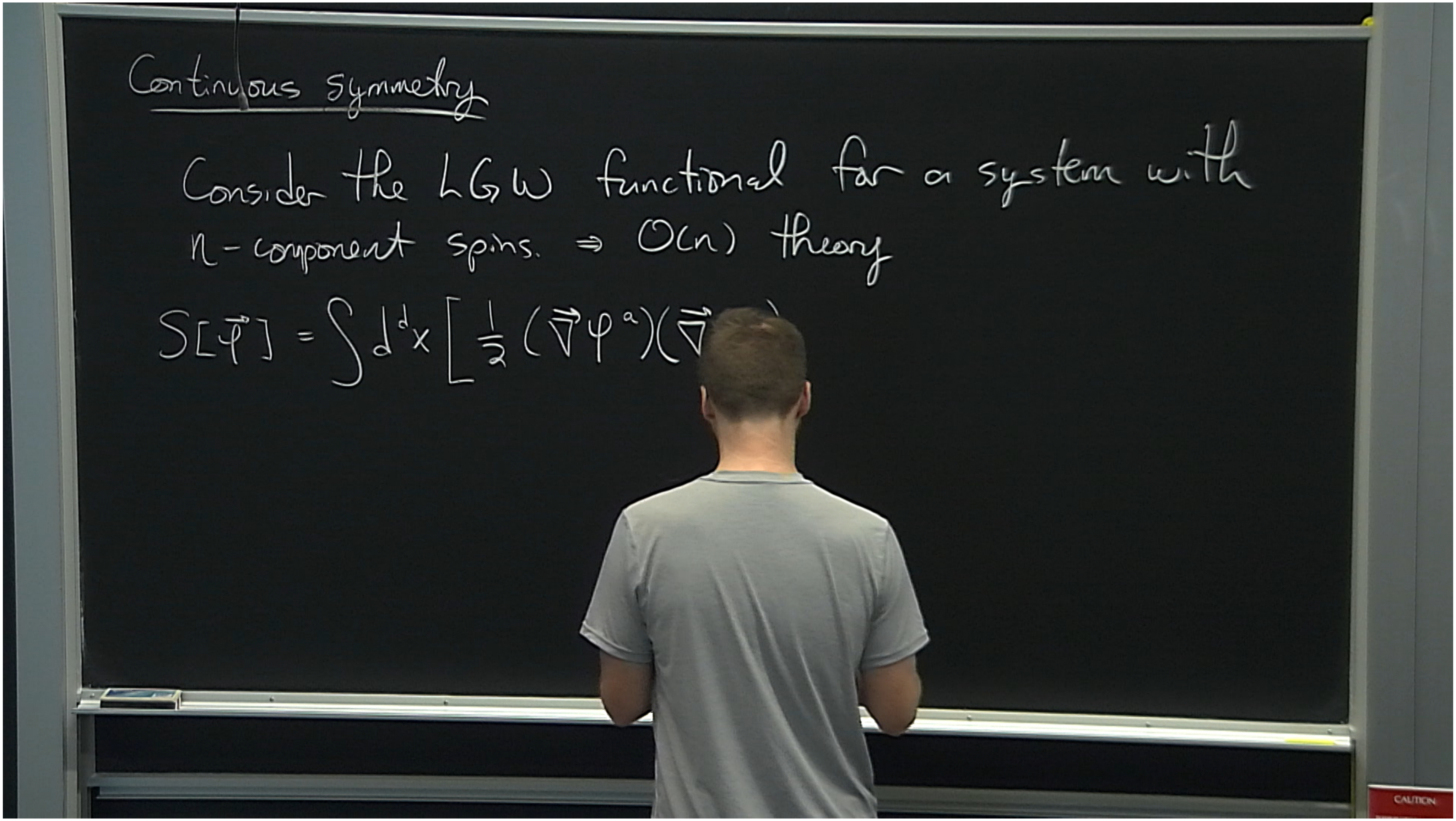
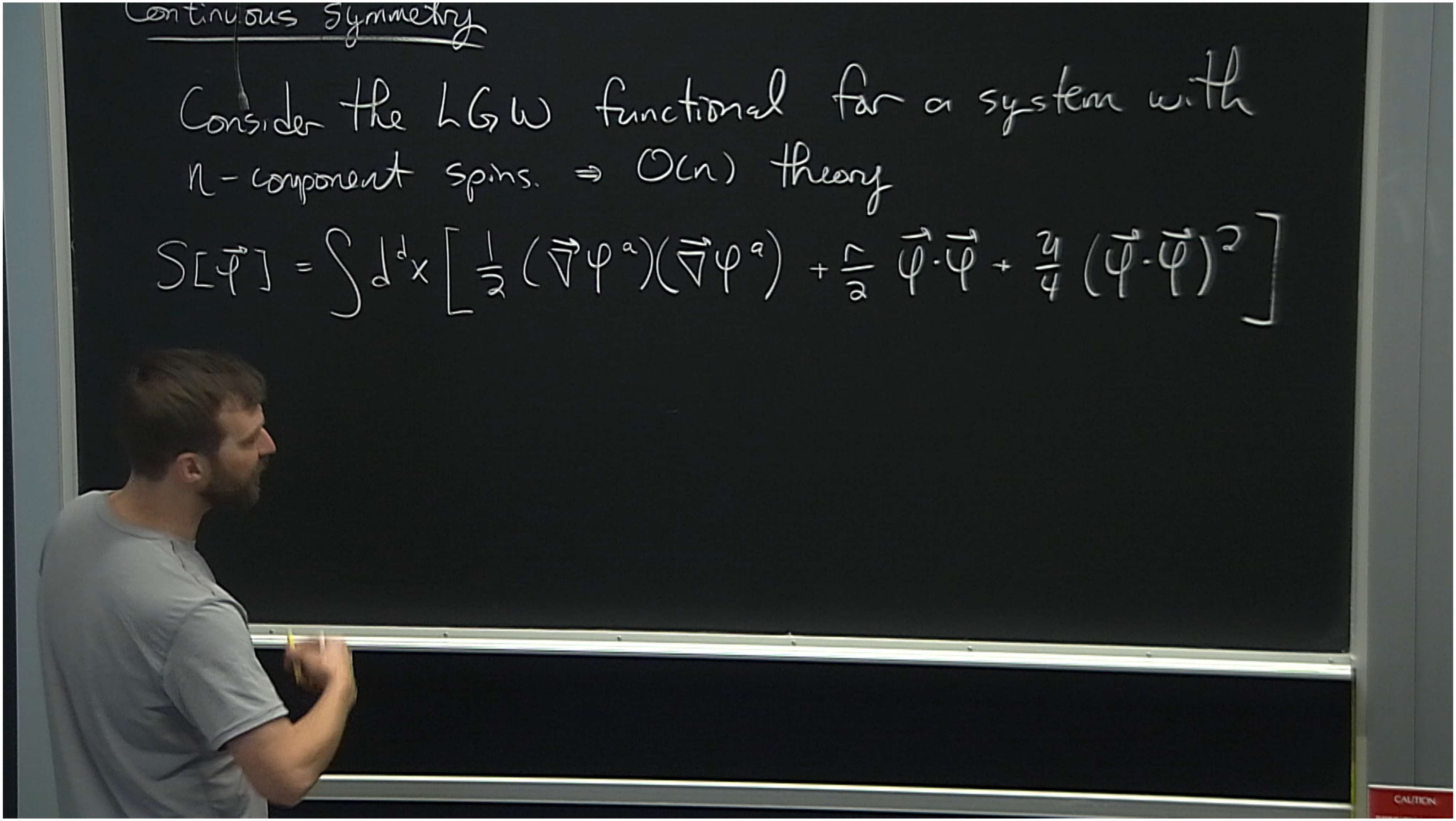
Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 19

Date: Mar 08, 2017 10:30 AM

URL: http://pirsa.org/17030003

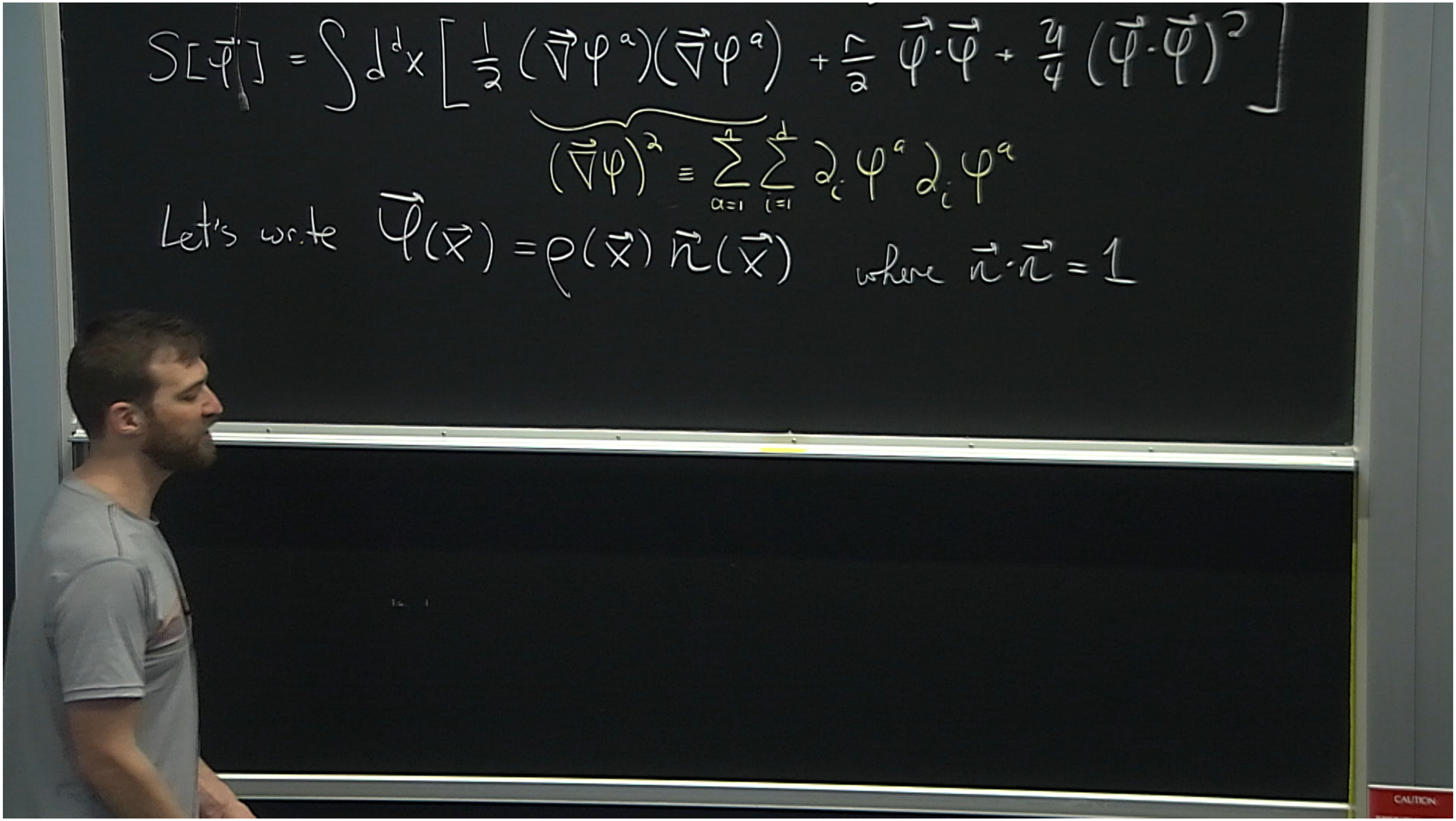
Abstract:


Continuous symmetry
Consider the LGW functional for a system with
n-component spins. ⇒ O(n) theory
$S[\vec{\varphi}] = \int d^dx \Big[\frac{1}{2} (\vec{\nabla}\varphi^a)(\vec{\nabla}$


Continuous Symmetry
Consider the LGW functional for a system with
n-component spins. ⇒ O(n) theory
$S[\vec{\varphi}] = \int d^dx \left[\frac{1}{2} (\vec{\nabla}\varphi^a)(\vec{\nabla}\varphi^a) + \frac{r}{2} \vec{\varphi}\cdot\vec{\varphi} + \frac{u}{4} (\vec{\varphi}\cdot\vec{\varphi})^2 \right]$

Consider the LGW functional for a system with n-component spins. $\Rightarrow$ $O(n)$ theory

$$S[\vec{\varphi}] = \int d^d x \left[\underbrace{\frac{1}{2} (\vec{\nabla}\varphi^a)(\vec{\nabla}\varphi^a)}_{(\vec{\nabla}\varphi)^2 \equiv \sum_{a=1}^{n} \sum_{i=1}^{d} \partial_i \varphi^a \partial_i \varphi^a} + \frac{r}{2} \vec{\varphi}\cdot\vec{\varphi} + \frac{u}{4} (\vec{\varphi}\cdot\vec{\varphi})^2 \right]$$


$S[\vec{\varphi}] = \int d^dx \left[\frac{1}{2} (\vec{\nabla}\varphi^a)(\vec{\nabla}\varphi^a) + \frac{r}{2} \vec{\varphi}\cdot\vec{\varphi} + \frac{u}{4} (\vec{\varphi}\cdot\vec{\varphi})^2 \right]$
$(\vec{\nabla}\varphi)^2 \equiv \sum_{a=1} \sum_{i=1}^{d} \partial_i \varphi^a \partial_i \varphi^a$
Let's write $\vec{\varphi}(\vec{x}) = \rho(\vec{x})\, \vec{n}(\vec{x})$ where $\vec{n}\cdot\vec{n} = 1$

$$(\vec{\nabla}\varphi)^2 \equiv \sum_{a=1}^{n}\sum_{i=1}^{d} \partial_i \varphi^a \partial_i \varphi^a$$

Let's write $\vec{\varphi}(\vec{x}) = \rho(\vec{x})\,\vec{n}(\vec{x})$ where $\vec{n}\cdot\vec{n} = 1$

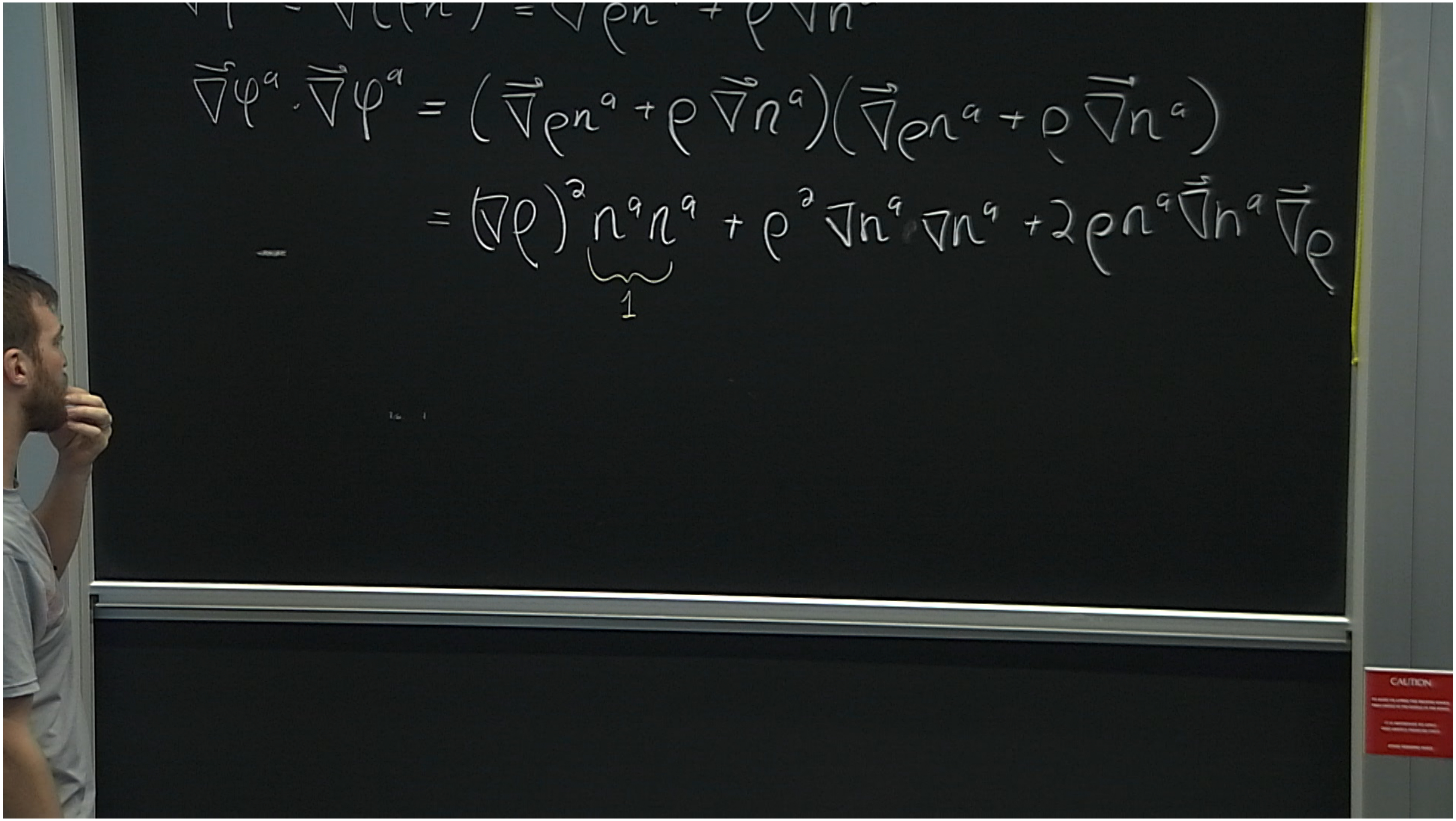
ρ is the magnitude of the order parameter

and $\vec{n}$ is its direction

$$\vec\nabla\varphi^a = \vec\nabla(\rho n^a) = \vec\nabla\rho\, n^a + \rho\vec\nabla n^a$$

$$\vec\nabla\varphi^a\cdot\vec\nabla\varphi^a = (\vec\nabla\rho\, n^a + \rho\vec\nabla n^a)(\vec\nabla\rho\, n^a + \rho\vec\nabla n^a)$$

$$= (\nabla\rho)^2 n^a n^a + \rho^2\nabla n^a\,\nabla n^a + 2\rho n^a\vec\nabla n^a\,\vec\nabla\rho$$

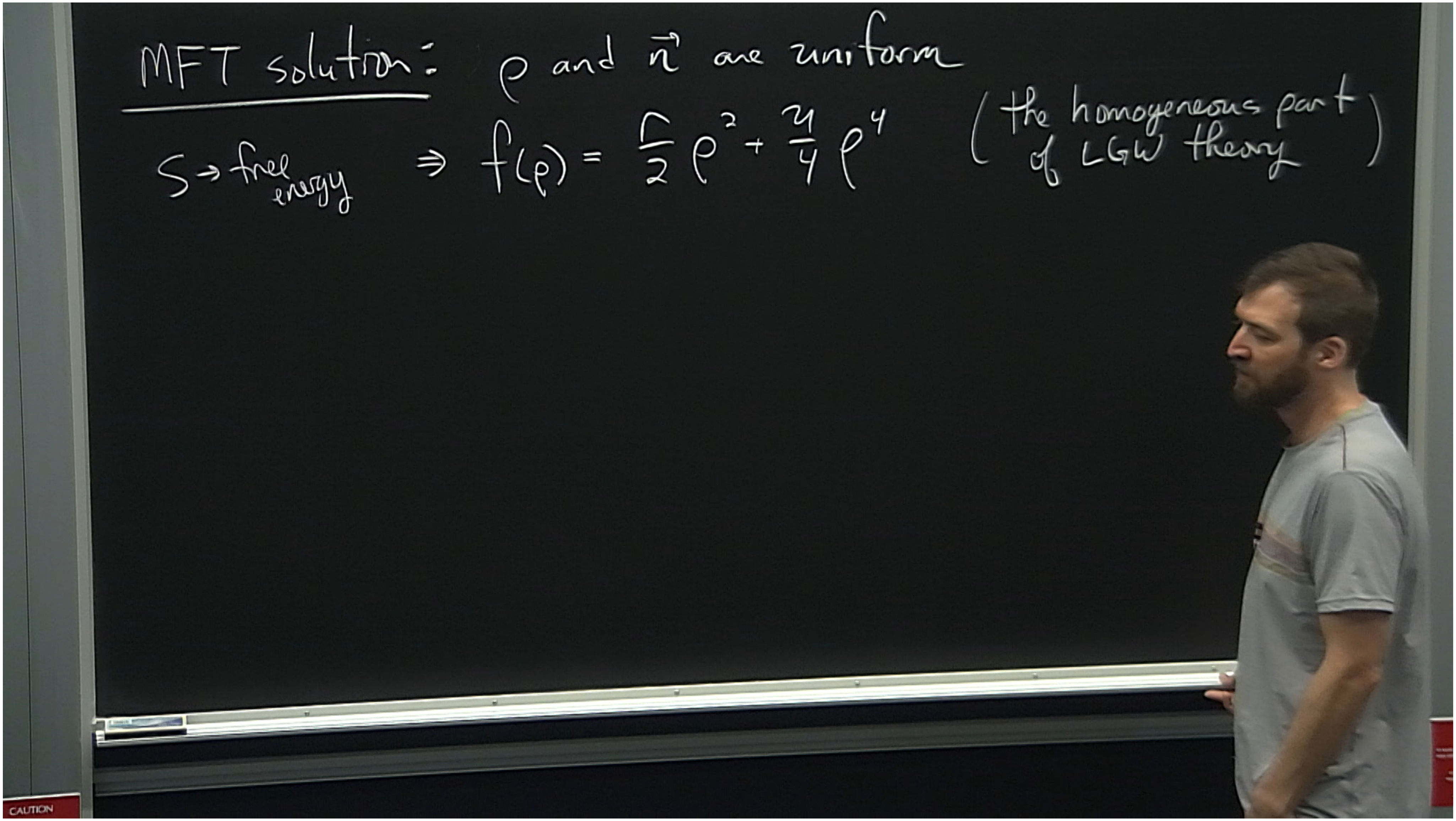

$\vec\nabla\varphi^a\cdot\vec\nabla\varphi^a = (\vec\nabla\rho n^a + \rho\vec\nabla n^a)(\vec\nabla\rho n^a + \rho\vec\nabla n^a)$
$= (\nabla\rho)^2 \underbrace{n^a n^a}_{1} + \rho^2 \nabla n^a \nabla n^a + 2\rho n^a \vec\nabla n^a \vec\nabla\rho$
CAUTION

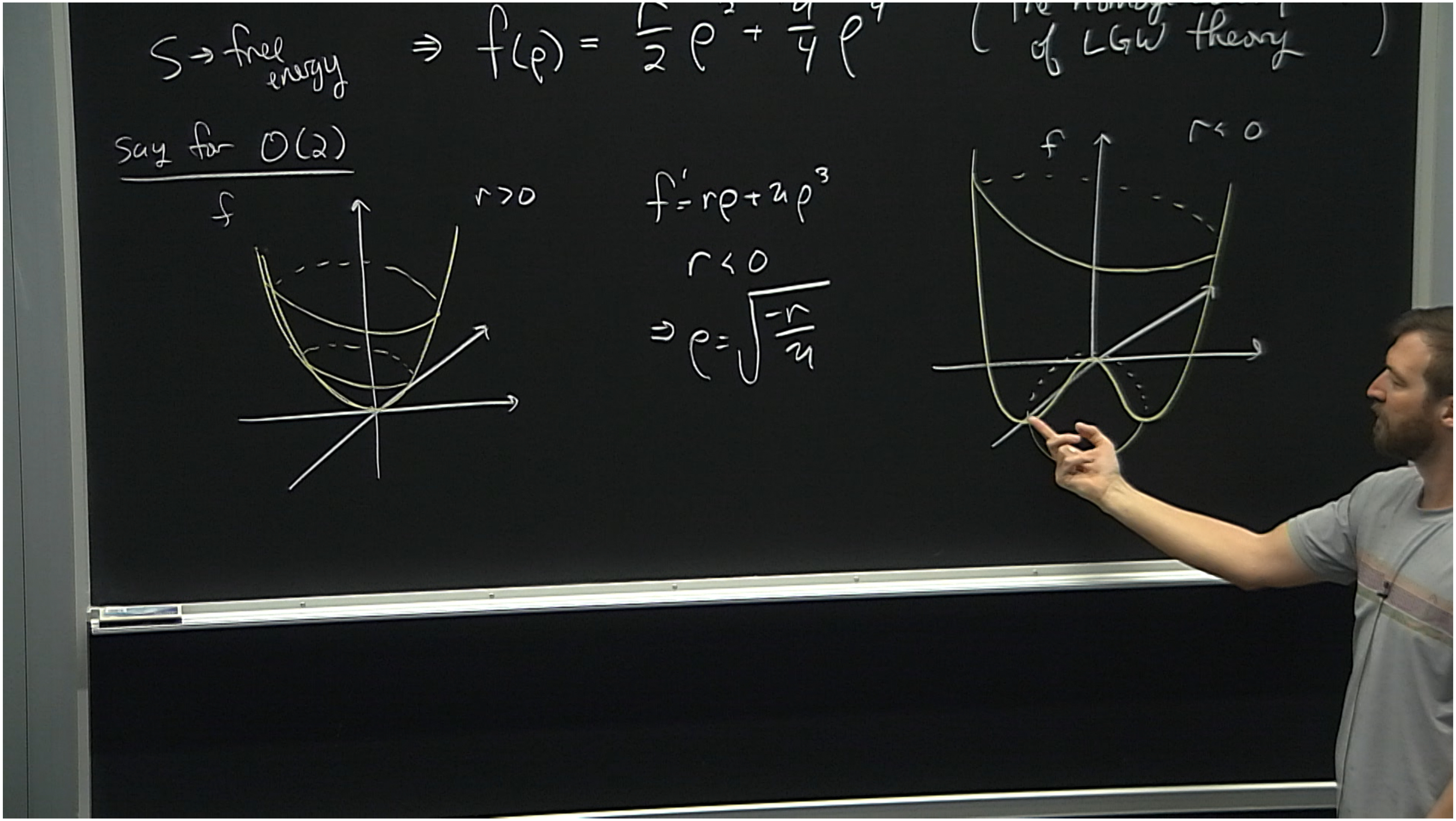
$$\vec\nabla\phi\cdot\vec\nabla\phi = (\nabla\rho\, n^a + \rho\nabla n^a)(\nabla\rho\, n^a + \rho\nabla n^a)$$

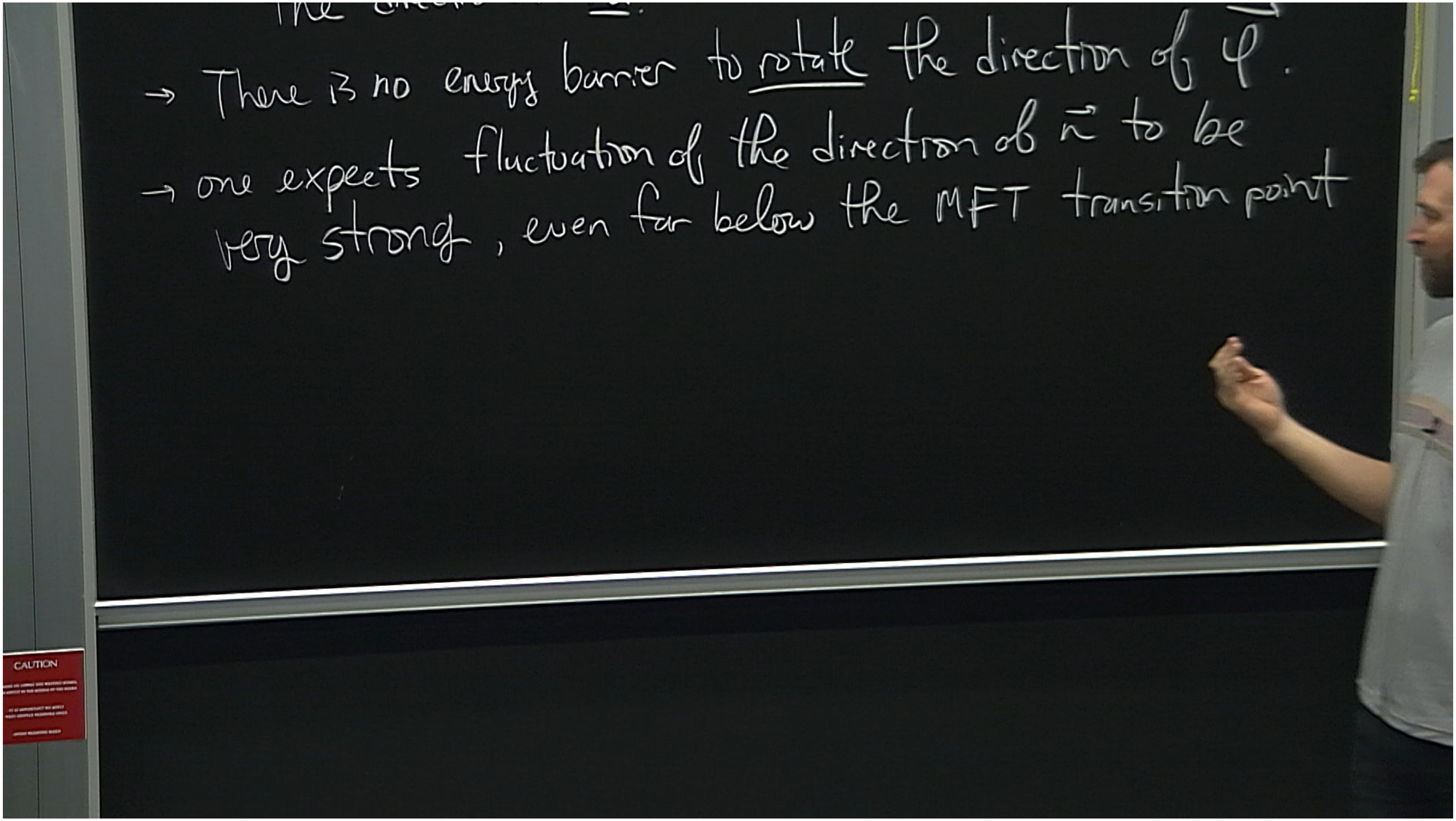
$$= (\nabla\rho)^2 \underbrace{n^a n^a}_{1} + \rho^2 \nabla n^a \nabla n^a + 2\rho \underbrace{n^a \vec\nabla n^a}_{0} \vec\nabla\rho$$

$$= (\vec\nabla\rho)^2 + \rho^2 \vec\nabla n^a \vec\nabla n^a$$

$$\Rightarrow S[\rho,\vec n] = \int d^d x \left[\frac{1}{2}\rho^2 \vec\nabla n^a \vec\nabla n^a + \frac{1}{2}(\vec\nabla\rho)^2 + \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4 \right]$$


MFT solution: ρ and $\vec{n}$ are uniform
$S \to$ free energy $\Rightarrow f(\rho) = \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4$
(the homogeneous part of LGW theory)
CAUTION


S → free energy
⇒ f(ρ) = r/2 ρ² + u/4 ρ⁴
of LGW theory
say for O(2)
f
r > 0
f' = rρ + uρ³
r < 0
⇒ ρ = √(−r/u)
f
r < 0


→ There is no energy barrier to rotate the direction of φ.
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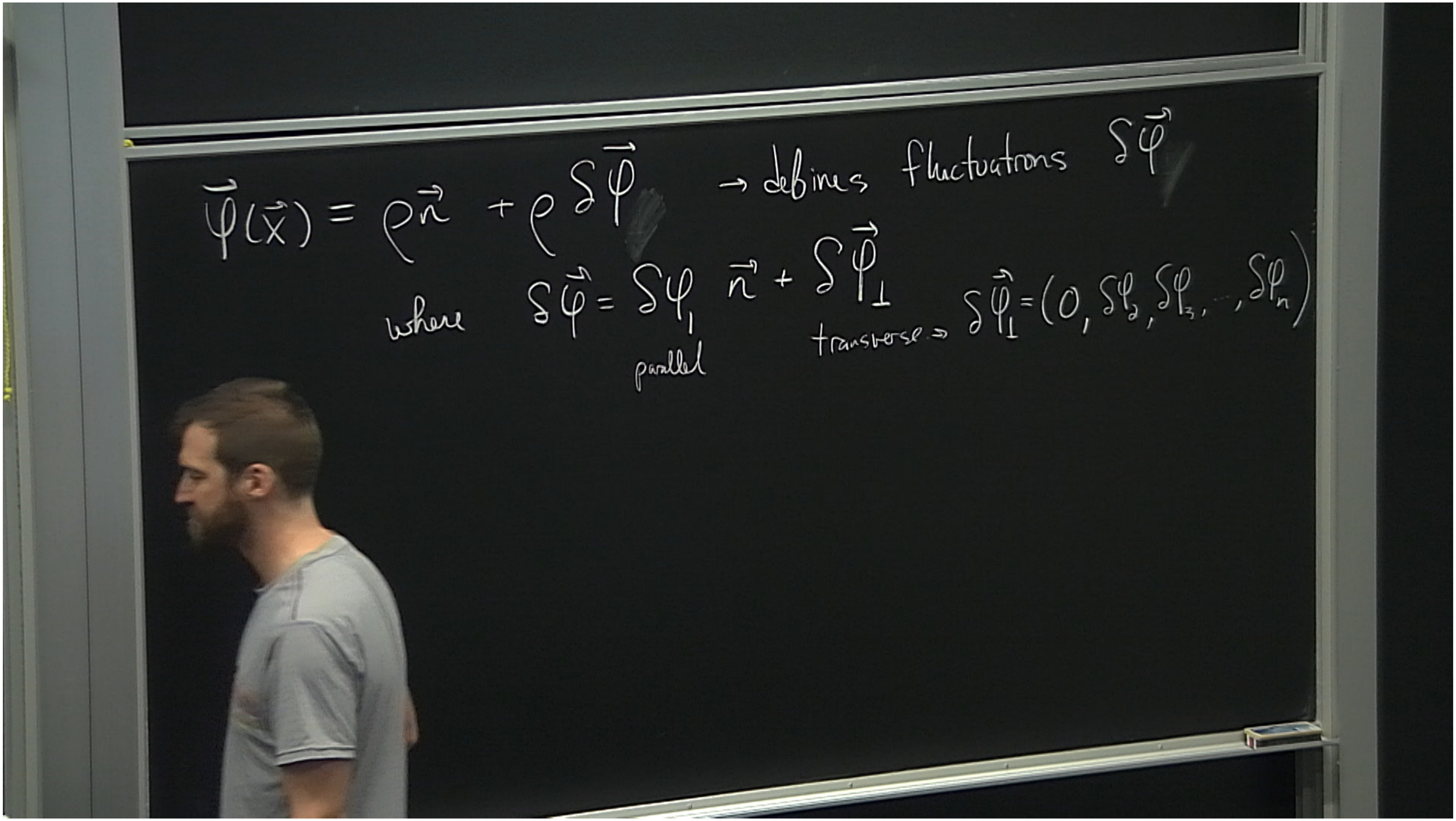
→ one expects fluctuation of the direction of $\vec{n}$ to be very strong, even far below the MFT transition point (where ρ is essentially fixed).

Consider small fluctuations around the MFT solution

$$\langle \vec{\varphi} \rangle = \rho \vec{n} \quad , \text{choose} \quad \vec{n} = (1, 0, 0, 0, \cdots, 0)$$

$$\vec{\varphi}(\vec{x}) = \rho\vec{n} + \rho\,\delta\vec{\varphi} \quad \rightarrow \text{defines fluctuations } \delta\vec{\varphi}$$

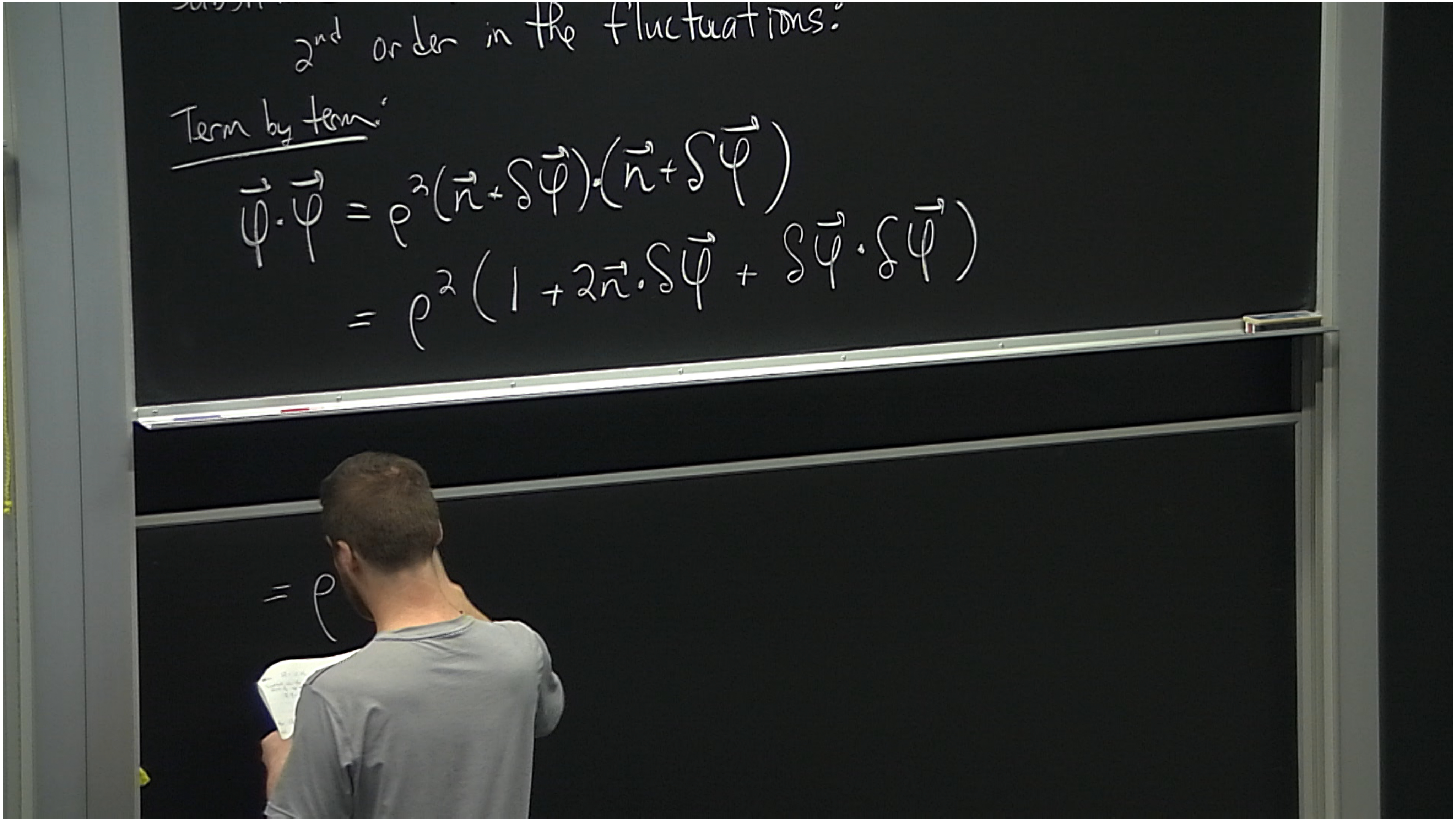
where $\delta\vec{\varphi} = \underbrace{\delta\varphi_{\parallel}}_{\text{parallel}}\,\vec{n} + \underbrace{\delta\vec{\varphi}_{\perp}}_{\text{transverse}}$


$\vec{\varphi}(\vec{x}) = \rho\vec{n} + \rho\,\delta\vec{\varphi}$ → defines fluctuations $\delta\vec{\varphi}$
where $\delta\vec{\varphi} = \delta\varphi_1\,\vec{n} + \delta\vec{\varphi}_\perp$
parallel
transverse ⇒ $\delta\vec{\varphi}_\perp = (0, \delta\varphi_2, \delta\varphi_3, \dots, \delta\varphi_n)$

$\varphi(\vec{x}) = \rho \vec{n} + \rho\, \delta\varphi$ ...

where $\delta\vec{\varphi} = \underbrace{\delta\varphi_1}_{\text{parallel}} \vec{n} + \delta\vec{\varphi}_\perp$ transverse $\Rightarrow$ $\delta\vec{\varphi}_\perp = (0, \delta\varphi_2, \delta\varphi_3, \dots, \delta\varphi_n)$

Substitute into the LGW function, leaving terms up to 2nd order in the fluctuations:



2^{nd} order in the fluctuations:
Term by term:
$\vec{\varphi}\cdot\vec{\varphi} = \rho^3(\vec{n}+\delta\vec{\varphi})\cdot(\vec{n}+\delta\vec{\varphi})$
$= \rho^2(1+2\vec{n}\cdot\delta\vec{\varphi}+\delta\vec{\varphi}\cdot\delta\vec{\varphi})$
$= \rho$

2nd order in the fluctuations:

Term by term:

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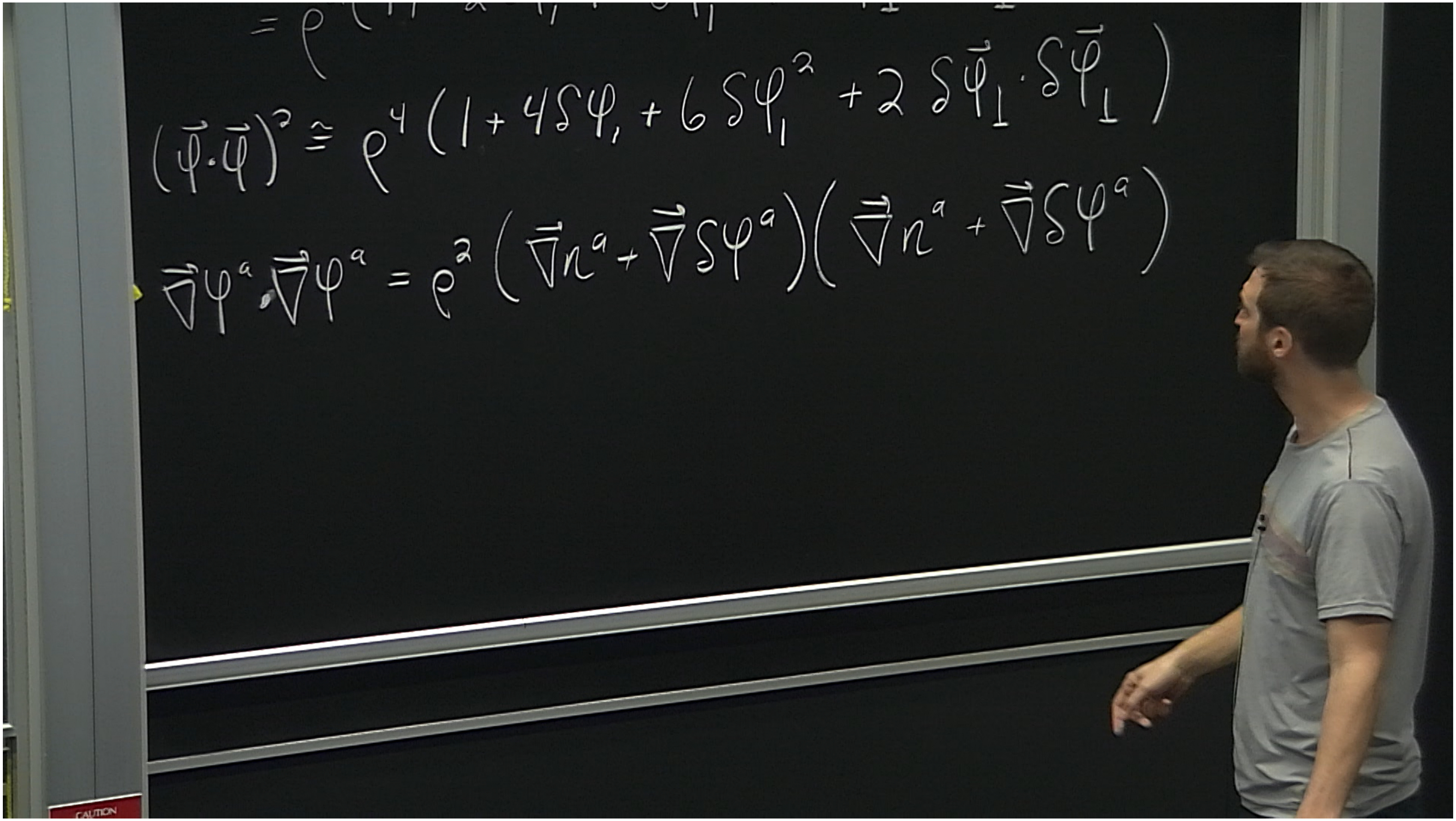
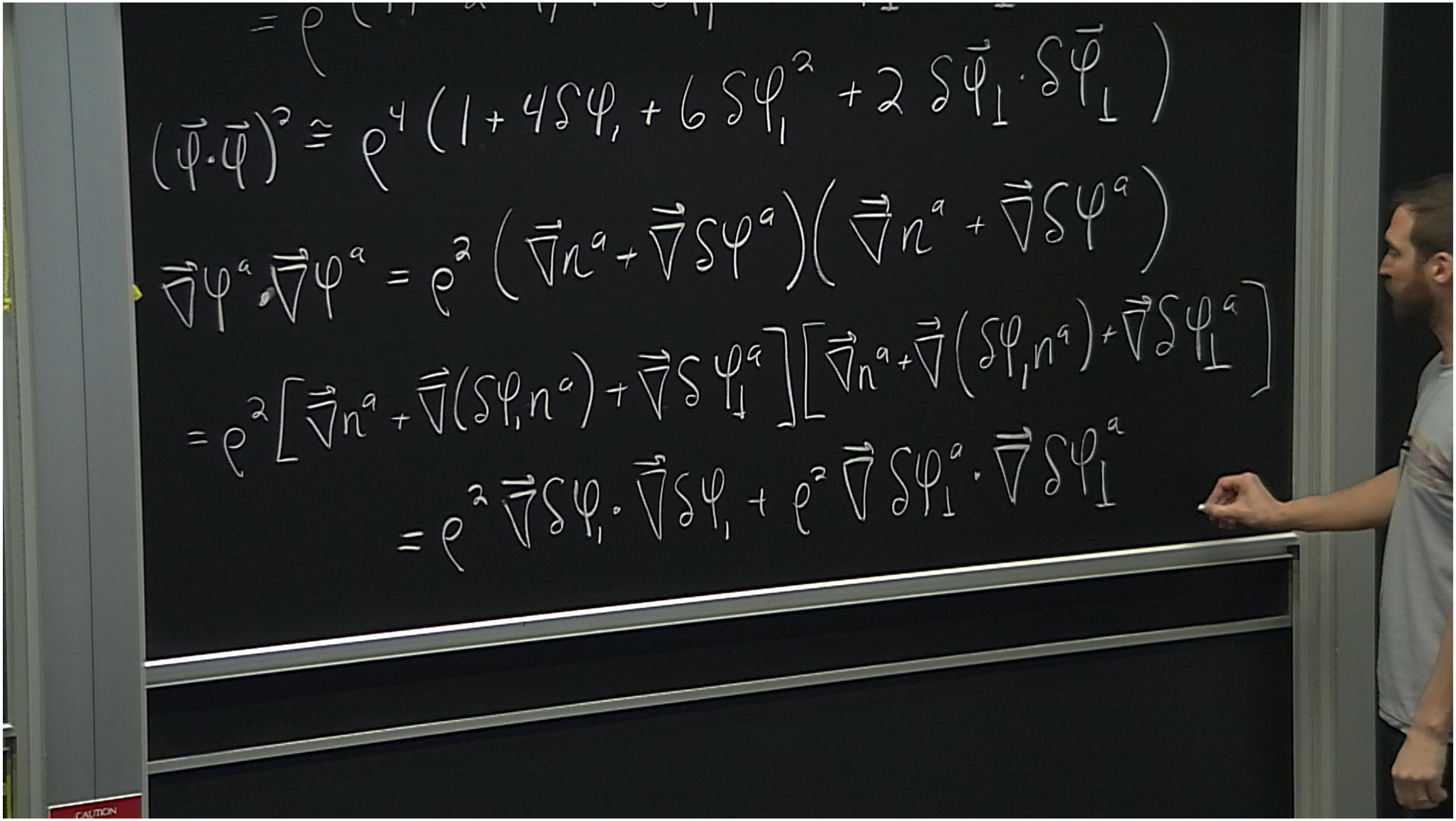
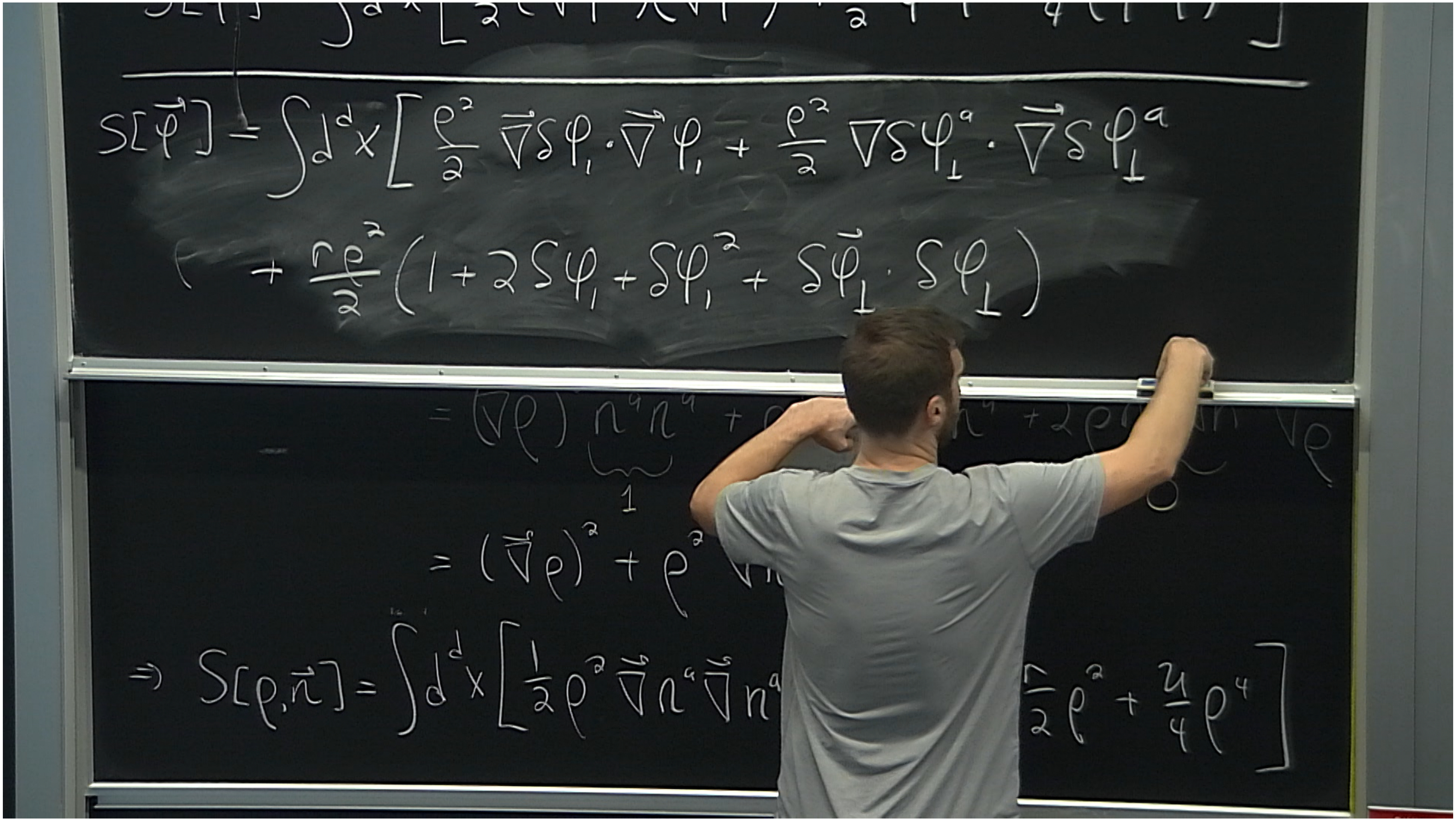
$$= \rho^2\left(1 + 2\vec{n}\cdot\delta\vec{\varphi} + \delta\vec{\varphi}\cdot\delta\vec{\varphi}\right)$$

$$= \rho^2\left(1 + 2\delta\varphi_1 + \delta\varphi_1^2 + \delta\vec{\varphi}_\perp\cdot\delta\vec{\varphi}_\perp\right)$$



$= \rho \, (\dots$
$(\vec{\varphi}\cdot\vec{\varphi})^2 \simeq \rho^4 (1 + 4\delta\varphi_1 + 6\delta\varphi_1^2 + 2\,\delta\vec{\varphi}_\perp \cdot \delta\vec{\varphi}_\perp)$
$\vec{\nabla}\varphi^a \cdot \vec{\nabla}\varphi^a = \rho^2 (\vec{\nabla} n^a + \vec{\nabla}\delta\varphi^a)(\vec{\nabla} n^a + \vec{\nabla}\delta\varphi^a)$


$(\vec{\varphi}\cdot\vec{\varphi})^2 \cong \rho^4(1+4\delta\varphi_1+6\delta\varphi_1^2+2\,\delta\vec{\varphi}_\perp\cdot\delta\vec{\varphi}_\perp)$
$\vec{\nabla}\varphi^a\cdot\vec{\nabla}\varphi^a=\rho^2(\vec{\nabla}n^a+\vec{\nabla}\delta\varphi^a)(\vec{\nabla}n^a+\vec{\nabla}\delta\varphi^a)$
$=\rho^2[\vec{\nabla}n^a+\vec{\nabla}(\delta\varphi_1 n^a)+\vec{\nabla}\delta\varphi_\perp^a][\vec{\nabla}n^a+\vec{\nabla}(\delta\varphi_1 n^a)+\vec{\nabla}\delta\varphi_\perp^a]$
$=\rho^2\vec{\nabla}\delta\varphi_1\cdot\vec{\nabla}\delta\varphi_1+\rho^2\vec{\nabla}\delta\varphi_\perp^a\cdot\vec{\nabla}\delta\varphi_\perp^a$



$S[\vec{\varphi}] = \int d^dx \left[\frac{\rho^2}{2} \vec{\nabla}\delta\varphi_1 \cdot \vec{\nabla}\varphi_1 + \frac{\rho^2}{2} \nabla\delta\varphi_\perp^a \cdot \vec{\nabla}\delta\varphi_\perp^a\right.$
$+ \frac{r\rho^2}{2}\left(1 + 2\delta\varphi_1 + \delta\varphi_1^2 + \delta\vec{\varphi}_\perp \cdot \delta\varphi_\perp\right)$
$= (\vec{\nabla}\rho)^2 + \rho^2$
$\Rightarrow S[\rho,\vec{n}] = \int d^dx \left[\frac{1}{2}\rho^2 \vec{\nabla}n^a \vec{\nabla}n^a\right.$
$\left.\frac{r}{2}\rho^2 + \frac{u}{4}\rho^4 \right]$

$$+\frac{u\rho^4}{4}\left(1+\underline{4\delta\varphi_1}+6\varphi_1^2+2\delta\vec{\varphi}_\perp\cdot\delta\vec{\varphi}_\perp\right)\Big]$$

<u>use</u> $\rho=\sqrt{\frac{-r}{u}}$

$\rightarrow$ Note the coefficient of $\delta\varphi_1$

$$r\rho^2+u\rho^4=-\frac{r^2}{u}+\frac{r^2}{u}=0$$

$\Rightarrow$ $\delta\vec{\varphi}_\perp\cdot\delta\vec{\varphi}_\perp$ has the <u>same</u> coefficient $=0$

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Then the coefficient of $\delta\varphi_1^2$ is

$$r\frac{\rho^2}{2}+\frac{3}{2}u\rho^4=-\frac{r^2}{2u}+\frac{3}{2}\frac{r^2}{u}=\frac{r^2}{u}=\rho^2|r|$$

$$S[\delta\vec{\varphi}]=\int d^dx\,\frac{\rho^2}{2}\Big[\vec{\nabla}\delta\varphi_1\cdot\vec{\nabla}\delta\varphi_1+\vec{\nabla}\delta\varphi_\perp^a\cdot\vec{\nabla}\delta\varphi_\perp^a$$
$$+2|r|\,\delta\varphi_1^2\Big]$$

of φ from $\vec{n}$ do not cost energy

Now rewrite in Fourier space

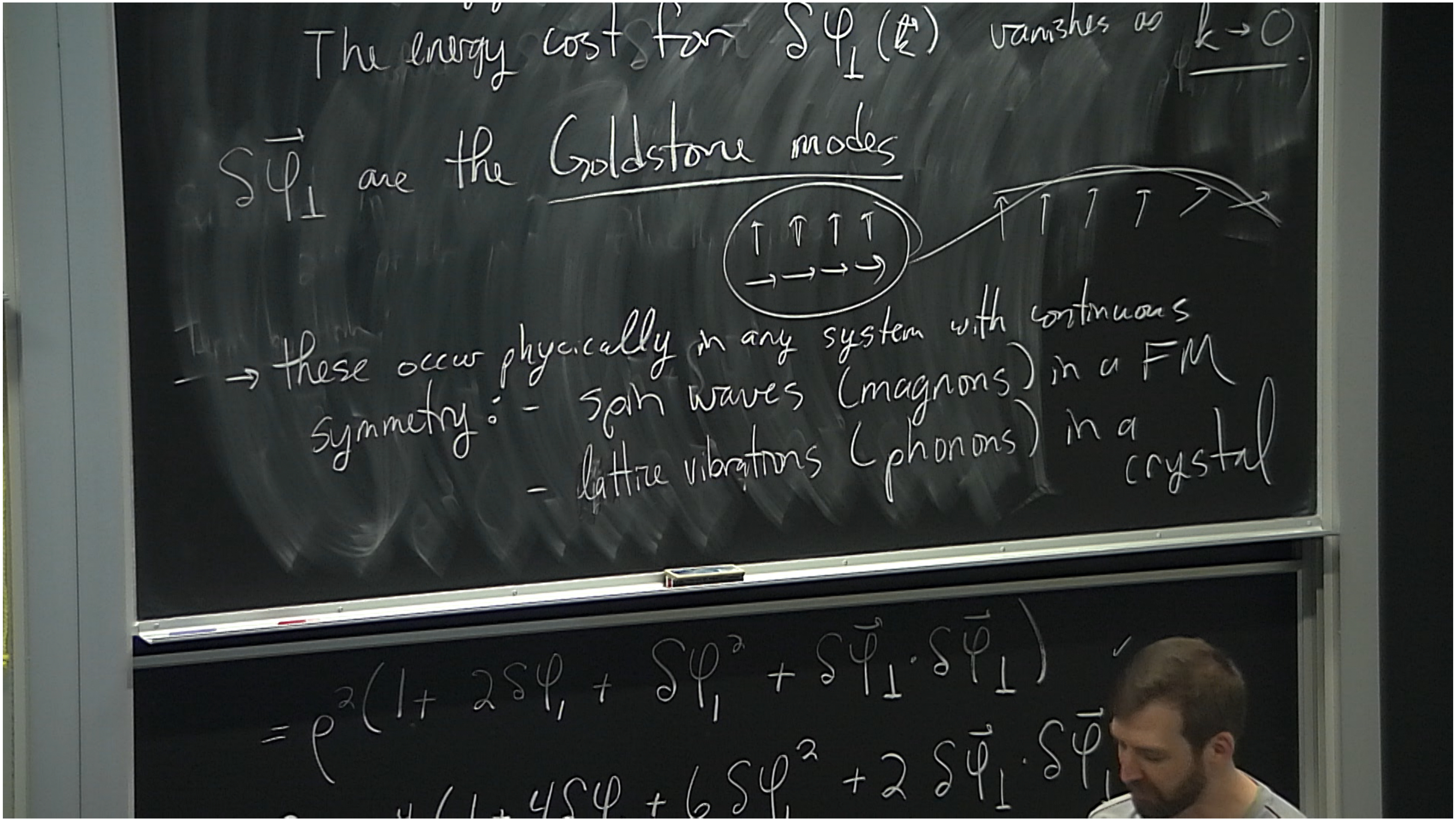
$$\delta\varphi_1(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} \delta\varphi_1(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

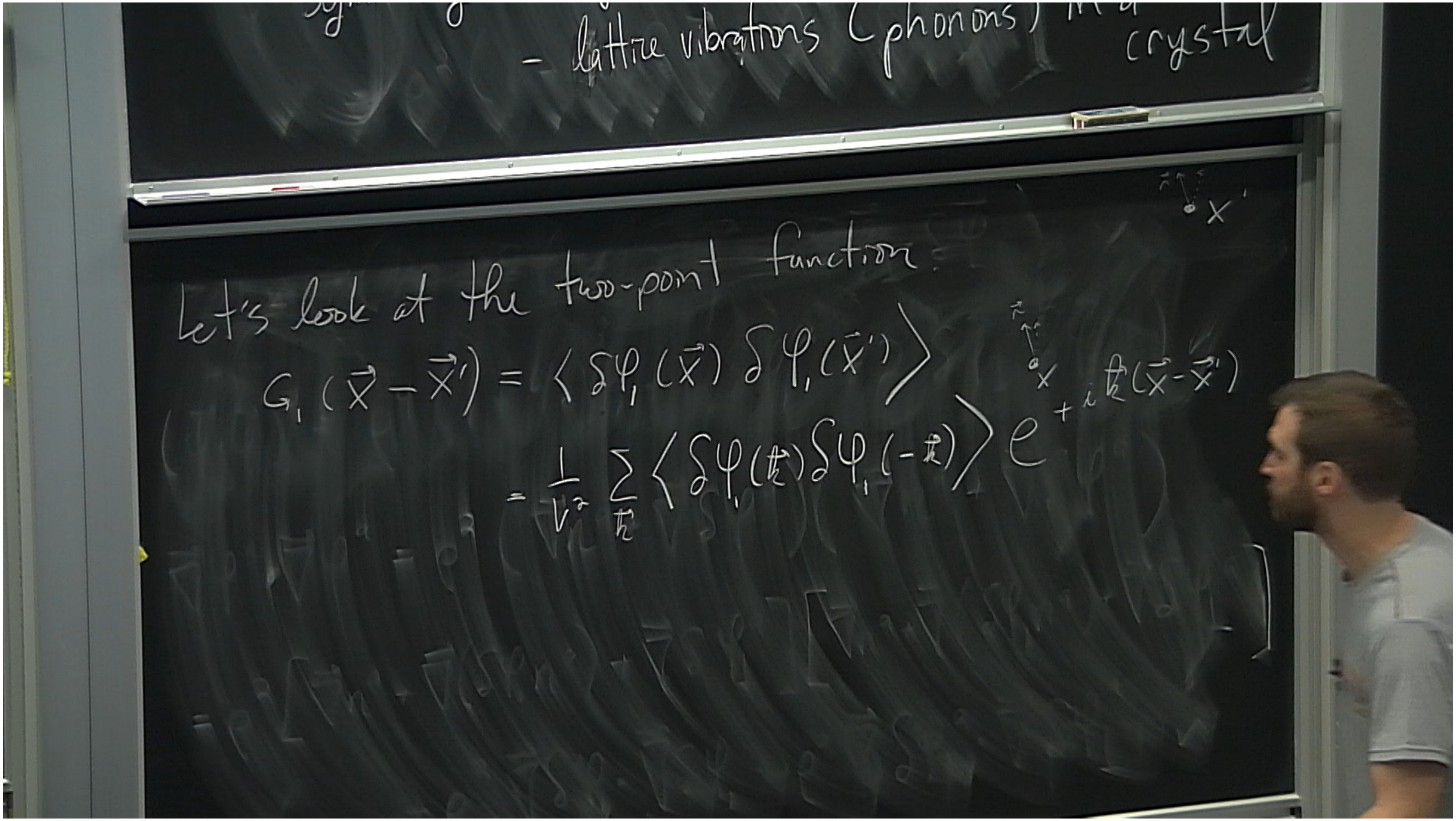
$$\delta\varphi_\perp^a(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} \delta\varphi_\perp^a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

$$\delta\varphi_1(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} \delta\varphi_1(\vec{k})\, e$$

$$\delta\varphi_\perp^a(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} \delta\varphi_\perp^a(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

$$S[\delta\vec{\varphi}] = \frac{\rho^2}{2V}\sum_{\vec{k}}\Big[k^2|\delta\varphi_1(\vec{k})|^2 + 2|r|\,|\delta\varphi_1(\vec{k})|^2 + k^2\,\delta\varphi_\perp^a(\vec{k})\,\delta\varphi_\perp^a(-\vec{k})\Big]$$


The energy cost for $\delta\varphi_\perp(k)$ vanishes as $k \to 0$.
$\delta\vec{\varphi}_\perp$ are the Goldstone modes
→ these occur physically in any system with continuous symmetry:
- spin waves (magnons) in a FM
- lattice vibrations (phonons) in a crystal
$= \rho^2(1 + 2\delta\varphi_1 + \delta\varphi_1^2 + \delta\vec{\varphi}_\perp \cdot \delta\vec{\varphi}_\perp)$
$+ 6\delta\varphi_1^2 + 2\delta\vec{\varphi}_\perp \cdot \delta\vec{\varphi}_\perp$


- lattice vibrations (phonons) in a crystal
Let's look at the two-point function:
$G_1(\vec{x}-\vec{x}') = \langle \delta\varphi_1(\vec{x})\, \delta\varphi_1(\vec{x}') \rangle$
$= \frac{1}{V^2} \sum_{\vec{k}} \langle \delta\varphi_1(\vec{k})\, \delta\varphi_1(-\vec{k}) \rangle e^{+i\vec{k}(\vec{x}-\vec{x}')}$

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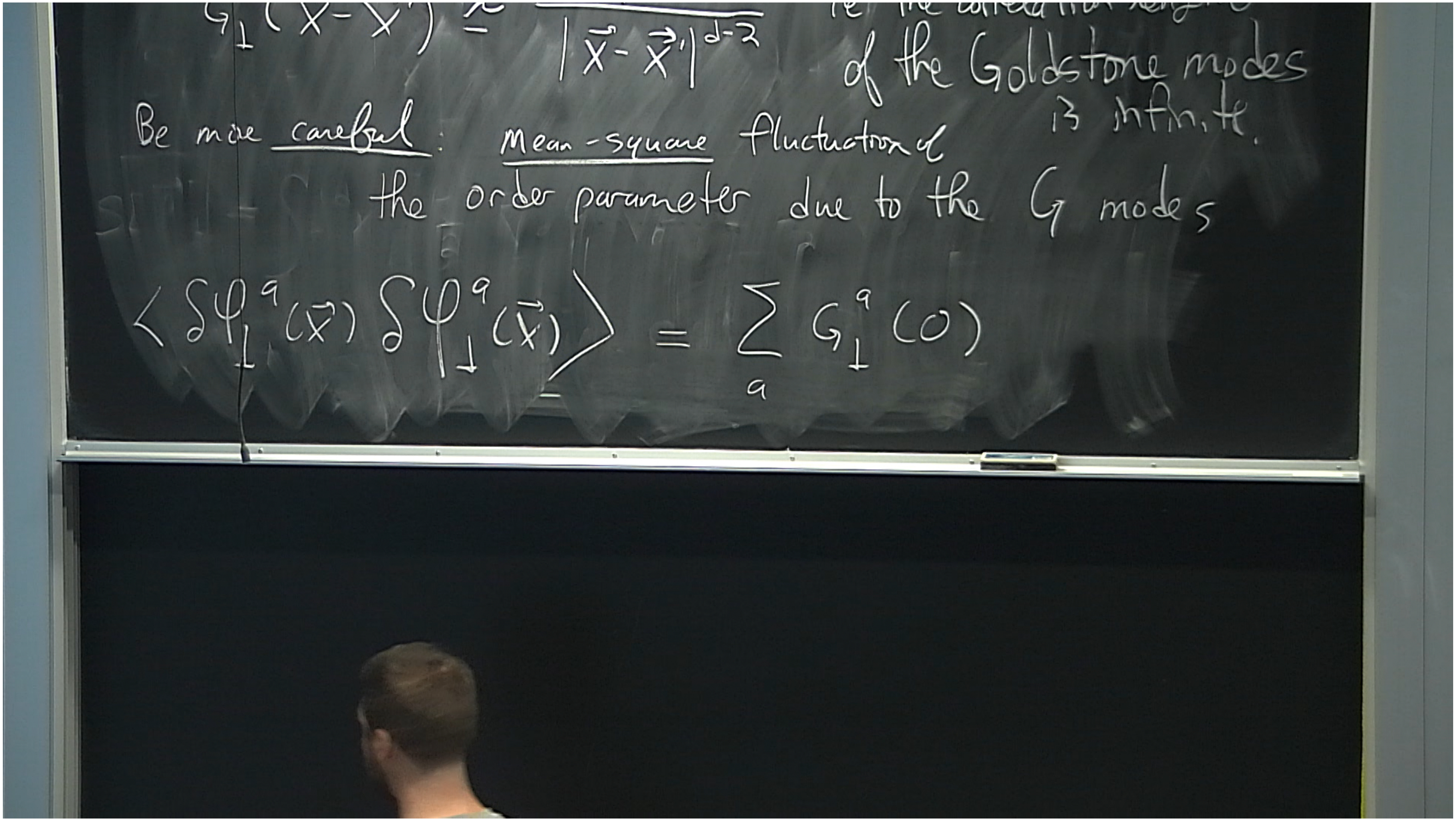
similar for $G_\perp(\vec{x}-\vec{x}')$

It follows that

$$G_{\parallel}(\vec{x}-\vec{x}\,') \simeq e^{-\frac{|\vec{x}-\vec{x}\,'|}{\xi}} \quad \text{where } \xi^2 = \frac{1}{2|r|}$$

$$G_{\perp}(\vec{x}-\vec{x}\,') \simeq \frac{1}{|\vec{x}-\vec{x}\,'|^{d-2}}$$

ie the correlation length of the Goldstone modes is infinite.


of the Goldstone modes
is infinite.
Be more careful: Mean-square fluctuation of the order parameter due to the G modes
$\langle \delta\varphi_\perp^a(\vec{x})\, \delta\varphi_\perp^a(\vec{x}) \rangle = \sum_a G_\perp^a(0)$

$$= \frac{n-1}{\rho^2} \frac{1}{V} \sum_{\vec{k}} \frac{1}{k^2} = \frac{n-1}{\rho^2} \int_0^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}$$

$$= \frac{n-1}{\rho^2} \frac{S_d}{(2\pi)^d} \int_0^{\Lambda} \frac{k^{d-1}\, dk}{k^2} = \frac{n-1}{\rho^2} \frac{S_d}{(2\pi)^d} \int_0^{\Lambda} dk\, k^{d-3}$$

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<u>for</u>: $d>2$: integral converges $\Rightarrow$ fluctuations are finite

$d \le 2$: integral diverges $\Rightarrow$ <u>no</u> LRO at $\overline{T=0}$

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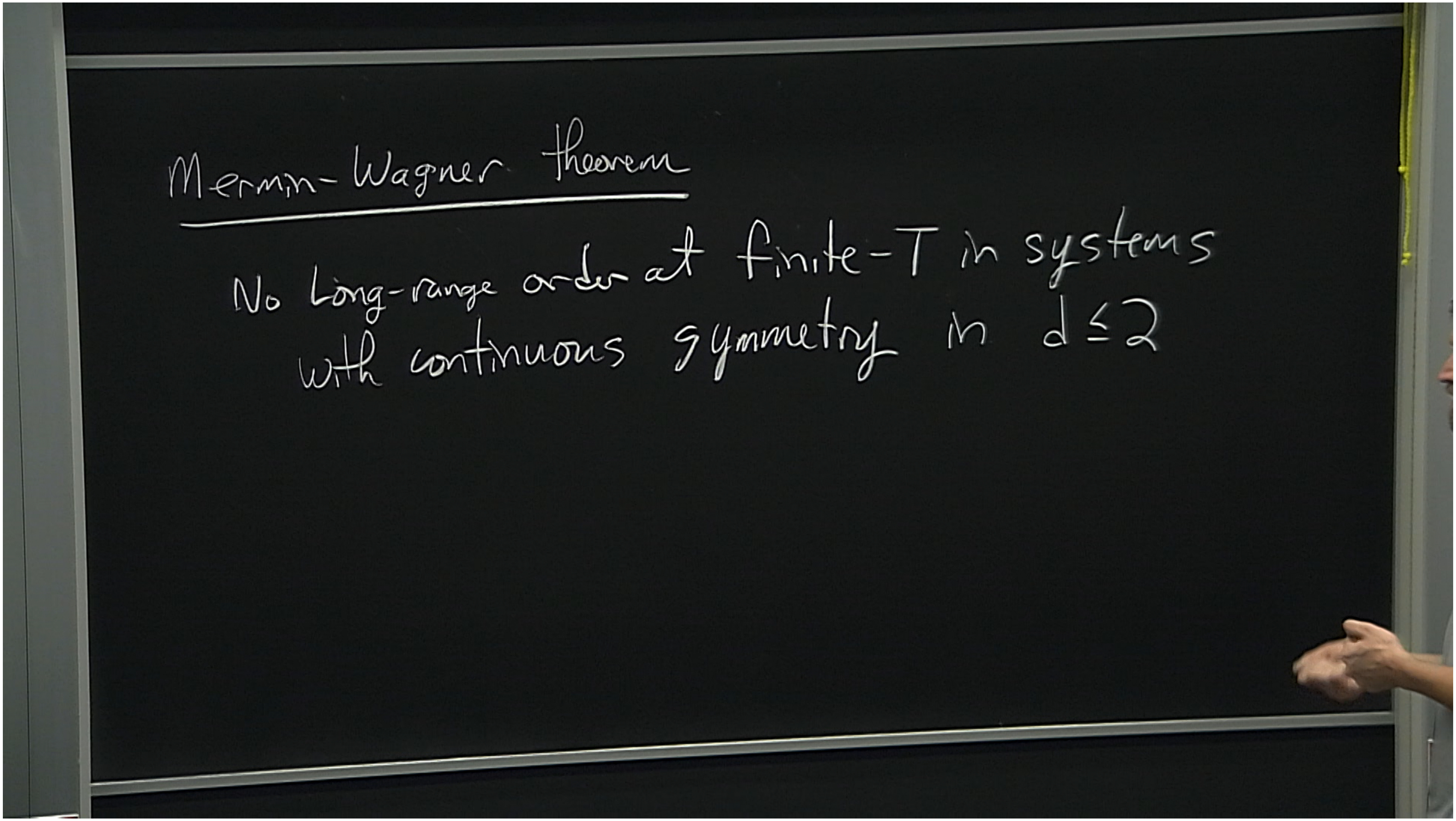
Mermin-Wagner theorem
No Long-range order at finite-T in systems
with continuous symmetry in d≤2

Mermin-Wagner theorem

No Long-range order at finite-T in systems with continuous symmetry in $d \leq 2$

$\Rightarrow$ the Goldstone modes are responsible for the Loss of LRO.

Mermin-Wagner theorem

No long-range order at finite-T in systems with continuous symmetry in $d \leq 2$

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