Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 20

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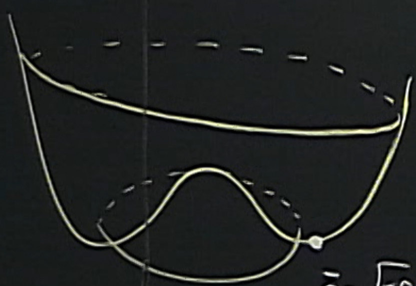
Abstract:

$$S[\vec{\varphi}] = \int d^dx \left[\frac{1}{2} (\vec{\nabla}\varphi^a)(\vec{\nabla}\varphi^a) + \frac{r}{2}\vec{\varphi}\cdot\vec{\varphi} + \frac{u}{4}(\vec{\varphi}\cdot\vec{\varphi})^2 \right]$$

$$\vec{\varphi}(\vec{x}) = \rho(\vec{x})\vec{n}(\vec{x}) \qquad \vec{n}\cdot\vec{n} = 1$$

$$S[\rho,\vec{n}] = \int d^dx \left[\frac{1}{2}\rho^2 \vec{\nabla}n^a\cdot\vec{\nabla}n^a + \frac{1}{2}(\vec{\nabla}\rho)^2 + \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4 \right]$$



$\bar{\rho} = \sqrt{\frac{-r}{u}}$

We'll now assume that we are far below the MFT T_c → magnitude ρ is fixed.

→ Keep only fluctuations of $\vec{n}$

→ set up a new ϵ-expansion ($d = 2+\epsilon$)

$(d = 2+\epsilon)$

$$S[\vec{n}] = \int d^dx \ \frac{1}{2}\rho^2 \vec{\nabla} n^a \cdot \vec{\nabla} n^a$$

measure T in units of ρ: $\rho^2 \equiv \frac{1}{T} = K$

$$= \frac{1}{2T}\int d^dx \left(\vec{\nabla} n(\vec{x})\right)^2$$

and $\vec{n}\cdot\vec{n} = 1$
(makes things highly non-linear)

$$Z = \int \mathcal{D}\vec{n}\ \delta(n^2 - 1)\, e^{-S[\vec{n}]}$$

NLσM

Allow fluctuations from $(1, 0, 0, \cdots, 0)$

then $n^2 = 1$

$$\vec{n} = (\sigma, \pi^1, \pi^2, \cdots, \pi^{n-1}) = (\sigma, \vec{\pi})$$

$\sigma^2 + \vec{\pi}^2 = 1$
or
$\sigma = \sqrt{1 - \vec{\pi}^2}$

$$S[\sigma, \vec{\pi}] = \frac{1}{2T} \int d^d x \left[(\vec{\nabla}\sigma)^2 + \vec{\nabla}\pi^a \cdot \vec{\nabla}\pi^a \right]$$

This Assumes $\sigma(\vec{x})$ has the same sign everywhere.

– this neglects important physics (topological defects, vortices, etc...)

write $S[\vec{\pi}]$ by using $(\vec{\nabla}\sigma)^2 = \vec{\nabla}\sqrt{1-\vec{\pi}^2} \cdot \vec{\nabla}\sqrt{1-\vec{\pi}^2}$

Also use the constraint to calculate the Jacobian

$$\int D\vec{n}\, \delta(n^2-1) = \int D\vec{\pi}\, D\sigma\, \delta(\sigma^2+\vec{\pi}^2-1)$$

use: $\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x+a)+\delta(x-a)\right]$

$$\Rightarrow \int D\vec{\pi} \int D\sigma \left[\frac{1}{2\sqrt{1-\pi^2}}\Big(\delta(\sigma+a)+\delta(\sigma-a)\Big)\right]$$

$$= \int D\vec{\pi}\, \frac{1}{2\sqrt{1-\pi^2}}(1+1) = \int \prod_i \frac{d\vec{\pi}_i}{\sqrt{1-\vec{\pi}_i^2}} \qquad \vec{\pi}(\vec{x})$$

Then $$Z = \int \prod_i \frac{d\vec{\pi}_i}{\sqrt{1-\pi_i^2}}\, e^{-S[\vec{\pi}]} \qquad \swarrow \; \frac{N}{V} = \frac{1}{a^d}$$

$$= \int \prod_i d\pi_i\, e^{-S[\vec{\pi}] - \frac{1}{2}\sum_i \ln(1-\vec{\pi}_i^2)}$$

$$= \int D\vec{\pi}\, e^{-S[\pi] - \frac{1}{2a^d}\int d^dx\, \ln(1-\vec{\pi}^2)}$$

Ie. write a <u>new</u> LGW functional

$$S[\vec{\pi}] = \frac{1}{2T}\int d^dx \left((\vec{\nabla}\vec{\pi})^2 + \left(\nabla\sqrt{1-\vec{\pi}^2}\right)^2 + \frac{T}{a^d}\ln(1-\vec{\pi}^2)\right)$$

Expand in terms of <u>powers</u> of $\vec{\pi}$, write $S[\vec{\pi}] = \underline{\underline{S_0}} + S_1 + S_2 + \dots$

<u>Gaussian:</u> non-interacting <u>Goldstone modes</u>

$$S_0 = \frac{1}{2T}\int d^dx \, (\vec{\nabla}\vec{\pi})^2$$

For S_1, expand in terms of $\vec{\pi}$

$\sqrt{1-\pi^2} \sim 1 - \frac{\pi^2}{2}$ <u>and</u> $\vec{\nabla}(\vec{\pi}\cdot\vec{\pi}) = \vec{\nabla}\vec{\pi}\cdot\pi + \vec{\pi}\cdot\vec{\nabla}\vec{\pi}$

and $\ln(1-\pi^2) \simeq -\pi^2$

giving $S_1 = \frac{1}{2T}\int d^d x \left[(\vec{\pi}\cdot\vec{\nabla}\vec{\pi})^2 - \frac{T}{a^d}\pi^2 \right]$

Try: dim analysis to show $\langle \pi^2 \rangle \sim T$

$S_0 \sim \mathcal{O}(1)\,,\; S_1 \sim \mathcal{O}(T)\,,\; S_2 \sim \mathcal{O}(T^2) \ldots$

F.T. the Goldstone mode: $\pi^a(\vec{x}) = \int \frac{d^d k}{(2\pi)^d}\, \pi^a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$

gives $S_0 = \frac{1}{2T}\int \frac{d^d k}{(2\pi)^d}\, k^2 \underbrace{|\vec{\pi}(\vec{k})|^2}_{\pi^a(\vec{k})\,\pi^a(-\vec{k})}$

$\underline{\text{use}}$: $\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x+a) + \delta(x-a)\right]_{x=\sigma}$

$$\Rightarrow \int D\vec{\pi} \left[\int D\sigma \left[\frac{1}{2\sqrt{1-\pi^2}}\left(\delta(\sigma+a) + \delta(\sigma-a)\right)\right]\right]$$

$$= \int D\vec{\pi} \, \frac{1}{2\sqrt{1-\pi^2}}(1+1) = \int \prod_i \frac{d\vec{\pi}_i}{\sqrt{1-(\vec{\pi}_i)^2}} \quad \vec{\pi}(\vec{x})$$

Then $Z = \int \prod_i \frac{d\vec{\pi}_i}{\sqrt{1-\pi_i^2}} e^{-S[\vec{\pi}]}$ $\quad \swarrow \frac{N}{V} = \frac{1}{a^d}$

$$= \int \prod_i d\pi_i \, e^{-S[\vec{\pi}] - \frac{1}{2}\sum_i \ln(1-\vec{\pi}_i^2)}$$

$$= \int D\vec{\pi} \, e^{-S[\pi] - \frac{1}{2a^d}\int d^dx \ln(1-\vec{\pi}^2)}$$

$$\ldots \vec{k}_2)\, \pi^b(\vec{k}_3)\, \pi^b(\vec{k}_4)\, e^{i \vec{x} \cdot (\vec{k}_{1} +_{2} +_{3} -_{4})}$$

and $\ln(1-\pi^2) \simeq -\pi^2$

giving $S_1 = \frac{1}{2T}\int d^dx \left[(\vec{\pi}\cdot\vec{\nabla}\vec{\pi})^2 - \frac{T}{a^d}\pi^2 \right]$

Try: dim analysis to show $\langle \pi^2 \rangle \sim T$

$S_0 \sim \mathcal{O}(1)$, $S_1 \sim \mathcal{O}(T)$, $S_2 \sim \mathcal{O}(T^2)$ ...

F.T. the Goldstone mode: $\pi^a(\vec{x}) = \int \frac{d^dk}{(2\pi)^d}\, \pi^a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$

gives $\underline{S_0} = \frac{1}{2T}\int \frac{d^dk}{(2\pi)^d}\, k^2 \underbrace{|\vec{\pi}(\vec{k})|^2}_{\pi^a(\vec{k})\pi^a(-\vec{k})}$

$$S_1 = -\frac{1}{2T}\int d^dx \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}\left(\vec{k}_1\cdot\vec{k}_3\right)$$

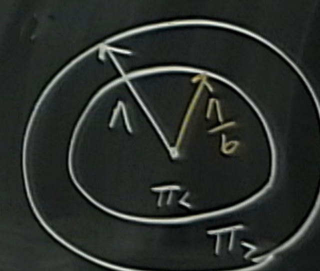
$$\pi^a(\vec{k}_1)\,\pi^a(\vec{k}_2)\,\pi^b(\vec{k}_3)\,\pi^b(\vec{k}_4)\,e^{i\vec{x}\cdot(\vec{k}_{1+2+3+4})}$$

$$-\frac{1}{2T}\,\frac{T}{a^d}\int\frac{d^dk}{(2\pi)^d}\left|\vec{\pi}(\vec{k})\right|^2 \Big\}\; S_1\,[\text{easy}]$$

NOTE: $$\int d^dx\, e^{i\vec{x}\cdot(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)} = (2\pi)^d\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

RG: split into fast $\pi_>$ and slow $\pi_<$

$$\pi^a(\vec{k}) = \pi^a_<(\vec{k}) + \pi^a_>(\vec{k})$$



In analogy to the Ising theory:

$$S'[\vec{\pi}_<] = S_0[\vec{\pi}_<] + \langle S_1[\vec{\pi}_<, \vec{\pi}_>] \rangle_{0>} + \ldots$$

Let's examine $\underline{S_1}$

$$S_1[\text{easy}] \Rightarrow \frac{-1}{2T}\frac{T}{a^d}\int \frac{d^dk}{(2\pi)^d} |\underbrace{\vec{\pi}(\vec{k})}|^2 \qquad \pi^a = \pi^a_< + \pi^a_>$$

$$\Rightarrow \frac{1}{a^d}(\vec{\pi}_<(\vec{x}))^2 + \underbrace{\langle (\pi_>(\vec{x}))^2 \rangle_{0>}}_{\text{additive constants.}}$$

$$S_1[\text{hard}] - \frac{1}{2T}\int_{k_1k_2k_3k_4} (2\pi)^d \delta(\vec{k}_1 \cdots \vec{k}_4)(\vec{k}_1 \cdot \vec{k}_3)[A+B+C+D+E]$$

where like before:

$A = \Pi_<^a(\vec{k}_1)\,\Pi_<^a(\vec{k}_2)\,\Pi_<^b(\vec{k}_3)\,\Pi_<^b(\vec{k}_4)$

B = all fast fields → additive constant

C, E = odd # fast modes → all zero.

D = two slow, two fast $\quad \Pi_<^a \Pi_<^a \Pi_>^b \Pi_>^b$

How to determine the spin component index, and k, to integrate over?

→ <u>need</u> both $\vec{k}_1$ and $\vec{k}_3$ to be <u>FAST</u> or <u>SLOW</u>

or fast integrals are <u>odd</u> → 0

BOTH FAST

$$-\frac{1}{2T}\int_{k_1\cdots k_4}(2\pi)^d\delta(\vec{k}_1\cdots\vec{k}_4)\,(\vec{k}_1\cdot\vec{k}_3)\,\Big\langle \pi_>^a(\vec{k}_1)\,\pi_>^b(\vec{k}_3)\Big\rangle_{0>}\,\pi_<^a(k_2)\,\pi_<^b(k_4)$$

need $$\Big\langle \pi^a(\vec{q})\,\pi^b(\vec{q}\,')\Big\rangle_0 = \delta_{ab}\underbrace{\frac{T}{q^2}}_{G_0(\vec{q})}(2\pi)^d\delta(\vec{q}+\vec{q}\,')$$

From the δ-functions: $\vec{k}_1 = -\vec{k}_3 \Rightarrow (\vec{k}_1\cdot\vec{k}_3) = \cancel{-k^2} = -q^2$

$$\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4) \Rightarrow \vec{k}_2 = -\vec{k}_4 = \vec{k}$$

$$S_1[\text{hard}] \text{ BOTH FAST } = \frac{1}{2T}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} |\vec{\pi}_<(k)|^2 \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\, q^2 G_0(\vec{q})$$

also for BOTH SLOW, the q^2 moves into the $<$ integral as k^2

For each case we need:

$$I_1 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d} G(\vec{q}) = \frac{S_d}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda} dq\, q^{d-1} \frac{T}{q^2}$$

$$= \frac{S_d T}{(2\pi)^d}\frac{1}{d-2}\left[\Lambda^{d-2} - \left(\Lambda/b\right)^{d-2}\right]$$

Use: $b = e^{\Delta l}$ $\Rightarrow$ $I_1 = \dfrac{S_d T \Lambda^{d-2}}{(2\pi)^d}\Delta l$

S_1[hard] BOTH FAST $= \frac{\ }{2T}\int_0 \frac{\ }{(2\pi)} \ldots \int_{\Lambda/b}$

also for BOTH SLOW, the q^2 moves into the < integral ... k

For each case we need: $I_1 = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G(\vec{q}) = \frac{S_d}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda} dq\, q^{d-1}\, \frac{T}{q^2}$

$$= \frac{S_d T}{(2\pi)^d}\frac{1}{d-2}\left[\Lambda^{d-2} - \left(\Lambda/b\right)^{d-2}\right]$$

use: $b = e^{\Delta l}$ $\Rightarrow$ $I_1 \approx \frac{S_d T \Lambda^{d-2}}{(2\pi)^d}\Delta l = T \cdot I_d(b) = I_1$

$\int^{b} \frac{d^d q}{\ } q^2 G(\vec{q}) = \frac{S_d T}{(2\pi)^d}\frac{\Lambda^d}{d}\left[1 - \left(\frac{1}{b}\right)^d\right]$

similarly: $I_2 = \int_{\Lambda/b}^{b} \frac{d^dq}{(2\pi)^d}\, q^2 G_0(\vec q) = \underbrace{\frac{S_d T}{(2\pi)^d}\frac{\Lambda^d}{d}}_{T\int \frac{d^dq}{(2\pi)^d} = T\frac{1}{a^d}}\left[1-\left(\frac{1}{b}\right)^d\right]$

Let's collect everything $S' = S_0 + \langle S_1\rangle_{0>}$

$$S' = \frac{1}{2T}\left[\overset{S_0}{1} + \overset{\text{[Both SLOW]}}{T\, I_d(b)}\right]\int_0^{\Lambda/b}\frac{d^dk}{(2\pi)^d}\, k^2\, |\vec\pi_<(\vec k)|^2$$

$$-\frac{1}{2a^d}\left[\overset{S_1\text{ (EASY)}}{1} - \overset{\text{(BOTH FAST)}}{(1-b^{-2})}\right]\int_0^{\Lambda/b}\frac{d^dk}{(2\pi)^d}\, |\vec\pi_<(\vec k)|^2 \qquad \text{"A"}$$

$$-\frac{1}{2T}\int_{k_1\ldots k_4}(2\pi)^d\delta(k_1\ldots k_4)\,\pi_<^a(\vec k_1)\,\pi_<^a(\vec k_2)\,\pi_<^b(\vec k_3)\,\pi_<^b(\vec k_4)\,(\vec k_1\cdot\vec k_3)$$

Next: $\vec{k}' = b\vec{k}$ and $\vec{\pi}' = Z^{-1}\vec{\pi}_<$

NOT = Ising scal. dim.

$$S' = \frac{1}{T}\left[1 + T I_1\right] b^{-d} b^{-2} Z^2 \int \frac{d^d k'}{(2\pi)^d}\, k'^2\, \pi'^a(\vec{k}')\, \pi'^a(-\vec{k}')$$

+ two terms that renormalize "trivially" (up to $\mathcal{O}(T^2)$)

The temperature is renormalized non-trivially

$$\frac{1}{T'} = \frac{1}{T}\,\frac{Z^2}{b^{d+2}}\left(1 + I_1\right) = \frac{1}{T}\left(\frac{Z}{b^d}\right)^2 b^{d-2}\left(1 + I_1\right)$$

$$= \frac{S_d T}{(2\pi)^d} \frac{1}{d-2} \left[\Lambda^{d-2} - \left(\Lambda/b\right)^{d-2} \right]$$

use: $b = e^{\Delta l}$ $\Rightarrow$ $\boxed{I_1 = \frac{S_d T \Lambda^{d-2}}{(2\pi)^d} \Delta l} = T \cdot I_d(b) = I_1$

similarly: $I_2 = \int_{\Lambda/b}^{b} \frac{d^d q}{(2\pi)^d} q^2 G_0(\vec{q}) = \frac{S_d T}{(2\pi)^d} \frac{\Lambda^d}{d} \left[1 - \left(\frac{1}{b}\right)^d \right]$

$$\underbrace{\phantom{G_0(\vec{q})}}_{T \int \frac{d^d q}{(2\pi)^d} = T \frac{1}{a^d}}$$

Let's collect everything. $S' = S_0 + \langle S_1 \rangle_{0>}$

$$S' = \frac{1}{2T} \left[\overset{S_0}{1} + \overset{\text{[Both slow]}}{T I_d(b)} \right] \int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d} k^2 \left| \vec{\pi}_< (k) \right|^2$$

(BOTH FAST)

"A"

Where (assignment 5 ?)

$$\boxed{\frac{Z}{b^d} = 1 - \frac{(n-1)}{2} I_1}$$

RG recursion:

$$\frac{1}{T'} = \frac{1}{T} b^{d-2} (1 + I_1)\left(1 - \frac{n-1}{2} I_2\right)^2$$

$$= \frac{1}{T} b^{d-2} \left(1 - (n-2) I_1 + O(I_1^2)\right)$$

$$b^{d-2} = e^{(d-2)\Delta l} \simeq 1 + \epsilon \Delta l$$

$$\epsilon = d-2$$

$$\frac{1}{T'} = \frac{1}{T} (1 + \epsilon \Delta l)\left(1 - (n-2) \frac{S_d \Lambda^\epsilon}{(2\pi)^d} T \Delta l\right)$$

$$= \frac{1}{T} + \frac{1}{T}\left[\epsilon - (n-2) \frac{S_d \Lambda^\epsilon}{(2\pi)^d} T\right] \Delta l + O(\Delta l^2)$$

defining $\tilde{T} = T\Lambda^{\epsilon}$, $K_d = \frac{S_d}{(2\pi)^d}$

$$\frac{d}{dl}\left(\frac{1}{\tilde{T}}\right) = \frac{1}{\tilde{T}}\left[\epsilon - (n-2)\tilde{T}K_d\right] \text{ or}$$

$$\boxed{\frac{d\tilde{T}}{dl} = -\epsilon\tilde{T} + (n-2)K_d\tilde{T}^2}$$

$b = e \quad \Rightarrow \quad I_1 = \frac{\cdots}{(2\pi)^d}$

Similarly: $I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\, q^2 G_0(\vec{q}) = \underbrace{\frac{S_d T}{(2\pi)^d}}_{T\int \frac{d^dq}{(2\pi)^d} = T\frac{1}{a^d}} \frac{\Lambda^d}{d}\left[1-\left(\frac{1}{b}\right)^d\right]$

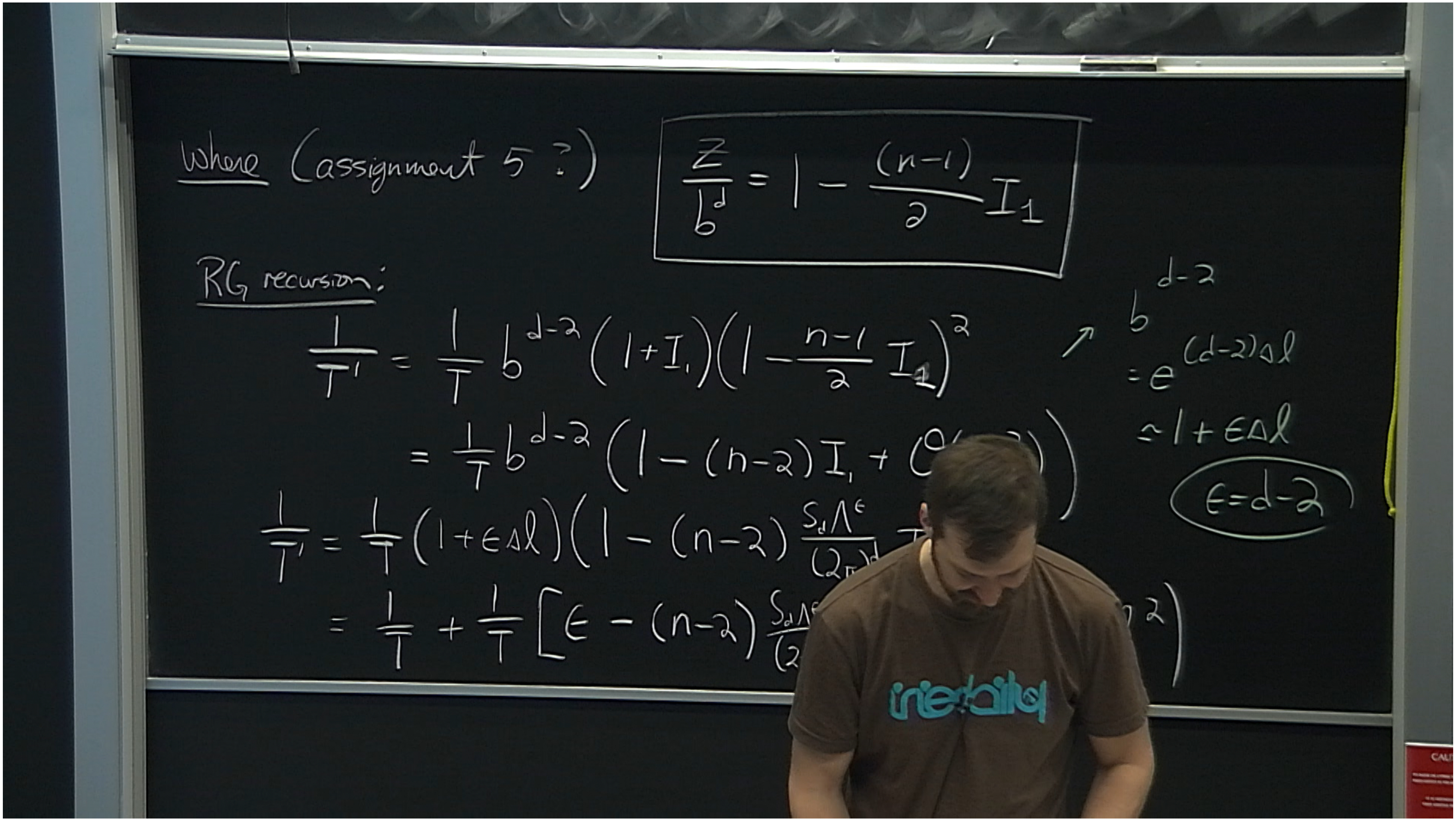
Let's collect everything. $S' = S_0 + \langle S_1 \rangle_{0>}$

$$S' = \frac{1}{2T}\Big[\overset{S_0}{1} + \overset{\text{[Both SLOW]}}{T\, I_d(b)}\Big] \int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d}\, k^2 \left|\vec{\pi}_<(\vec{k})\right|^2$$

$$- \frac{1}{2a^d}\Big[\overset{S_1\text{ [EASY]}}{1} - \overset{\text{(BOTH FAST)}}{(1-b^{-2})}\Big] \int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} \left|\vec{\pi}_<(\vec{k})\right|^2$$

"A"

$$- \frac{1}{2T}\int_{k_1 \ldots k_4} (2\pi)^d \delta(k_1 \ldots k_4)\, \pi_<^a(\vec{k}_1)\, \pi_<^a(\vec{k}_2)\, \pi_<^b(\vec{k}_3)\, \pi_<^b(\vec{k}_4)\, (\vec{k}_1 \cdot \vec{k}_3)$$


Where (assignment 5 ?)
$\boxed{\frac{Z}{b^d} = 1 - \frac{(n-1)}{2} I_1}$
RG recursion:
$\frac{1}{T'} = \frac{1}{T} b^{d-2} (1+I_1)\left(1 - \frac{n-1}{2} I_1\right)^2$
$= \frac{1}{T} b^{d-2} \left(1 - (n-2) I_1 + \mathcal{O}(\right.$
$\frac{1}{T'} = \frac{1}{T}(1+\epsilon\Delta\ell)\left(1-(n-2)\frac{S_d\Lambda^\epsilon}{(2\pi)^d}\right.$
$= \frac{1}{T} + \frac{1}{T}\left[\epsilon - (n-2)\frac{S_d\Lambda}{(2\right.$
b^{d-2}
$= e^{(d-2)\Delta\ell}$
$\simeq 1+\epsilon\Delta\ell$
$\epsilon = d-2$