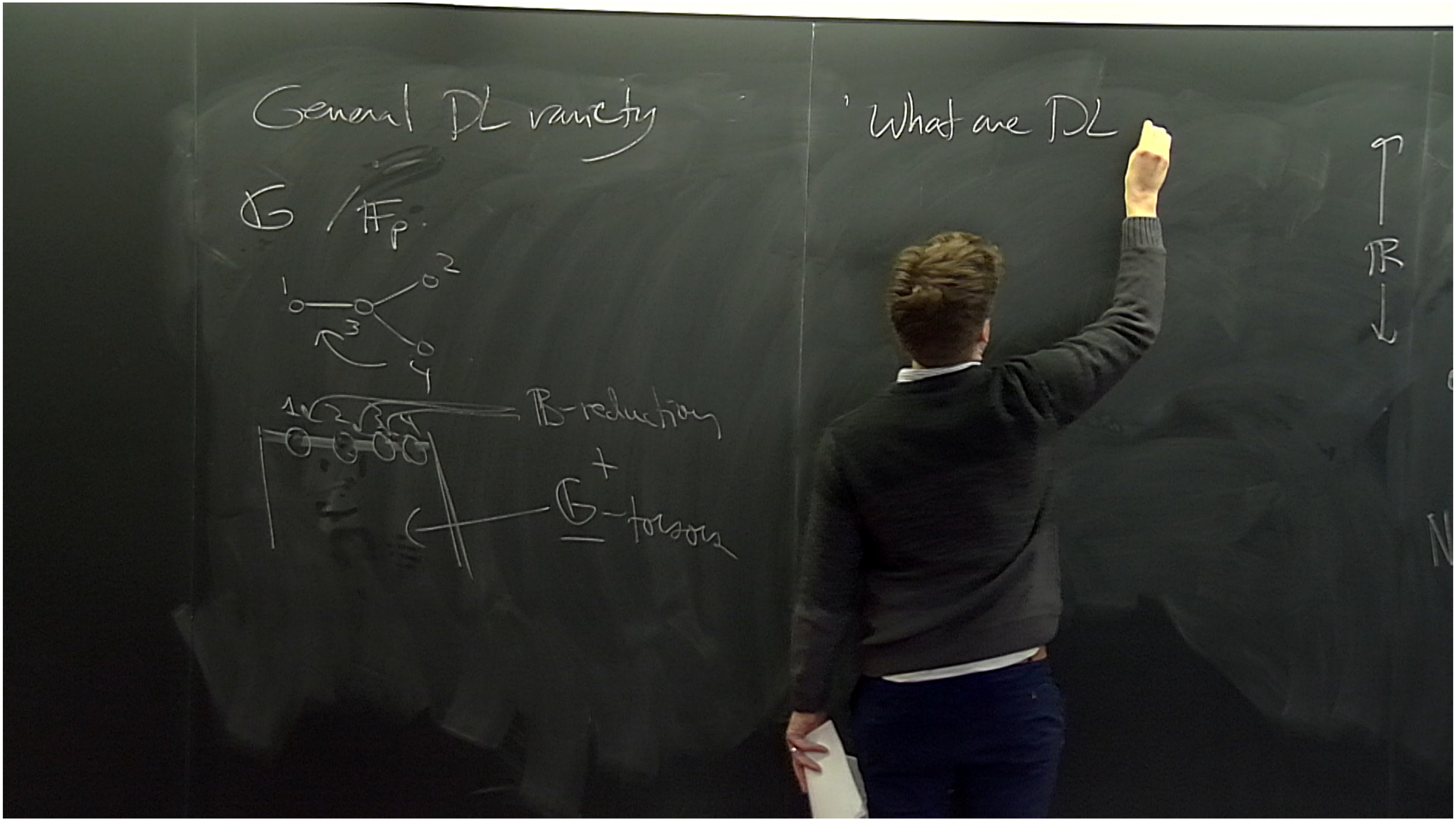
$L = \begin{pmatrix} x \\ 1 \end{pmatrix}$.

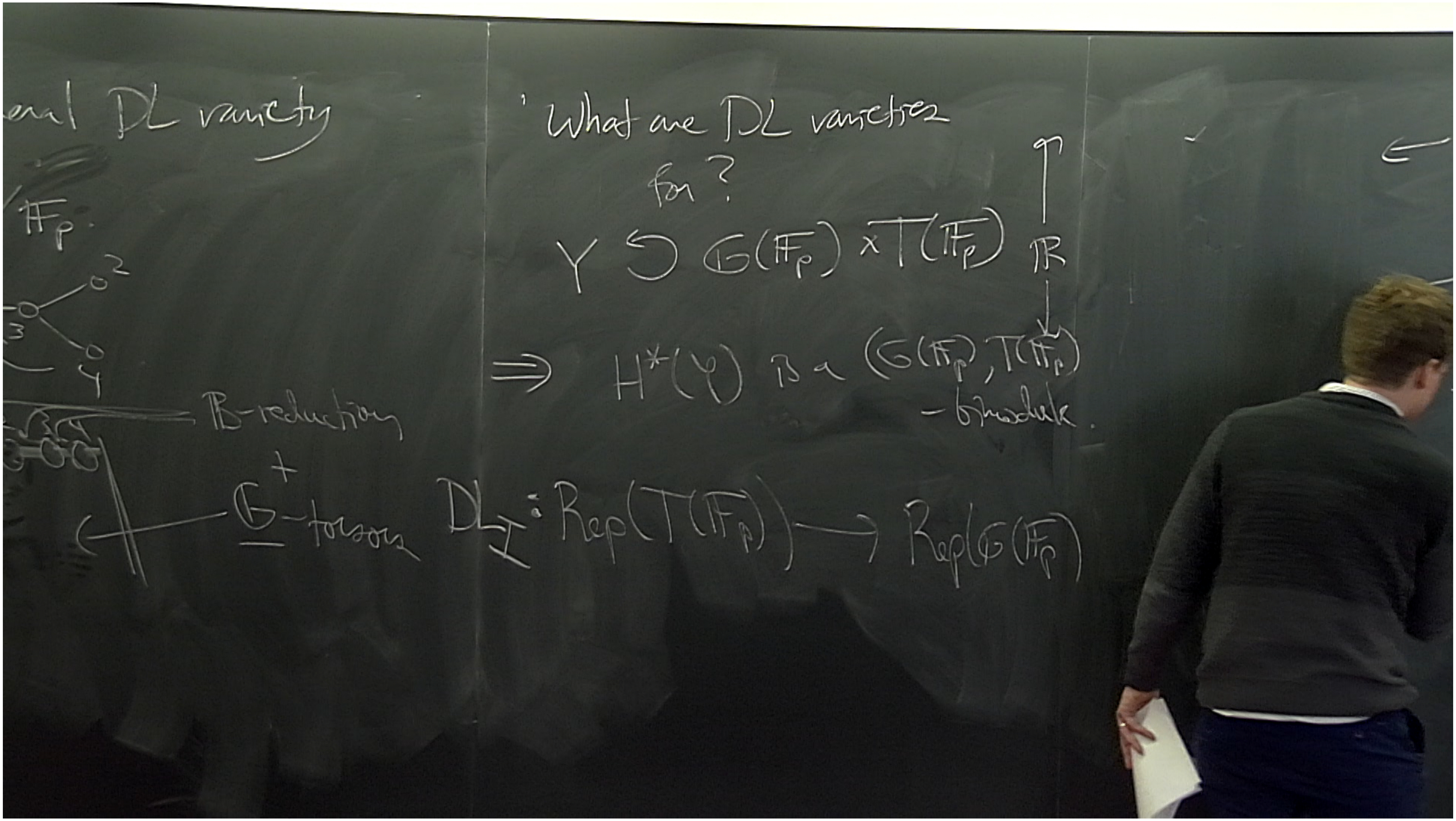
$X \neq X^p$.

Drinfeld curve

$$\frac{\mathbb{P}^1(R) - \mathbb{P}^1(\mathbb{F}_p)}{GL_2(\mathbb{F}_p)} \cong \frac{\{(x,y) \in R^2 \mid xy^p - x^p y = 1\}}{GL_2(\mathbb{F}_p) \times GL_1(\mathbb{F}_{p^2})}$$

eg: $p=2$ The numerator is the elliptic w/ $j=0$.


General DL variety
G / Fp
B-reduction
+
G-torsor
What are DL
R


al DL variety
What are DL varieties for?
Y ⟲ G(F_p) × T(F_p)
R
⟹ H*(Y) is a (G(F_p), T(F_p))-bimodule.
B-reduction
+
G-torsors
DL_I : Rep(T(F_p)) → Rep(G(F_p))

Essential property of DL_I

$$\langle DL_I(\varphi), DL_J(\psi) \rangle = 0$$ unless $I, J \mapsto$ same twisted conj class in Weyl.

How to prove this?

Study $\dfrac{Y_I \times Y_J}{G(\mathbb{F}_p)} = \coprod_{\text{strata}} \mathbb{G}_m^n.$

$\mathbb{R}$

$G(\mathbb{F}_p)$

$\mathrm{Frob}(L)$ L

$\rightarrow V \cong k \otimes (\mathbb{F}_p^2)$

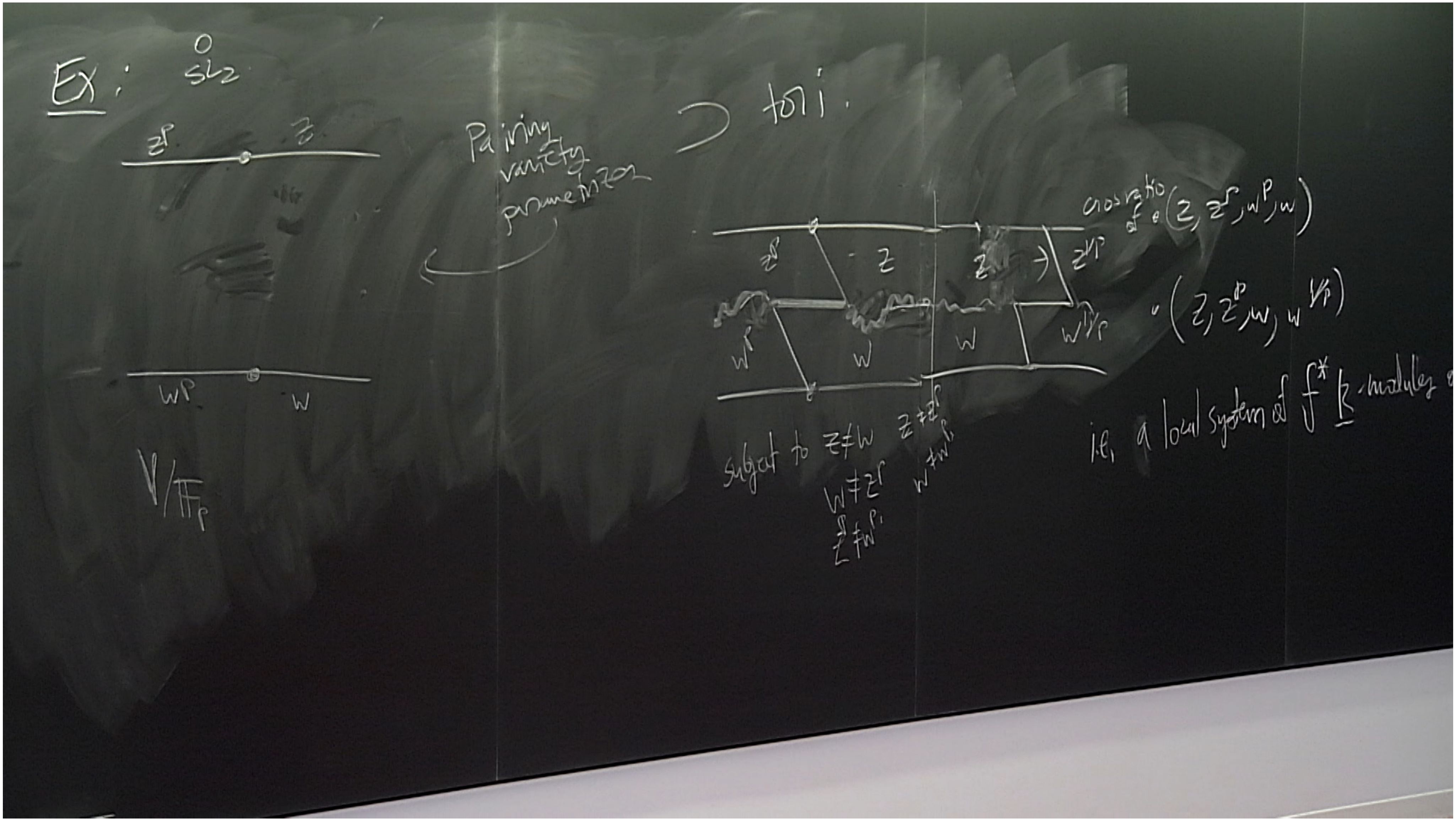
$\cong k^2$

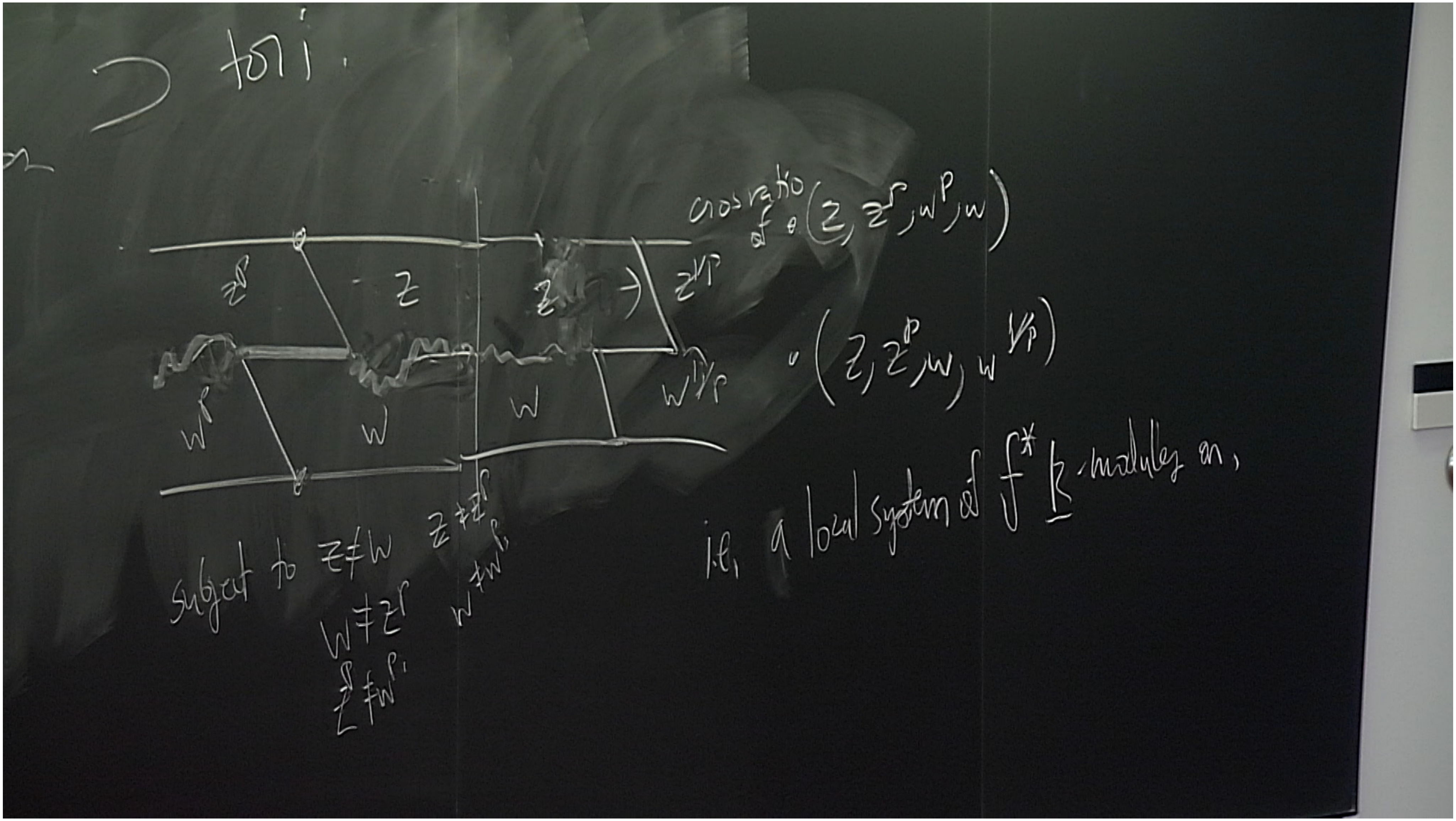
0

0

$L + \mathrm{Fr}$

$L \neq$


Ex: sl2
Pairing variety parametrizes
⊃ tori.
cross ratio of (z, z^p, w^p, w)
(z, z^p, w, w^1/p)
i.e, a local system of f* k-modules
subject to z≠w z≠z^p w≠z^p w≠w^p z^p≠w^p
V/F_p


⊃ tori.
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i.e., a local system of f* k-modules on,
subject to z≠w z≠z^p
w≠z^p w≠w^p
z^p≠w^p