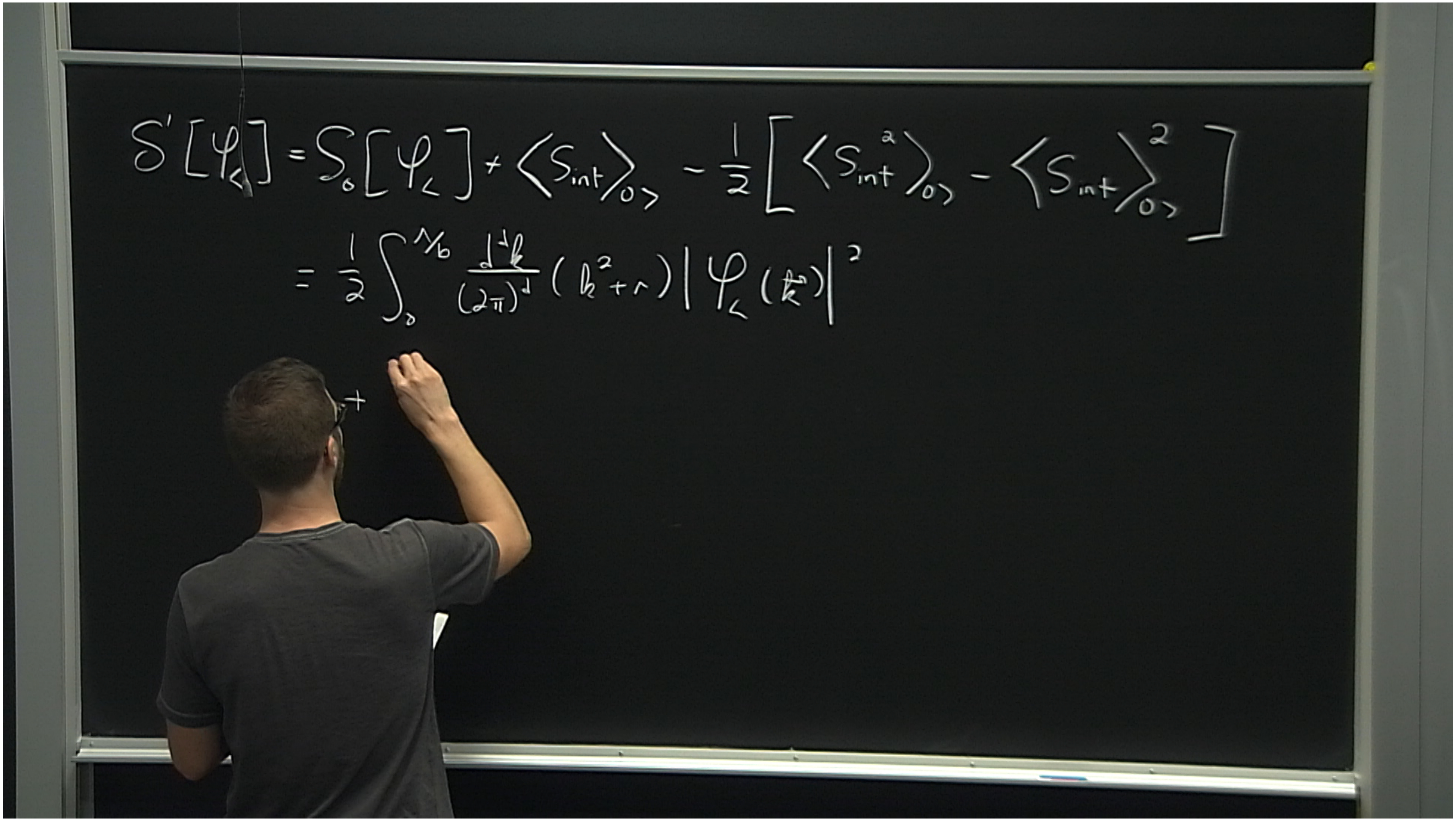
Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 17

Date: Mar 01, 2017 10:30 AM

URL: http://pirsa.org/17030001

Abstract:



$S'[\varphi_<] = S_0[\varphi_<] + \langle S_{int} \rangle_{0>} - \frac{1}{2}\left[\langle S_{int}^2 \rangle_{0>} - \langle S_{int} \rangle_{0>}^2 \right]$
$= \frac{1}{2} \int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} (k^2 + r) |\varphi_<(k)|^2$
$+$

$$S'[\varphi_<] = S_0[\varphi_<] + \langle S_{int}\rangle_{0>} - \frac{1}{2}\Big[\langle S_{int}^2\rangle_{0>} - \langle S_{int}\rangle_{0>}^2\Big]$$

$$= \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d}\,(k^2+r)\,|\varphi_<(k)|^2$$

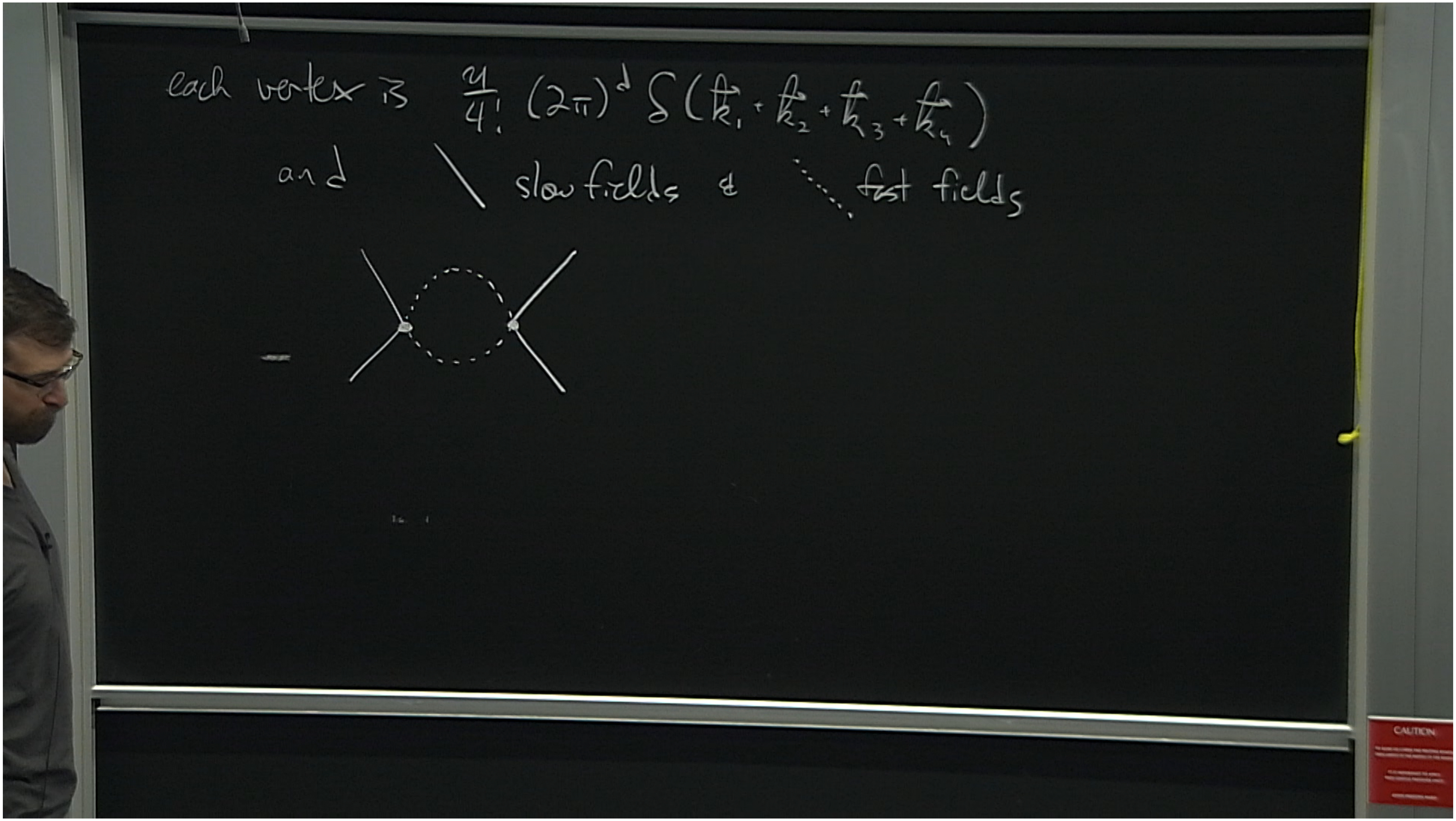
$$+ \frac{3u}{2}\, I_1 \int_0^{\Lambda} \frac{d^dk}{(2\pi)^d}\,|\varphi_<(k)|^2 \qquad \text{"D term"}$$

$$-\frac{1}{2}\Big\langle \;\;\; + \;\;\; + \cdots \Big\rangle_{0>} \qquad (25\ \text{terms})$$

fast

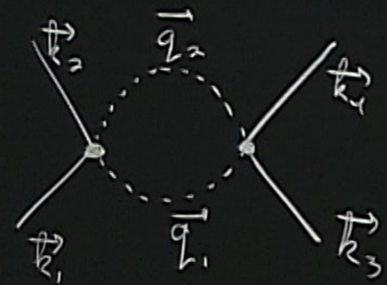
slow

$$+\frac{1}{2}\Big\langle \;\; + \;\; \cdots \Big\rangle_{0>}\Big\langle \;\; + \;\; + \cdots \Big\rangle_{0>} \qquad (5\ \text{in each})$$


each vertex is $\frac{\lambda}{4!}(2\pi)^d\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$
and slow fields & fast fields
−

each vertex is $\frac{y}{4!}(2\pi)^d\,\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$

and $\diagdown$ slow fields & $\cdots$ fast fields

$-$ 

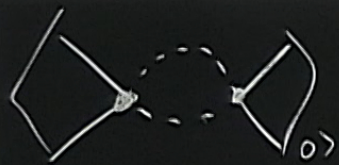
combinatorial factor of 36

δ-functions say

$$\left.\begin{aligned}\vec{k}_1+\vec{k}_2+\vec{q}_1+\vec{q}_2&=0\\ \vec{k}_3+\vec{k}_4-\vec{q}_1-\vec{q}_2&=0\end{aligned}\right\}\quad \vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4=0$$

<u>let</u>: $\vec{q}_1=\vec{q}$

$\vec{q}_2=-\vec{q}-\vec{k}_1-\vec{k}_2$



$$\Big\rangle_0 = \left(\frac{u}{4}\right)^2 \int_{k_1 \cdots k_4} (2\pi)^d \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4) \cdot$$

$$\cdot\, \varphi_<(\vec{k}_1)\,\varphi_<(\vec{k}_2)\,\varphi_<(\vec{k}_3)\,\varphi_<(\vec{k}_4)$$

$$\cdot \underbrace{\int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d}\, G_0(\vec{q})\, G_0(-\vec{k}_1-\vec{k}_2-\vec{q})}_{I_2}$$

Note:

$$\Big|_{0>} = \left(\frac{u}{4}\right)^2 \int_{k_1 \cdots k_4} (2\pi)^d \delta\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4\right) \cdot$$

$$\cdot\, \varphi_<(\vec{k}_1)\, \varphi_<(\vec{k}_2)\, \varphi_<(\vec{k}_3)\, \varphi_<(k_4)$$

$$\cdot\, \underbrace{\int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d}\, G_0(\vec{q})\, G_0(-\vec{k}_1 - \vec{k}_2 - \vec{q})}_{I_2}$$

Note:

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(r + q^2)} \frac{1}{\left[r + (-\vec{k}_1 - \vec{k}_2 - \vec{q})^2\right]}$$

$$\int_{k_1 \cdots k_4}$$

$$\cdot\, \varphi_<(\vec{k}_1)\,\varphi_<(\vec{k}_2)\,\varphi_<(\vec{k}_3)\,\varphi_<(k_4)$$

$$\cdot\, \underbrace{\int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\, G_0(\vec{q})\, G_0(-\vec{k}_1-\vec{k}_2-\vec{q})}_{I_2}$$

Note:

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\, \frac{1}{(r+q^2)}\, \frac{1}{\left[r+(-\vec{k}_1-\vec{k}_2-\vec{q})^2\right]}$$

$$\underbrace{\int_{\Lambda/b} \frac{\ }{(2\pi)^d} \cdots}_{I_2}$$

Note:

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(r+q^2)} \frac{1}{\left[r + (-\vec{k}_1 - \vec{k}_2 - \vec{q})^2\right]}$$

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(r+q^2)^2} \left[1 - \frac{2\vec{q}(-\vec{k}_1 - \vec{k}_2) + (-\vec{k}_1 - \vec{k}_2)^2}{(r+q^2)} + \ldots\right]$$

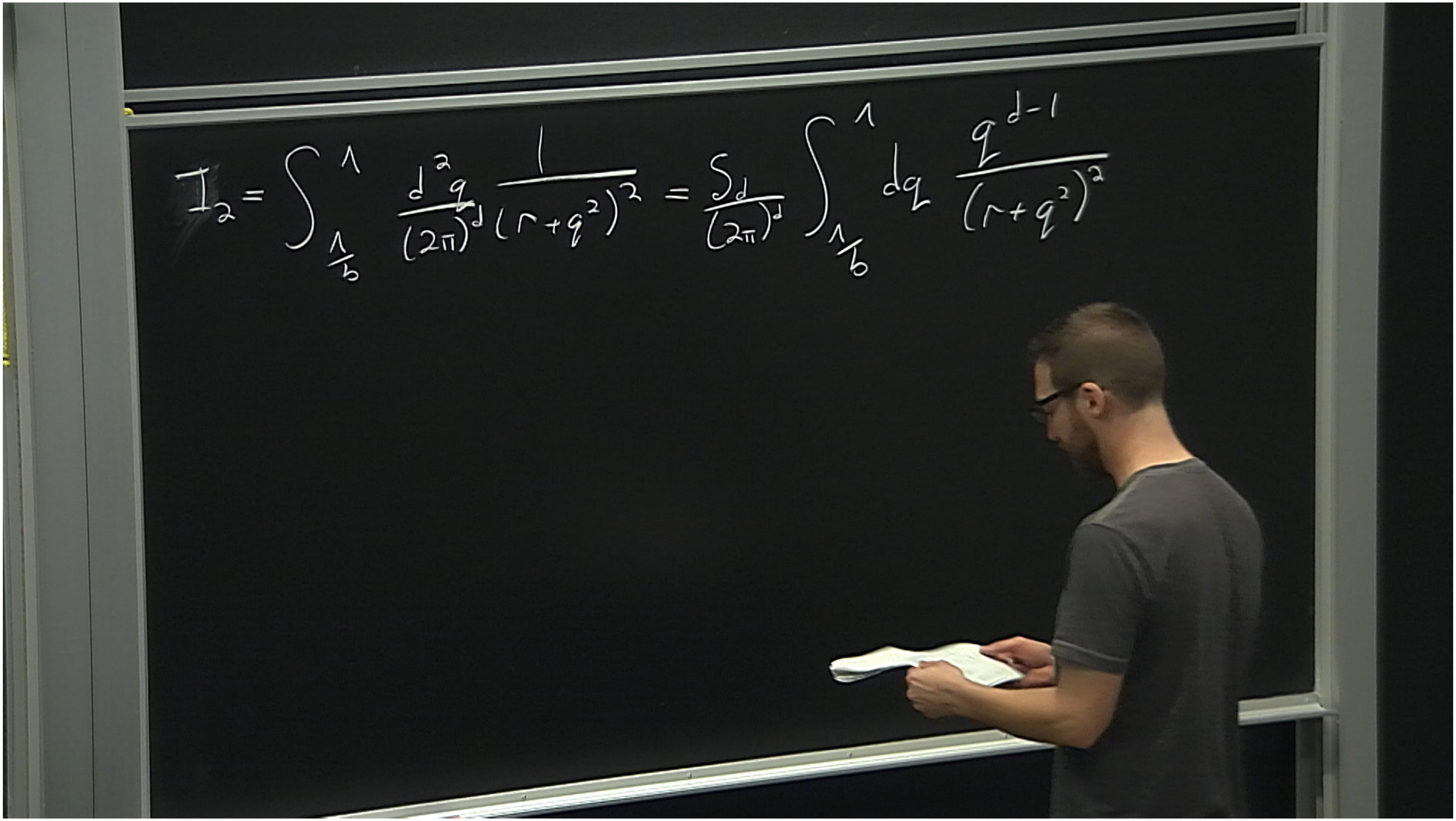
$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\,\frac{1}{(r+q^2)^2}\left[1 - \underbrace{\frac{2\vec{q}\,(-\vec{k}_1-\vec{k}_2) + (-\vec{k}_1-\vec{k}_2)^2}{(r+q^2)} + \dots}\right]$$

after F.T. back into real space
these give you additional terms
past φ^4: $\varphi^2(\nabla\varphi)^2$, $\varphi^2\nabla^2\varphi^2$
etc.

so call

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\,\frac{1}{(r+q^2)^2}$$

and ignore for now the
fact that parameter space
isn't closed at φ^4



$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^2 q}{(2\pi)^d} \frac{1}{(r+q^2)^2} = \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} dq \, \frac{q^{d-1}}{(r+q^2)^2}$

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(r+q^2)^2} = \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} dq \, \frac{q^{d-1}}{(r+q^2)^2}$$

$$= \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} dq \, \frac{q^{d-1}}{q^4} \left[1 - \frac{2r}{q^2} + \cdots \right]$$

$$= \frac{S_d}{(2\pi)^d} \int_{\Lambda/b} dq \, \frac{q^{d-1}}{q^4} \left[1 - \frac{2r}{q^2} + \dots \right]$$

$$= \frac{S_d}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} dq \left[q^{d-5} - 2r\, q^{d-7} + \dots \right]$$

$$= \frac{S_d}{(2\pi)^d} \left[\frac{\Lambda^{d-4}}{d-4} \left(1 - \left(\frac{1}{b}\right)^{d-4} \right) - 2r \, \frac{\Lambda^{d-6}}{d-6} \left(1 - \left(\frac{1}{b}\right)^{d-6} \right) + \dots \right]$$

use again $1 - \left(\frac{1}{b}\right)^n = 1 - e^{-n\Delta l} \cong n\Delta l$

$$I_2 = \frac{S_d}{(2\pi)^d}\left[\Lambda^{d-4} - 2r\Lambda^{d-6} + \cdots\right]\Delta l$$

$$= \frac{S_d}{(2\pi)^d}\frac{\Lambda^d \Delta l}{(r+\Lambda^2)^2}$$

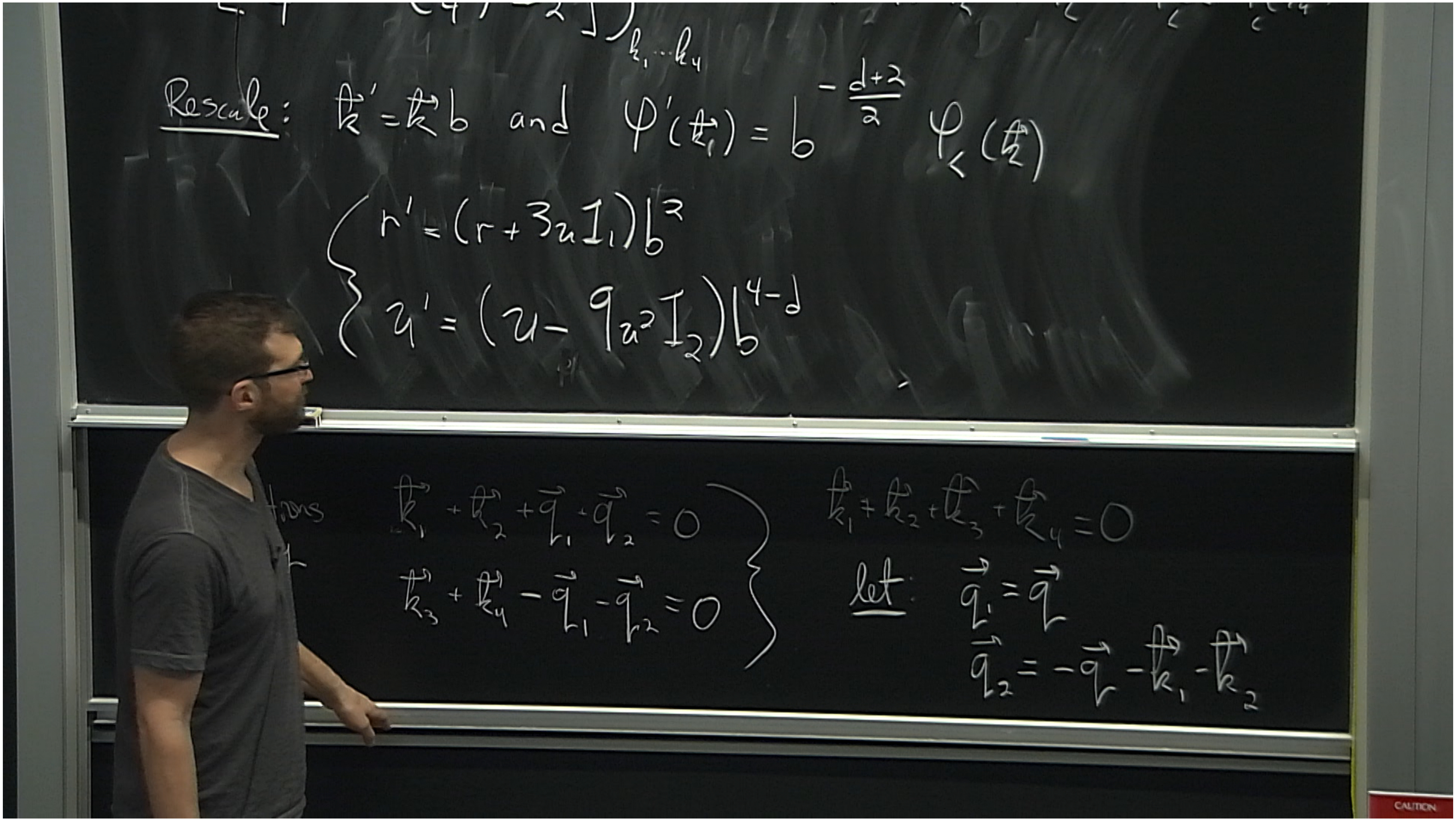
use again $1-\left(\frac{1}{b}\right)^{n} = 1 - e^{-n\delta l} \cong n\delta l$

$$I_2 = \frac{S_d}{(2\pi)^d}\left[\Lambda^{d-4} - 2r\Lambda^{d-6} + \cdots\right]\delta l$$

$$= \frac{S_d}{(2\pi)^d}\,\frac{\Lambda^d\,\delta l}{\left(r+\Lambda^2\right)^2}$$

 $= \left(\frac{u}{4}\right)^2 I_2 \int_{k_1\cdots k_4} (2\pi)^d\,\delta(\vec{k}_1+\cdots+\vec{k}_4)\,\varphi_<(\vec{k}_1)\,\varphi_<(\vec{k}_2)\,\varphi_<(\vec{k}_3)\,\varphi(\vec{k}_4)$

Collect all terms up to second order in u to get S'


Rescale: $\vec{k}' = \vec{k} b$ and $\varphi'(\vec{k}_1) = b^{-\frac{d+2}{2}} \varphi_<(\vec{k})$
$r' = (r + 3u I_1) b^2$
$u' = (u - 9u^2 I_2) b^{4-d}$
$\vec{k}_1 + \vec{k}_2 + \vec{q}_1 + \vec{q}_2 = 0$
$\vec{k}_3 + \vec{k}_4 - \vec{q}_1 - \vec{q}_2 = 0$
$\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = 0$
let: $\vec{q}_1 = \vec{q}$
$\vec{q}_2 = -\vec{q} - \vec{k}_1 - \vec{k}_2$

In differential form

$$\begin{cases} \dfrac{dr}{dl} = 2r + 3u \dfrac{S_d}{(2\pi)^d} \dfrac{\Lambda^d}{(r+\Lambda^2)} \\[2ex] \dfrac{du}{dl} = (4-d)u - 9u^2 \dfrac{S_d}{(2\pi)^d} \dfrac{\Lambda^d}{(r+\Lambda^2)^2} \end{cases}$$

In differential form

$$\begin{cases} \dfrac{dr}{dl} = 2r + 3u \dfrac{S_d}{(2\pi)^d} \dfrac{\Lambda^d}{(r+\Lambda^2)} \\ \dfrac{du}{dl} = (4-d)u - 9u^2 \dfrac{S_d}{(2\pi)^d} \dfrac{\Lambda^d}{(r+\Lambda^2)^2} \end{cases}$$

solving $\dfrac{dr}{dl} = \dfrac{du}{dl} = 0$

$$u^* \cong \frac{(4-d)}{9} \frac{(2\pi)^d}{S_d} \Lambda^{4-d} \quad \text{near } d=4$$

In differential form

$$\begin{cases} \dfrac{dr}{dl} = 2r + 3u\,\dfrac{S_d}{(2\pi)^d}\,\dfrac{\Lambda^d}{(r+\Lambda^2)} \\[2ex] \dfrac{du}{dl} = (4-d)u - 9u^2\,\dfrac{S_d}{(2\pi)^d}\,\dfrac{\Lambda^d}{(r+\Lambda^2)^2} \end{cases}$$

solving $\dfrac{dr}{dl} = \dfrac{du}{dl} = 0$

$$u^* \cong \frac{(4-d)}{9}\,\frac{(2\pi)^d}{S_d}\,\Lambda^{4-d} \cong \frac{\varepsilon}{9}\,\frac{(2\pi)^4}{S_4} \qquad \frac{\text{near}}{d=4}$$

In differential form

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solving $\dfrac{dr}{dl} = \dfrac{du}{dl} = 0$

$$u^* \cong \frac{(4-d)}{9} \frac{(2\pi)^d}{S_d} \Lambda^{4-d} \cong \frac{\varepsilon}{9} \frac{(2\pi)^4}{S_4} \qquad \frac{\text{near}}{d=4}$$

also $S_4 = \frac{2\pi^{4/2}}{\Gamma(4/2)} = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$

$$u^* = \frac{\varepsilon}{9}\,\frac{16\pi^4}{2\pi^2} = \frac{8}{9}\,\varepsilon\pi^2$$

$$= \frac{\ }{6}\Lambda$$

$$\boxed{r^* = -\frac{\varepsilon}{6}\Lambda^2}$$

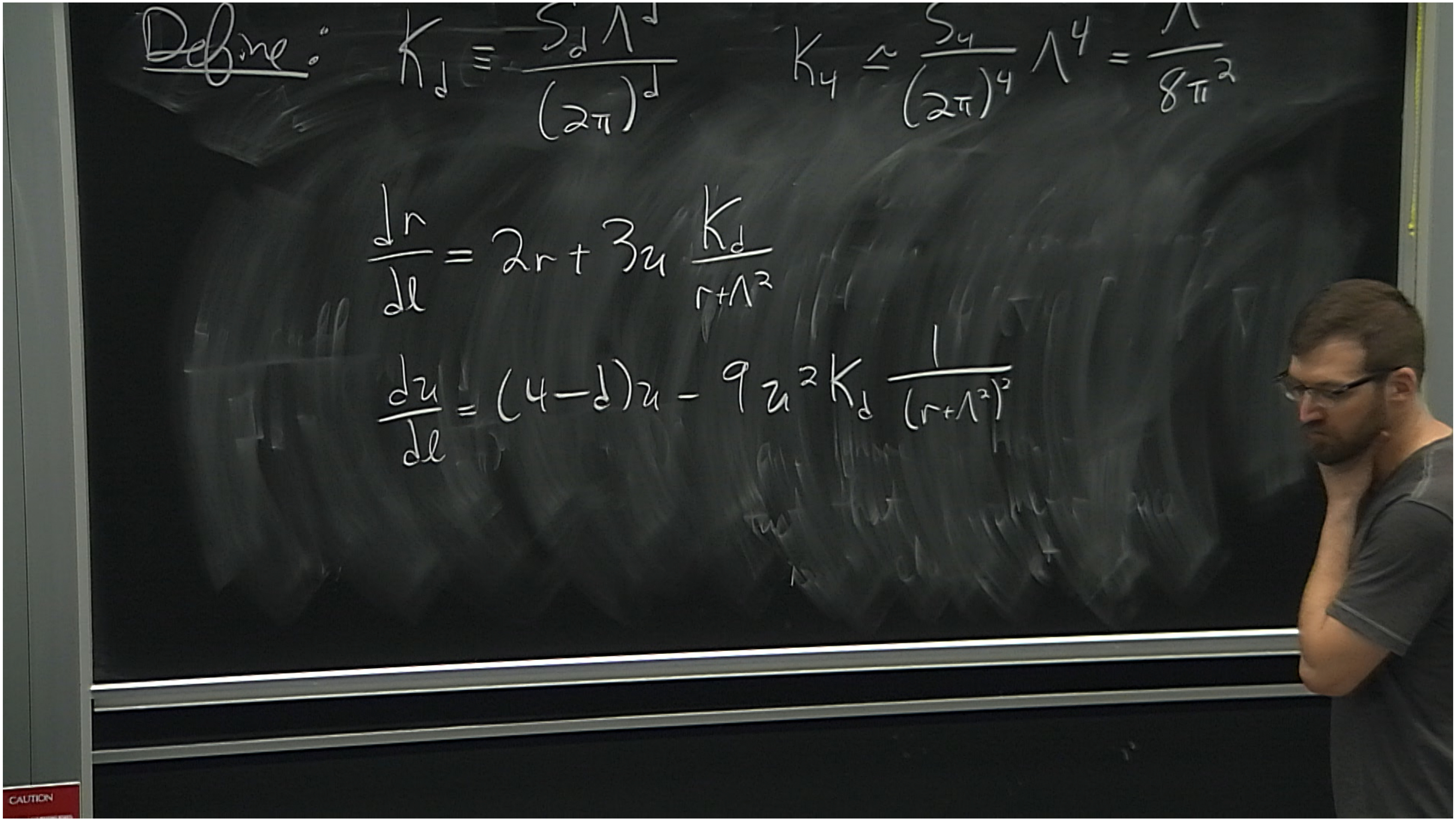
fixed point

after F.T. back into real space these give you additional terms past φ^4: $\varphi^2(\nabla\varphi)^2$, $\varphi^2\nabla^2\varphi^2$ etc.

so call

$$I_2 = \int_{\Lambda/b}^{\Lambda} \frac{d^dq}{(2\pi)^d}\frac{1}{(r+q^2)^2}$$

and ignore for now the fact that parameter space isn't closed at φ^4


Define:
$K_d \equiv \frac{S_d \Lambda^d}{(2\pi)^d}$
$K_4 \simeq \frac{S_4}{(2\pi)^4}\Lambda^4 = \frac{\Lambda^4}{8\pi^2}$
$\frac{dr}{dl} = 2r + 3u \frac{K_d}{r+\Lambda^2}$
$\frac{du}{dl} = (4-d)u - 9u^2 K_d \frac{1}{(r+\Lambda^2)^2}$
CAUTION

Define: $K_d \equiv \frac{S_d \Lambda^d}{(2\pi)^d}$ $\qquad K_4 \simeq \frac{S_4}{(2\pi)^4}\Lambda^4 = \frac{\Lambda^4}{8\pi^2}$

$$\begin{cases} \dfrac{dr}{dl} = 2r + 3u\,\dfrac{K_d}{r+\Lambda^2} \\[2ex] \dfrac{du}{dl} = (4-d)u - 9u^2 K_d \dfrac{1}{(r+\Lambda^2)^2} \end{cases}$$

$$\delta K' = b\ \delta K \qquad \text{using} \quad b =$$

$$\delta K' = e^{y\delta l}\,\delta K \simeq (1 + y\delta l)\,\delta K$$

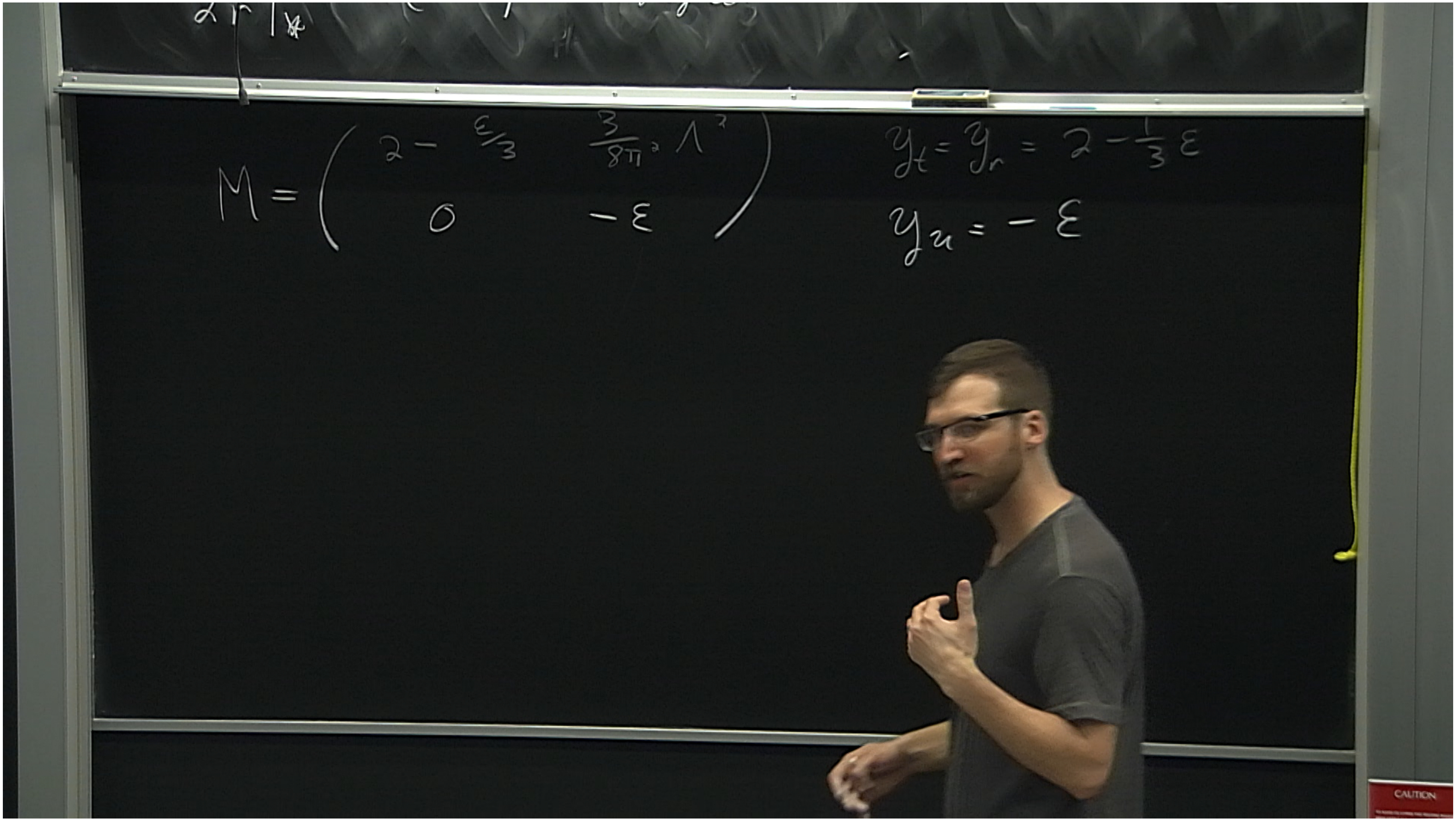
$$\frac{\delta K' - \delta K}{\delta l} = y\,\delta K \Rightarrow \boxed{\frac{d}{dl}\delta K = y\,\delta K}$$

So y is the eigenvalue of a "M" matrix

$$\frac{d}{d\ell}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix} = M\begin{pmatrix}\delta r\\ \delta u\end{pmatrix}$$

where: define $\frac{dr}{d\ell} = \beta_r$ and $\frac{du}{d\ell} = \beta_u$

and $$M = \begin{pmatrix}\frac{\partial \beta_r}{\partial r} & \frac{\partial \beta_r}{\partial u}\\ \frac{\partial \beta_u}{\partial r} & \frac{\partial \beta_u}{\partial u}\end{pmatrix}_{r^*, u^*}$$


$M = \begin{pmatrix} 2 - \varepsilon/3 & \frac{3}{8\pi^2}\Lambda^2 \\ 0 & -\varepsilon \end{pmatrix}$
$y_t = y_r = 2 - \frac{1}{3}\varepsilon$
$y_u = -\varepsilon$
CAUTION

$$M = \begin{pmatrix} 2 - \varepsilon/3 & \frac{3}{8\pi^2}\Lambda^2 \\ 0 & -\varepsilon \end{pmatrix}$$

$y_t = y_r = 2 - \frac{1}{3}\varepsilon$

$y_u = -\varepsilon$

check: $\vec{e}^{\,t} = (1,0)$, $\vec{e}^{\,u} = \left(-\frac{3\Lambda^2}{16\pi^2}, 1\right)$

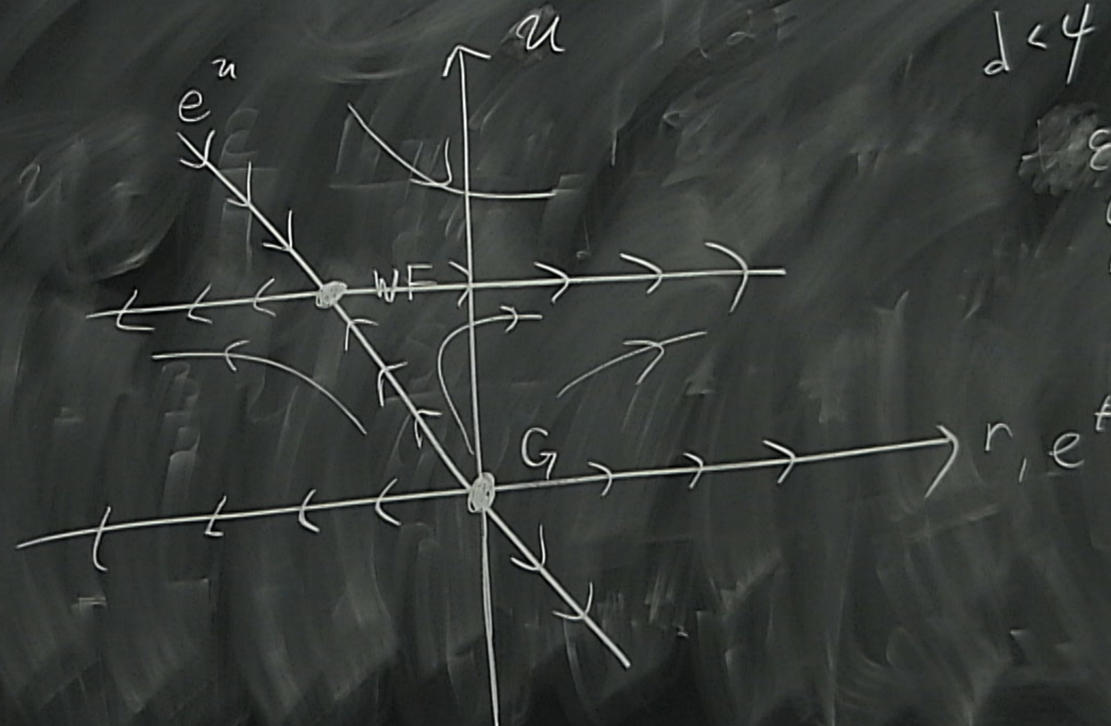
these should be the same eigenvectors as the 1st cumulant case.

Let's look at $d<4$: Gaussian F.P. at $u^* = r^* = 0$

W.F. at $(r^*, u^*) = \left(-\frac{1}{6}\varepsilon\Lambda^2, \frac{8}{9}\varepsilon\pi^2\right)$

Define: $K = S_d \Lambda^d$ $\quad K \;\; S_4 \;\; \Lambda^4 \quad \Lambda$

$$(r^*, u^*) = \frac{8}{9}\,\varepsilon\pi^2\left(-\frac{3}{16}\,\frac{\Lambda^2}{\pi^2}, 1\right) = \frac{8}{9}\,\varepsilon\pi^2\,\vec{e}^{\,u}$$



$d < 4$

$\varepsilon = 4 - d > 0$

y_t : relevant

y_u : irrelevant

Basin of attraction

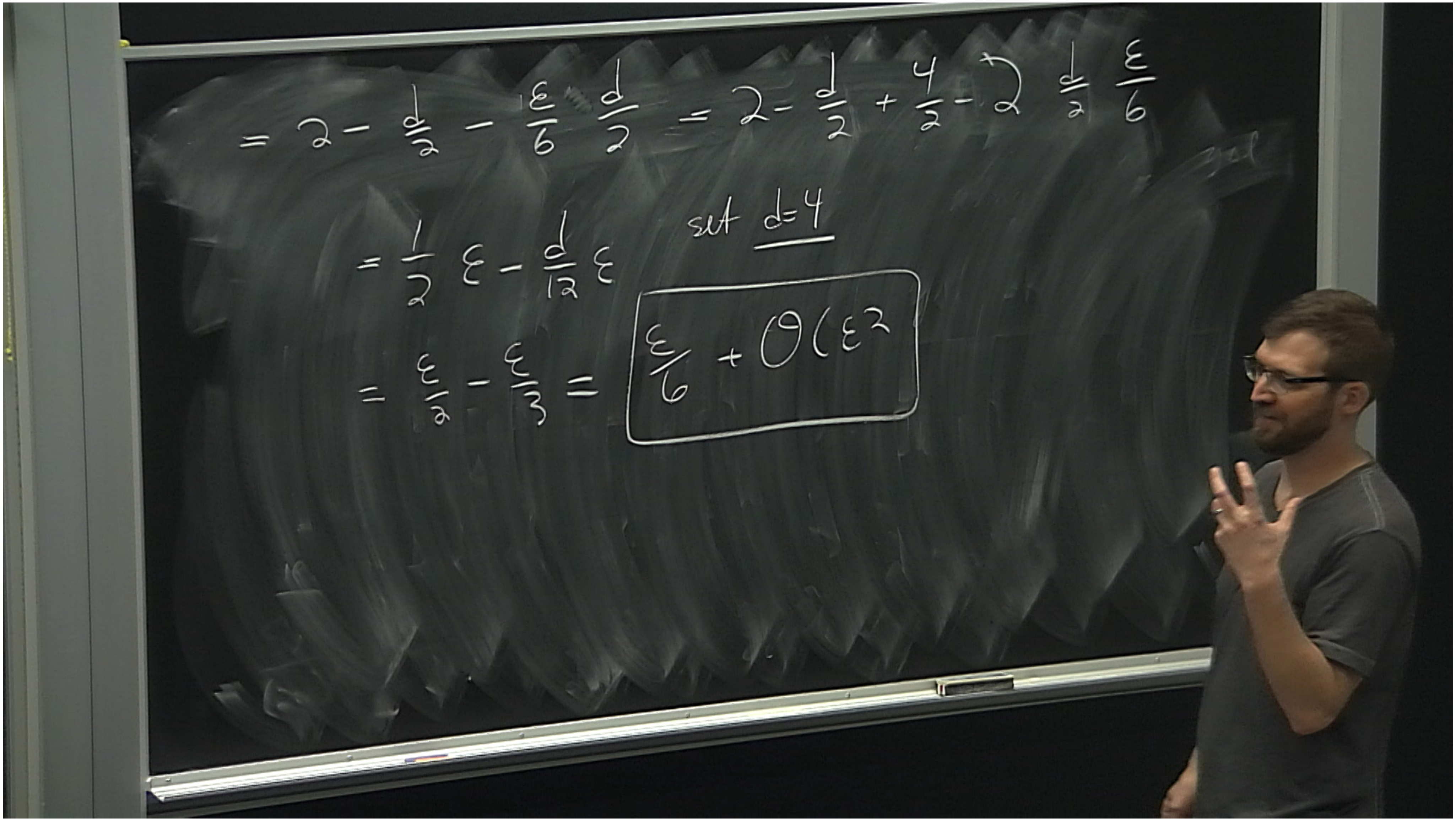
All LGW functions with (r, u) along the $\boxed{\text{line}}$ joining the G and WF fixed points flow towards the WF.

→ Other points flow to $\underset{\text{PM}}{r \to \infty}$ or $\underset{\text{FM}}{r \to -\infty}$

Critical exponents

$$\nu = \frac{1}{y_t} = \frac{1}{2 - \frac{1}{3}\varepsilon} = \frac{1}{2}\left(\frac{1}{1 - \frac{\varepsilon}{6}}\right) \simeq \frac{1}{2} + \frac{\varepsilon}{12} + \mathcal{O}(\varepsilon^2)$$

$$\alpha = 2 - \frac{d}{y_t} = 2 - d\,\frac{1}{2 - \frac{\varepsilon}{3}} = 2 - \frac{d}{2}\left(\frac{1}{1 - \frac{\varepsilon}{6}}\right) \simeq 2 - \frac{d}{2}\left(1 + \frac{\varepsilon}{6}\right)$$

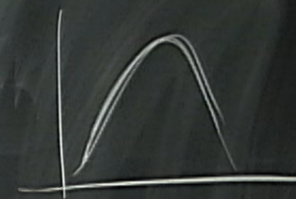

set d=4

$$= 2 - \frac{d}{2} - \frac{\varepsilon}{6}\frac{d}{2} = 2 - \frac{d}{2} + \frac{4}{2} - 2\,\frac{d}{2}\,\frac{\varepsilon}{6}$$

$$= \frac{1}{2}\varepsilon - \frac{d}{12}\varepsilon$$

set $\underline{d=4}$

$$= \frac{\varepsilon}{2} - \frac{\varepsilon}{3} = \boxed{\frac{\varepsilon}{6} + O(\varepsilon^2}$$



For many other exponents you need a magnetic field

$$S[\varphi] = \int d^d x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4 - B\varphi \right]$$

$$h' = h\, b^{1+\frac{d}{2}} \quad \text{(exercise)}$$

$$y_h = 1 + \frac{d}{2} = 1 + \frac{d}{2} - \frac{4}{2} + \frac{4}{2} = 3 - \frac{1}{2}\epsilon$$

$$\gamma = \frac{2y_h - d}{y_t} = \frac{6 + \frac{d}{2} - d}{2 - \frac{\varepsilon}{3}} = \left(6 - \frac{d}{2}\right)\frac{1}{2}\left(1 + \frac{\varepsilon}{6}\right) = 1 + \frac{\varepsilon}{6}$$

$$\beta = \frac{d - y_h}{y_t} = \frac{1}{2} - \frac{\varepsilon}{6} \quad \text{(check)}$$

$$\delta = \frac{y_h}{d - y_h} = 3 + \varepsilon$$

$$h' = h\, b^{1+\frac{d}{2}} \quad \text{(exercise)}$$

$$y_h = 1 + \frac{d}{2} = 1 + \frac{d}{2} - \frac{4}{2} + \frac{4}{2} = 3 - \frac{1}{2}\epsilon$$

$$\gamma = \frac{2y_h - d}{y_t} = \frac{6 + \frac{d}{2} - d}{2 - \frac{\epsilon}{3}} = \left(6 - \frac{d}{2}\right)\frac{1}{2}\left(1 + \frac{\epsilon}{6}\right) = 1 + \frac{\epsilon}{6}$$

$$\beta = \frac{d - y_h}{y_t} = \frac{1}{2} - \frac{\epsilon}{6} \quad \text{(check)}$$

$$\delta = \frac{y_h}{d - y_h} = 3 + \epsilon$$