Title: PSI 2016/2017 Introduction to Quantum Integrable Systems - Lecture 4

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Abstract:

$$E\left(S_z=\frac{L}{2}-N\right)=\sum_{j=1}^{N}\underbrace{\frac{\lambda}{u_j^2+\frac{1}{4}}}_{\epsilon(p_j)}$$

$\epsilon(p_j)$ energy of each magnon

Where u_j are quantized through Bethe equations

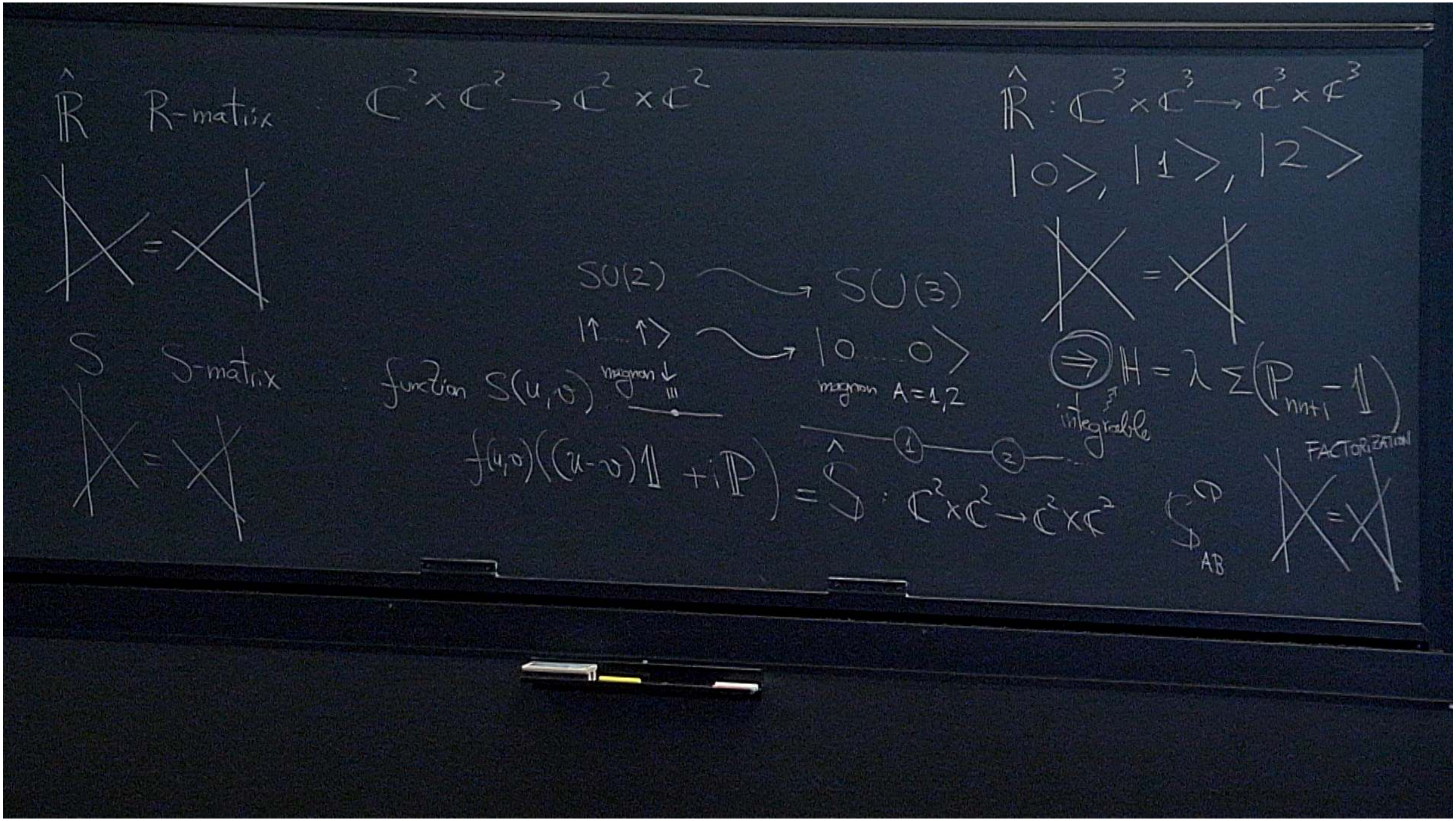
$S(p_j,p_k)$

$$\underbrace{\left(\frac{u_j+i/2}{u_j-i/2}\right)^L}_{e^{ip_jL}}\prod_{k\neq j}^{N}\overbrace{\frac{u_j-u_k-i}{u_j-u_k+i}}^{S(p_j,p_k)}=1=$$

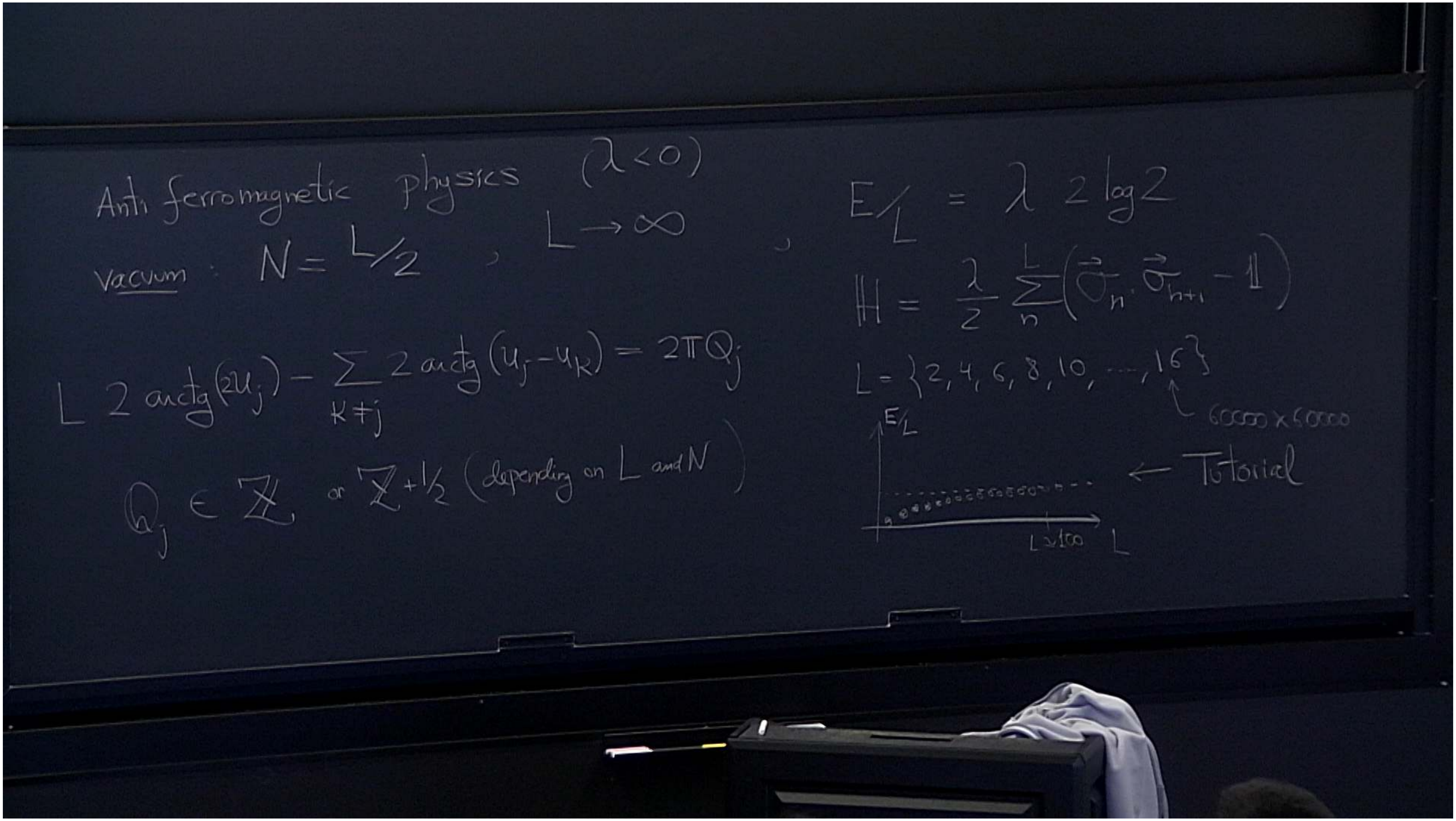
FACTORIZATION

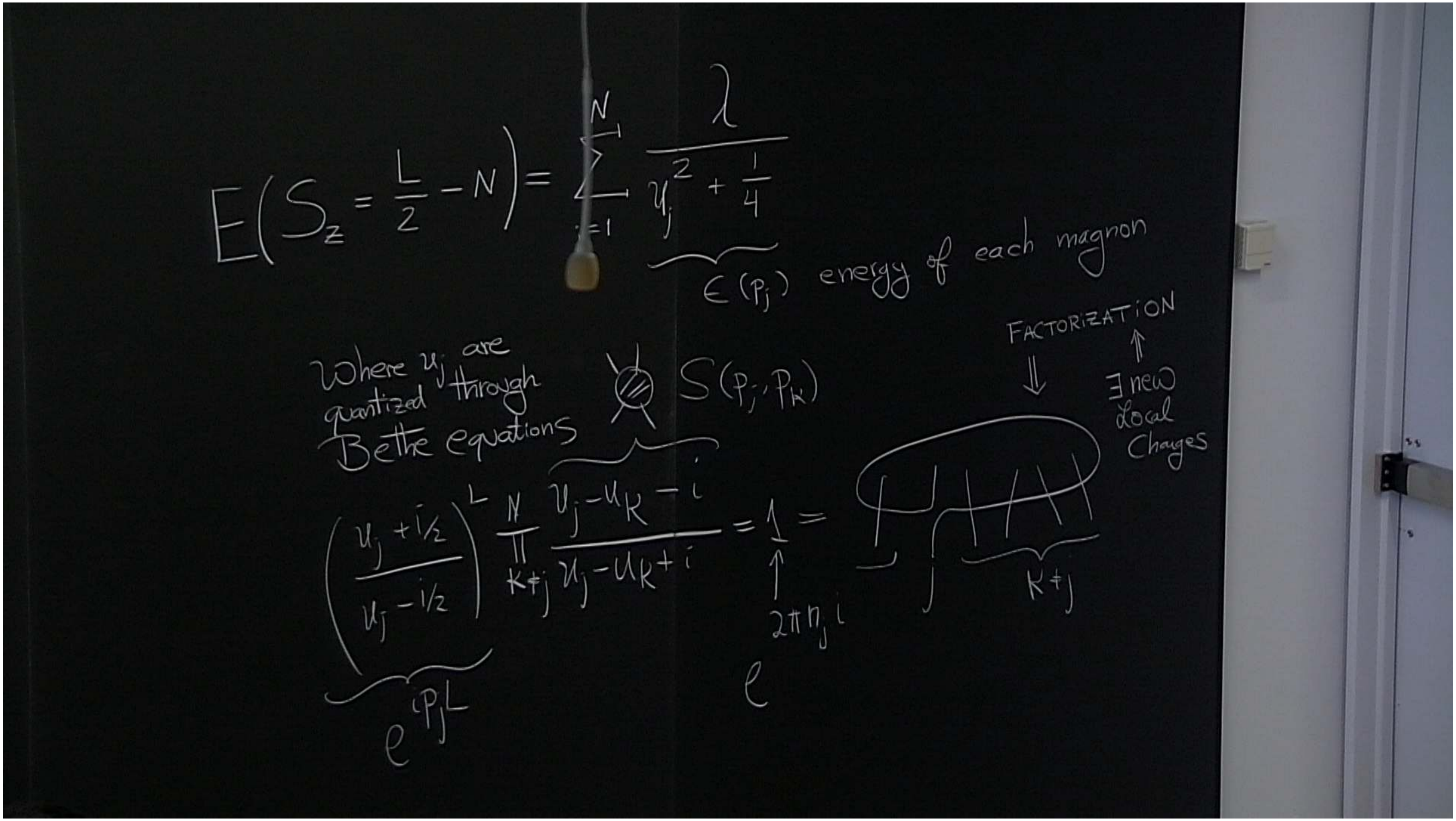
$\Downarrow$ $\Uparrow$

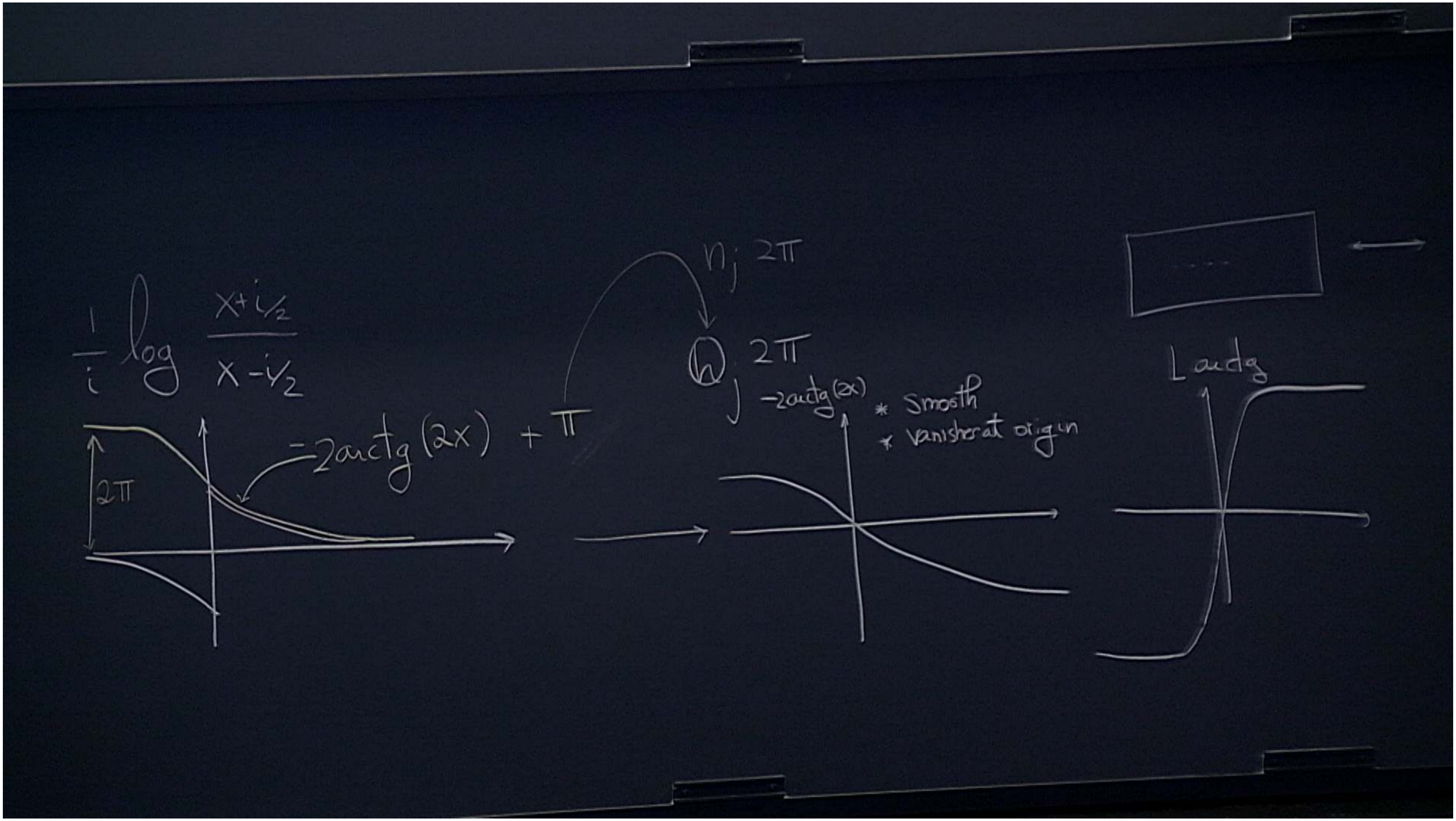
$\exists$ new Local Charges



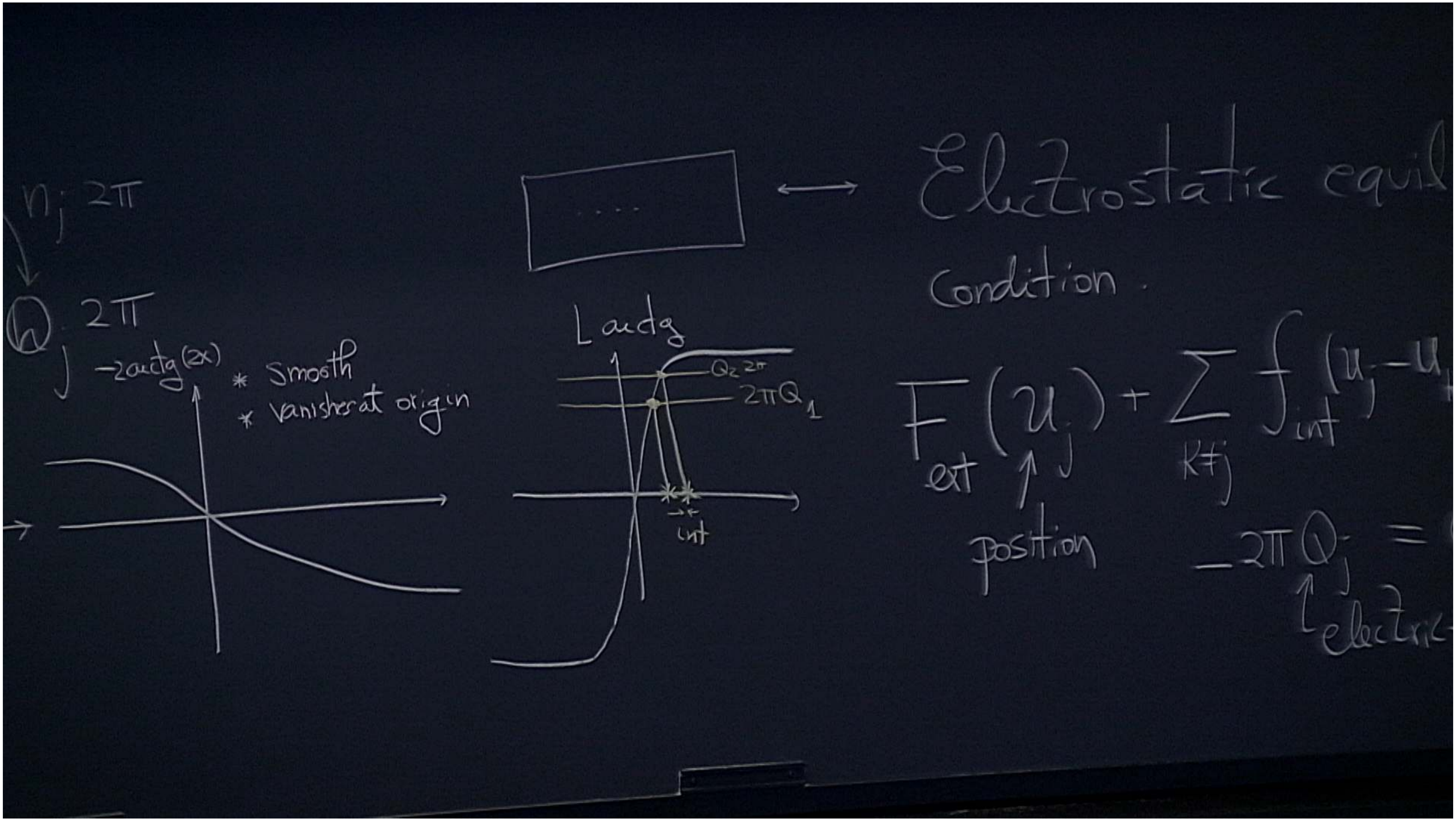
$\hat{\mathbb{R}}$ R-matrix : $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$
$\hat{\mathbb{R}} : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3 \times \mathbb{C}^3$
$|0\rangle, |1\rangle, |2\rangle$
$SU(2) \rightsquigarrow SU(3)$
$|\uparrow \ldots \uparrow\rangle \rightsquigarrow |0 \ldots 0\rangle$
S S-matrix
function $S(u,v)$
magnon
magnon $A=1,2$
$\Rightarrow H = \lambda \sum (\mathbb{P}_{n\,n+1} - \mathbb{1})$
integrable
$f(u,v)\big((u-v)\mathbb{1} + i\mathbb{P}\big) = \hat{S} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$
S_{AB}^{CD}
FACTORIZATION

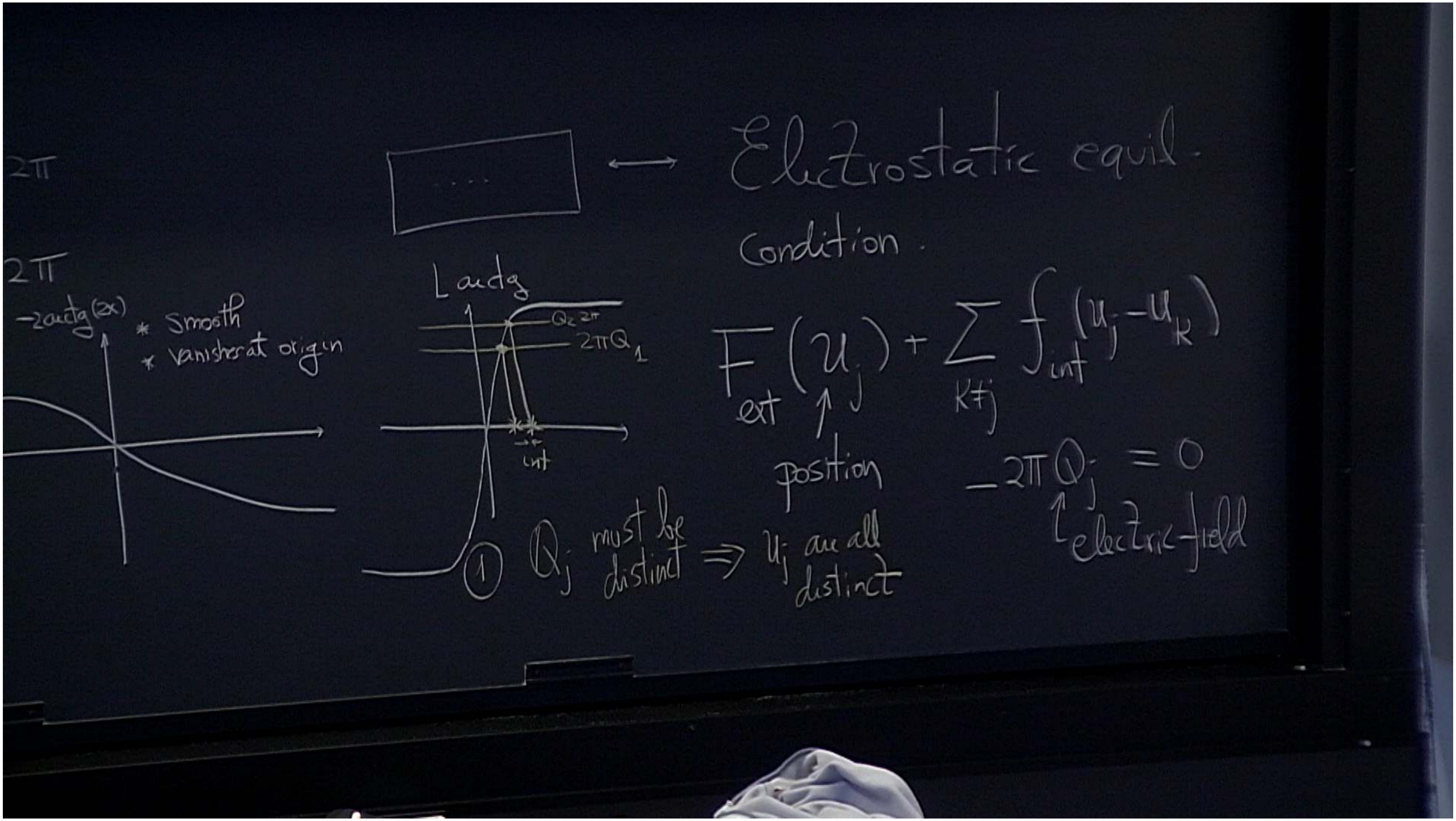

Anti ferromagnetic physics $(\lambda < 0)$
vacuum : $N = L/2$, $L \to \infty$
$L \, 2 \, \mathrm{arctg}(2u_j) - \sum_{k \neq j} 2 \, \mathrm{arctg}(u_j - u_k) = 2\pi Q_j$
$Q_j \in \mathbb{Z}$ or $\mathbb{Z} + 1/2$ (depending on L and N)
$E/L = \lambda \, 2 \log 2$
$\mathbb{H} = \frac{\lambda}{2} \sum_{n}^{L} \left(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - \mathbb{1} \right)$
$L = \{2, 4, 6, 8, 10, \ldots, 16\}$
60000 x 60000
E/L
← Tutorial
$L \gtrsim 100$
L

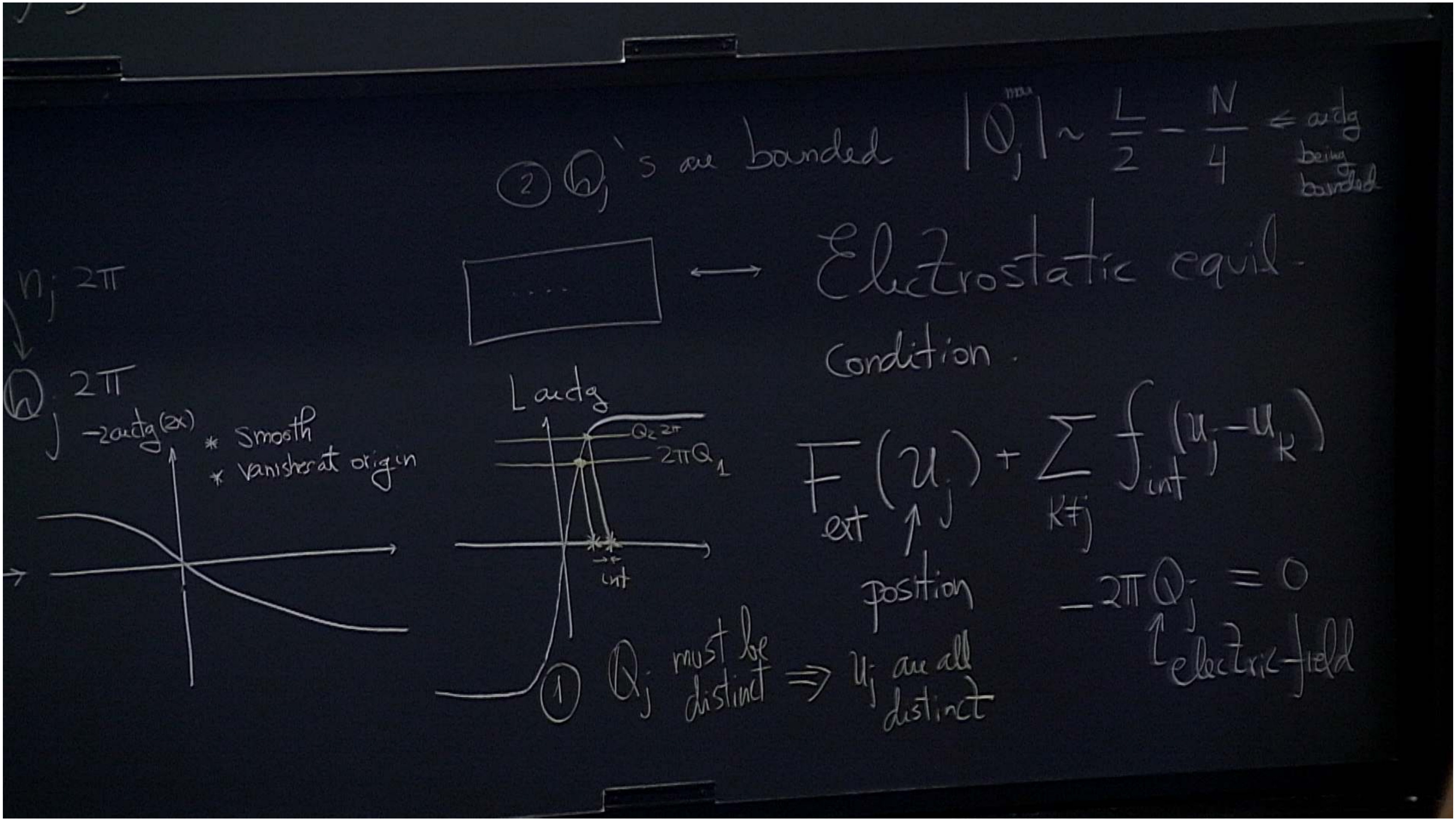

$E(S_z = \frac{L}{2} - N) = \sum_{j=1}^{N} \frac{\lambda}{u_j^2 + \frac{1}{4}}$
$\epsilon(p_j)$ energy of each magnon
FACTORIZATION
∃ new Local Charges
Where u_j are quantized through Bethe equations
$S(p_j, p_k)$
$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L \prod_{k \neq j}^{N} \frac{u_j - u_k - i}{u_j - u_k + i} = 1 =$
e^{ip_jL}
$e^{2\pi n_j i}$
$k \neq j$



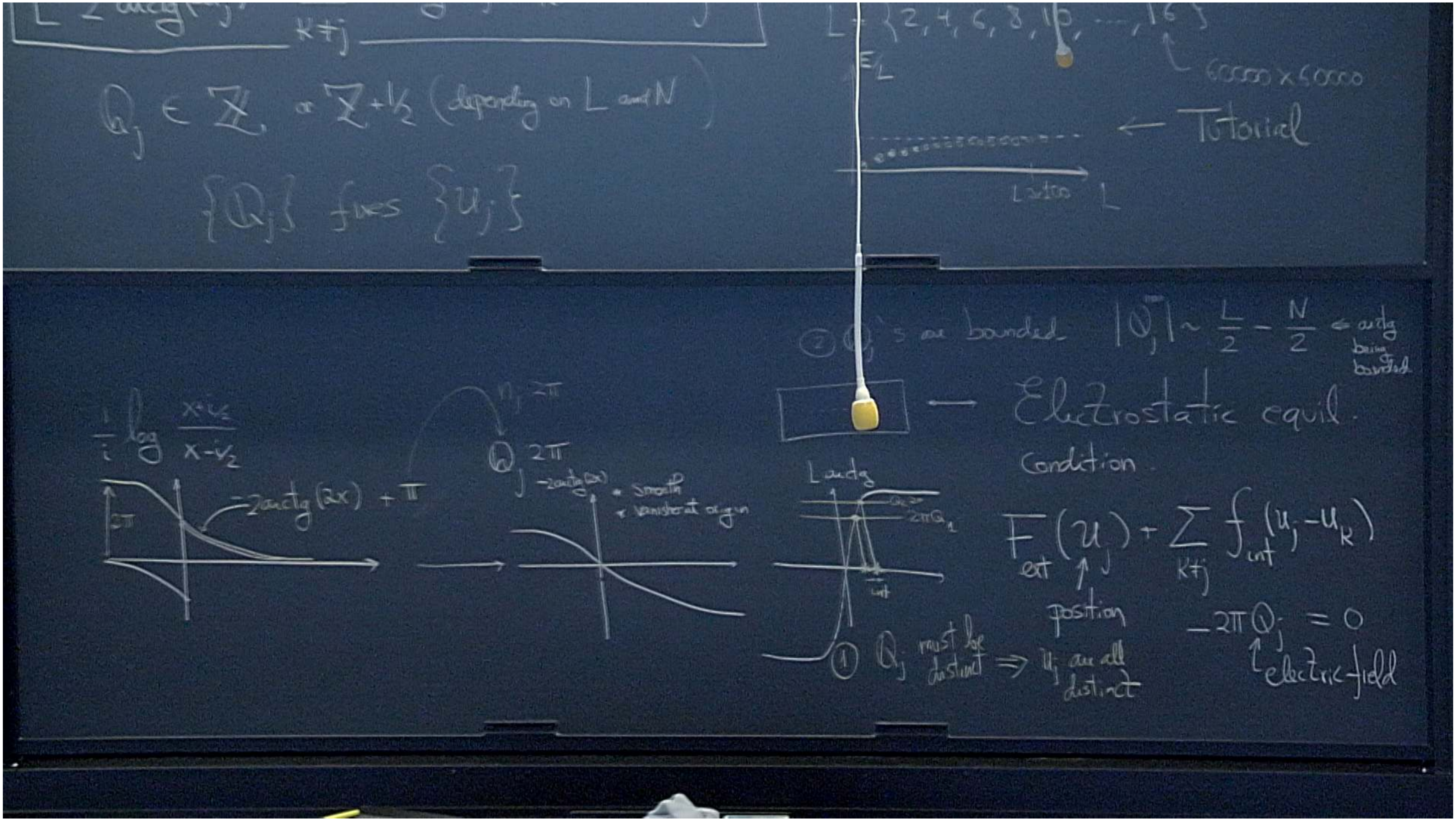
$\frac{1}{i} \log \frac{x+i/2}{x-i/2}$
$-2\operatorname{arctg}(2x) + \pi$
2π
$n_j \, 2\pi$
$h_j \, 2\pi$
$-2\operatorname{arctg}(2x)$
* Smooth
* vanisher at origin
L arctg


n_j 2π
h_j 2π
−2arctg(2x)
* smooth
* vanishes at origin
L arctg
Q_2 2π
2πQ_1
int
Electrostatic equil
condition.
F_ext(u_j) + Σ_{k≠j} f_int(u_j − u
position
− 2πQ_j =
electric


Electrostatic equil.
Condition.
Larctg
* Smooth
* vanishes at origin
$-2\arctan(2x)$
$2\pi Q_1$
int
$F_{ext}(u_j) + \sum_{k\neq j} f_{int}(u_j - u_k) - 2\pi Q_j = 0$
position
electric field
(1) Q_j must be distinct $\Rightarrow$ u_j are all distinct



2 Q_j 's are bounded |Q_j^max| ~ L/2 − N/4 ⇐ arctg being bounded
Electrostatic equil. condition.
n_j 2π
Q_j 2π
−2arctg(2x)
* smooth
* vanishes at origin
L arctg
Q₂ 2π
2πQ₁
int
F_ext(u_j) + Σ_{k≠j} f_int(u_j − u_k) − 2πQ_j = 0
position
electric field
1 Q_j must be distinct ⇒ u_j are all distinct


k≠j
L = {2, 4, 6, 8, 10, ..., 16}
60000 x 60000
E/L
Q_j ∈ ℤ or ℤ+1/2 (depending on L and N)
← Tutorial
L ≈ 100
L
{Q_j} fixes {u_j}
② Q_j's are bounded |Q_j^max| ~ L/2 − N/2 ⇐ arctg being bounded
← Electrostatic equil. condition.
1/i log (x+i/2)/(x−i/2)
n_j 2π
Q_j 2π
−2arctg(2x) + π
2π
−2arctg(2x)
* smooth
* vanish at origin
L arctg
2πQ_2
2πQ_1
F_ext(u_j) + Σ_{k≠j} f_int(u_j − u_k)
position
− 2π Q_j = 0
electric field
① Q_j must be distinct ⇒ u_j are all distinct

$B(\infty) \propto S_-$ ✓

$$E\left(S_z = \frac{L}{2} - N\right) = \sum_{j=1}^{N} \underbrace{\frac{\lambda}{u_j^2 + \frac{1}{4}}}_{\epsilon(p_j)}$$

energy of each magnon

Where u_j are quantized through Bethe equations

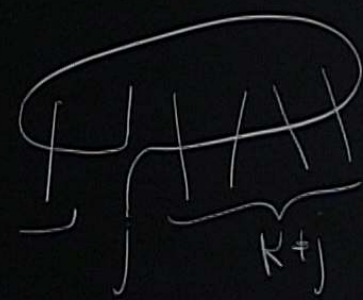
 $S(p_j, p_k)$

FACTORIZATION

$\Downarrow$ $\Uparrow$

$\exists$ new Local Charges

$$\underbrace{\left(\frac{u_j + i/2}{u_j - i/2}\right)^L}_{e^{ip_jL}} \prod_{k\neq j}^{N} \overbrace{\frac{u_j - u_k - i}{u_j - u_k + i}}^{S(p_j,p_k)} = \underset{\uparrow}{1} = $$

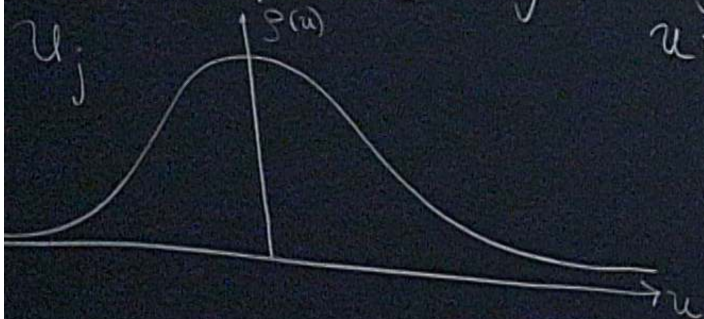


$e^{2\pi n_j i}$

$$n_j = \left\{ -\frac{L}{4} + \frac{1}{2},\ -\frac{L}{4} + \frac{3}{2},\ \ldots\ldots,\ \frac{L}{4} - \frac{1}{2} \right\}$$

$$n_j = -\frac{L}{4} - \frac{1}{2} + j \qquad , \quad j = 1 \ldots N = L/2$$

u_j



$u \equiv u_2 \longleftrightarrow j_2$

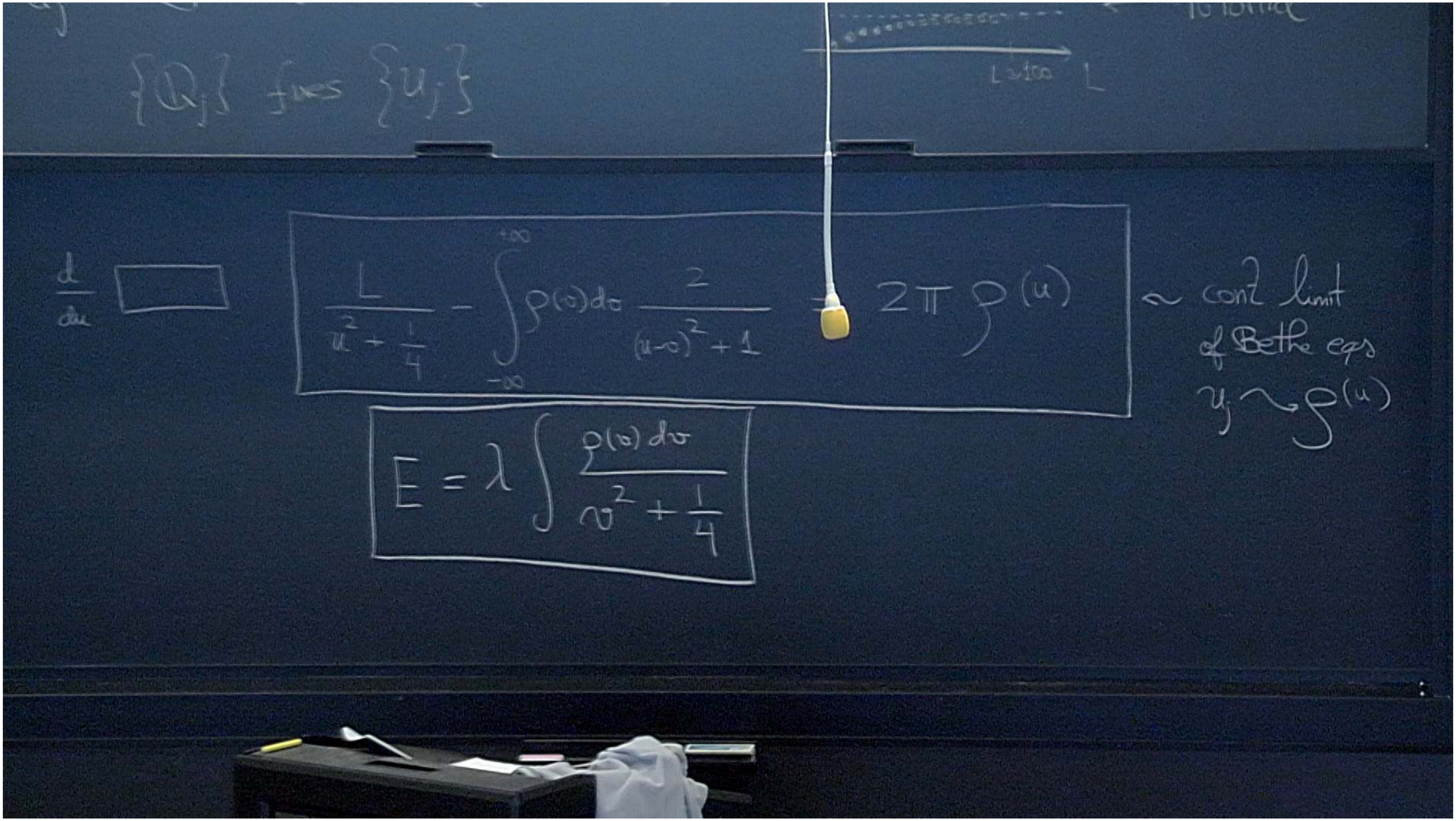
$$\int \rho(u)\, du = \#\ \text{roost between } u_2 \text{ and } u_1 = j_2 - j_1$$

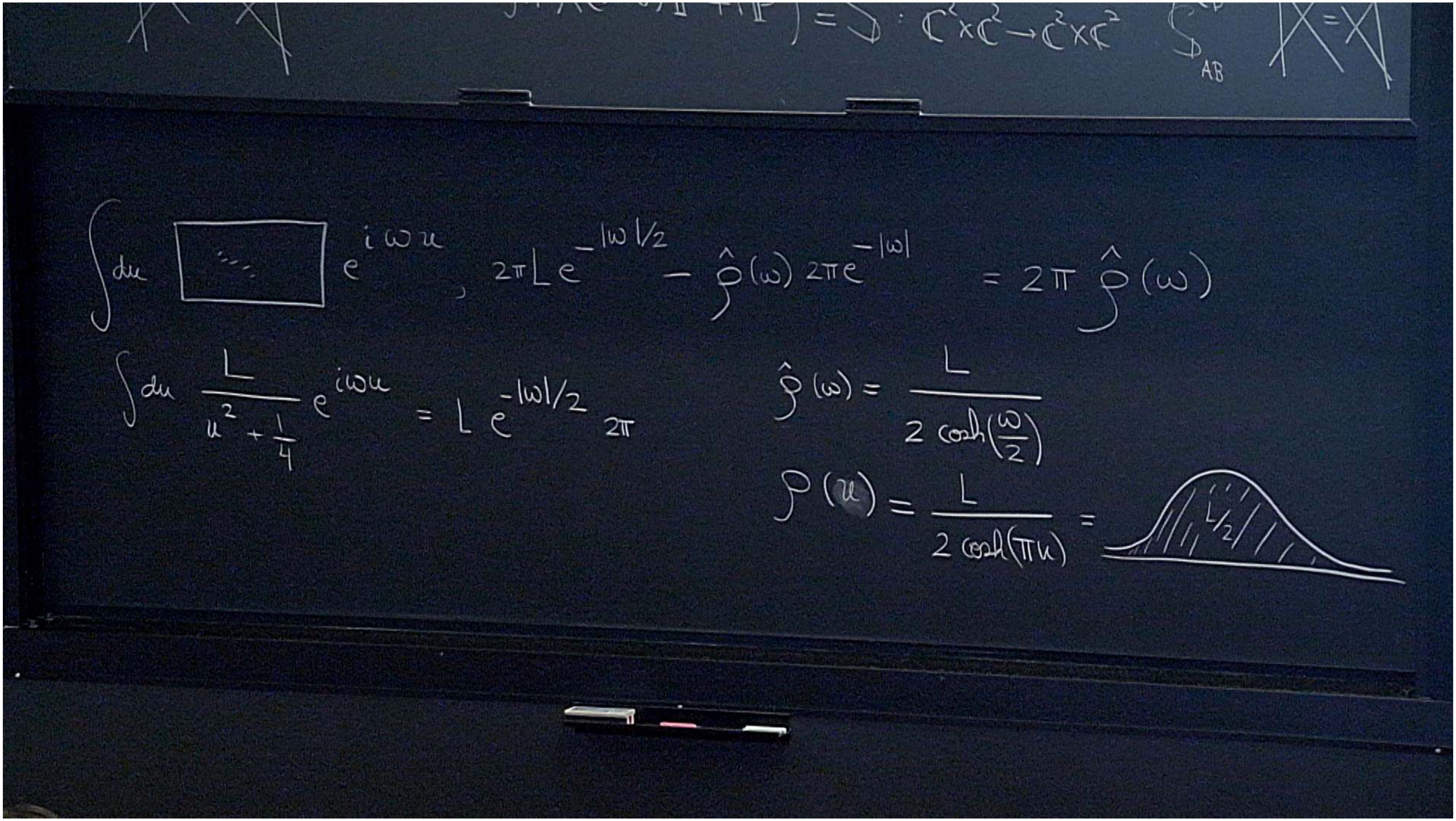
$u_1 \longleftrightarrow j_1$

$j(u)$

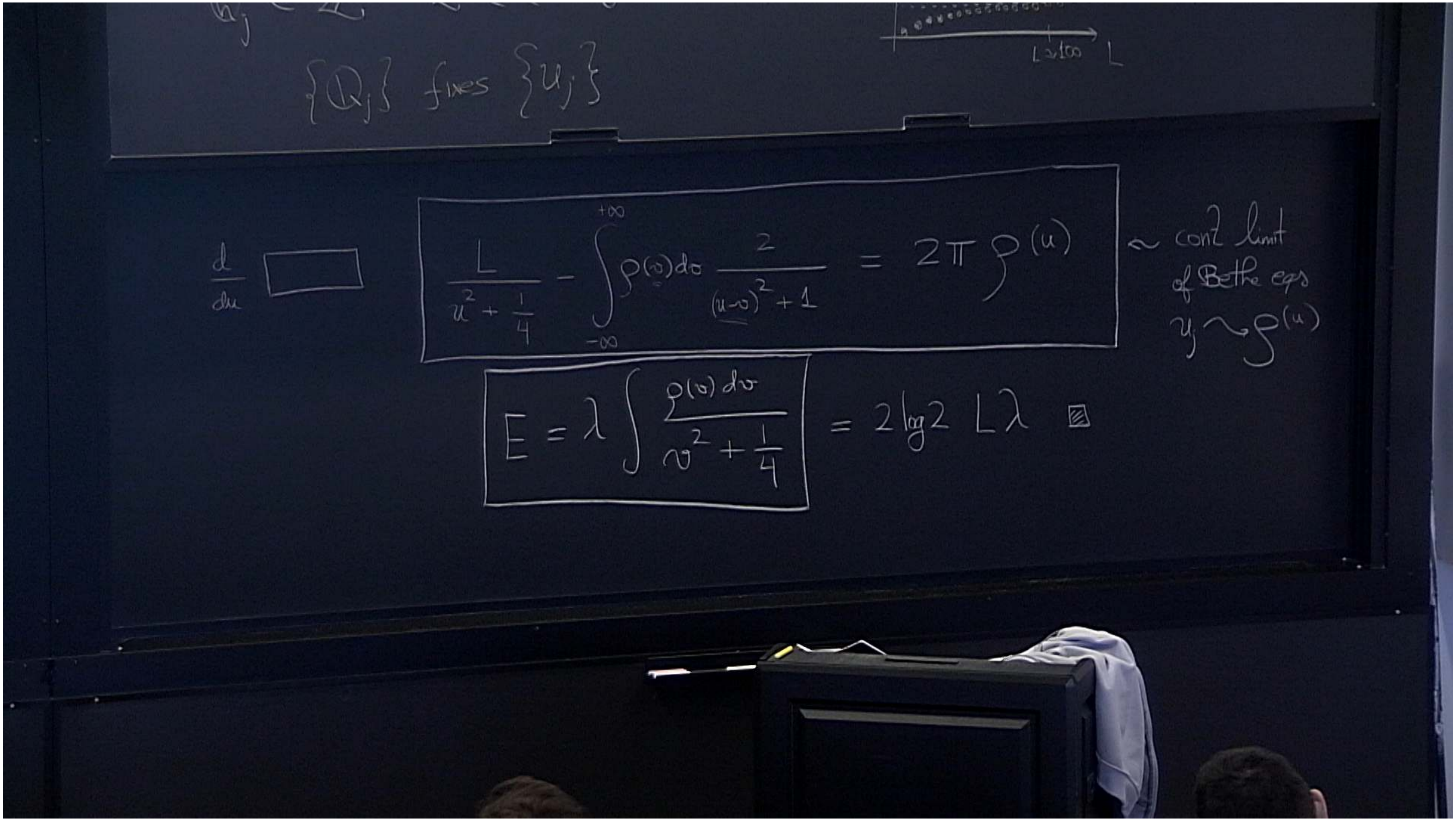
$u_j \to u(j)$

$$\rho(u) = \frac{d}{du}\, j(u)$$


$\{Q_j\}$ fixes $\{u_j\}$
$L \sim 100$
L
$\frac{d}{du}$
$\frac{L}{u^2+\frac{1}{4}} - \int_{-\infty}^{+\infty} \rho(v)\,dv \, \frac{2}{(u-v)^2+1} = 2\pi \rho(u)$
cont limit of Bethe eqs
$u_j \sim \rho(u)$
$E = \lambda \int \frac{\rho(v)\,dv}{v^2+\frac{1}{4}}$



$S : \mathbb{C}^2\times\mathbb{C}^2 \to \mathbb{C}^2\times\mathbb{C}^2$
S_{AB}
$\int du \; \boxed{\;\cdots\;} \; e^{i\omega u}, \quad 2\pi L e^{-|\omega|/2} - \hat{\rho}(\omega)\, 2\pi e^{-|\omega|} = 2\pi\, \hat{\rho}(\omega)$
$\int du \frac{L}{u^2+\frac{1}{4}} e^{i\omega u} = L\, e^{-|\omega|/2}\, 2\pi$
$\hat{\rho}(\omega) = \frac{L}{2\cosh\left(\frac{\omega}{2}\right)}$
$\rho(u) = \frac{L}{2\cosh(\pi u)} =$
$L/2$


$\{Q_j\}$ fixes $\{u_j\}$
$L \sim 100$ L
$\frac{d}{du}$
$\frac{L}{u^2+\frac{1}{4}} - \int_{-\infty}^{+\infty} \rho(v)\,dv\, \frac{2}{(u-v)^2+1} = 2\pi\,\rho(u)$
cont. limit of Bethe eqs
$u_j \sim \rho(u)$
$E = \lambda \int \frac{\rho(v)\,dv}{v^2+\frac{1}{4}} = 2\log 2\, L\lambda$

$$u_j \simeq \frac{2}{\pi}\,\mathrm{arcth}\left(\mathrm{tg}\left(\frac{Q_j \pi}{L}\right)\right)$$

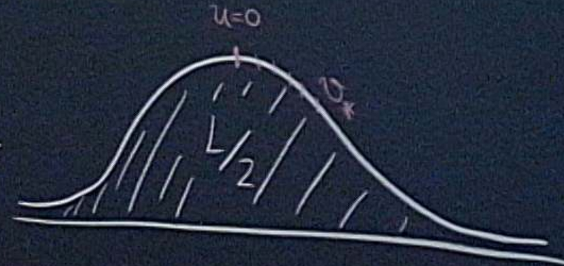
$$2\pi L e^{-|\omega|/2} - \hat{\rho}(\omega)\, 2\pi e^{-|\omega|} = 2\pi\, \hat{\rho}(\omega)$$

$$\int_0^{v_*} \rho(v)\, dv = n_*$$

$$e^{-|\omega|/2} \quad 2\pi$$

$$\hat{\rho}(\omega) = \frac{L}{2\cosh\left(\frac{\omega}{2}\right)}$$

$$\rho(u) = \frac{L}{2\cosh(\pi u)} =$$



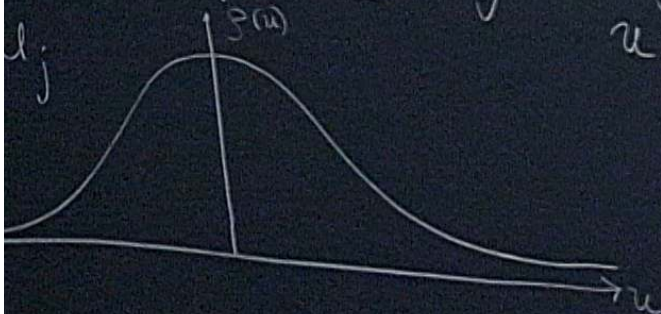
(-1)

$$\lambda_j \in \Big\{ -\frac{L}{4} + \cancel{\tfrac{1}{2}},\ -\frac{L}{4} + \cancel{\tfrac{3}{2}}\,1,\ \ldots\ldots,\ \underbrace{\frac{L}{4} - \cancel{\tfrac{1}{2}}}_{} \Big\}$$

$L/2$ choices $\longrightarrow$ $L/2 + 1$ choices

$$\lambda_j = -\frac{L}{4} - \frac{1}{2} + j \quad , \quad j = 1 \ldots N = L/2$$

u_j



$u \equiv u_2 \longleftrightarrow j_2$

$\int \rho(u)\,du = \#$ roots between u_2 and u_1 $= j_2 - j_1$

$u_1 \longleftrightarrow j_1$

$j(u)$

$u_j \downarrow u(j)$

○'s not used

$$\rho(u) = \frac{d}{du} j(u) = \frac{1}{u_{j+1} - u_j}$$

$\mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C}$

if $N = L/2 \longrightarrow L/2 - 1$

Single choice: $h_j \in \left\{ -\frac{L}{4} + \cancel{\frac{1}{2}},\ -\frac{L}{4} + \cancel{\frac{3}{2}}1, \dots\dots, \underbrace{\frac{L}{4} - \cancel{\frac{1}{2}}}_{} \right\}$

$L/2$ choices $\longrightarrow L/$

$L/2 - 1$ h's to put in a lattice of $L/2 + 1$ options

$$h_j = -\frac{L}{4} - 1 + j + \Theta(j - j_1) + \Theta(j - j_2), \quad j = 1 \dots N = L/2$$

$u \equiv u_2 \longleftrightarrow j_2$

$j(u)$

$\int \rho(u)\,du$ = # roost between u_2 and u_1 $= j_2 - j_1$

$u_1 \longleftrightarrow j_1$

$$\rho(u) = \frac{d}{du} j(u) = \frac{1}{u_{j+1} - u}$$

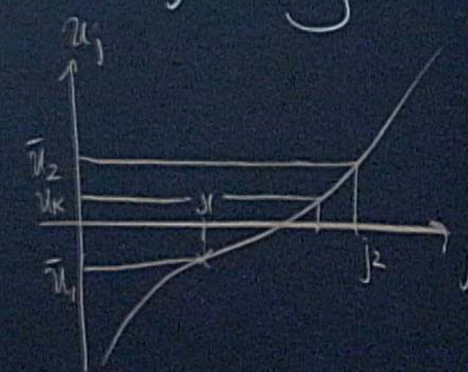
$$\frac{L}{u^2+\frac{1}{4}} - \int_{-\infty}^{+\infty} \rho(v)\,dv\,\frac{2}{(u-v)^2+1} = 2\pi\rho(u) + 2\pi\delta(u-\bar{u}_1) + 2\pi\delta(u-\bar{u}_2)$$

$\sim$ cont. limit of Bethe eqs

$u_j \rightsquigarrow \rho(u)$

$$\boxed{E = \lambda \int \frac{\rho(v)\,dv}{v^2+\frac{1}{4}}} = 2\log 2\, L\lambda \qquad \blacksquare$$

$u_1\, u_2\, u_3 \ldots\ldots$



$$u_1 \simeq \frac{2}{\pi}\,\mathrm{arcth}\left(\mathrm{tg}\left(\frac{\omega_1''}{L}\right)\right),$$

ωu

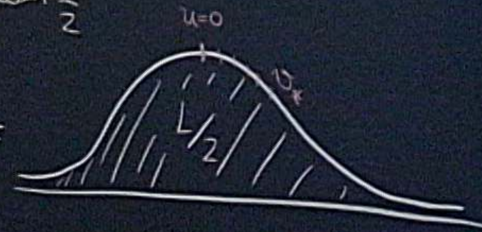
$$2\pi L e^{-|\omega|/2} - \hat{\rho}(\omega)\, 2\pi e^{-|\omega|} = 2\pi\, \hat{\rho}(\omega) + 2\pi e^{i\omega \bar{u}_1} + 2\pi e^{i\omega \bar{u}_2}$$

$$= L e^{-|\omega|/2}\; 2\pi$$

$$\hat{\rho}(\omega) = \frac{L}{2\cosh\left(\frac{\omega}{2}\right)} + \frac{e^{i\omega \bar{u}_1}}{2\cosh\frac{\omega}{2}} + (\bar{u}_2)$$

$$\int_0^{v_*} \rho(v)\,dv = n_*$$

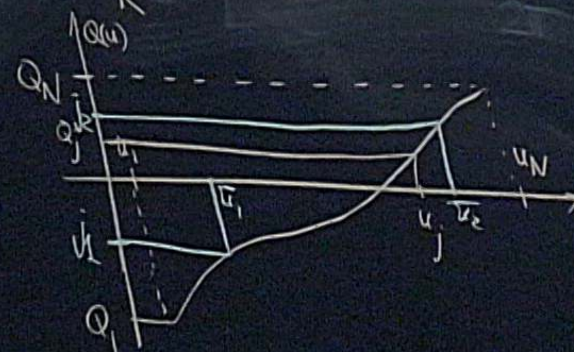
$$\rho(u) = \frac{L}{2\cosh(\pi u)} =$$



$$\int du \; \boxed{\;\cdots\;} \; e^{i\omega u}, \quad 2\pi L e^{-|\omega|/2} - \hat{\rho}(\omega)\, 2\pi e^{-|\omega|} = 2\pi$$

$$2\pi Q(u) \equiv L\, 2\,\mathrm{arctg}(2u) - \sum_{k} 2\,\mathrm{arctg}(u - u_k)$$

$$Q(u_j) = Q_j$$



$$\hat{\rho}(\omega) = \frac{L}{2\cosh\left(\frac{\omega}{2}\right)} +$$

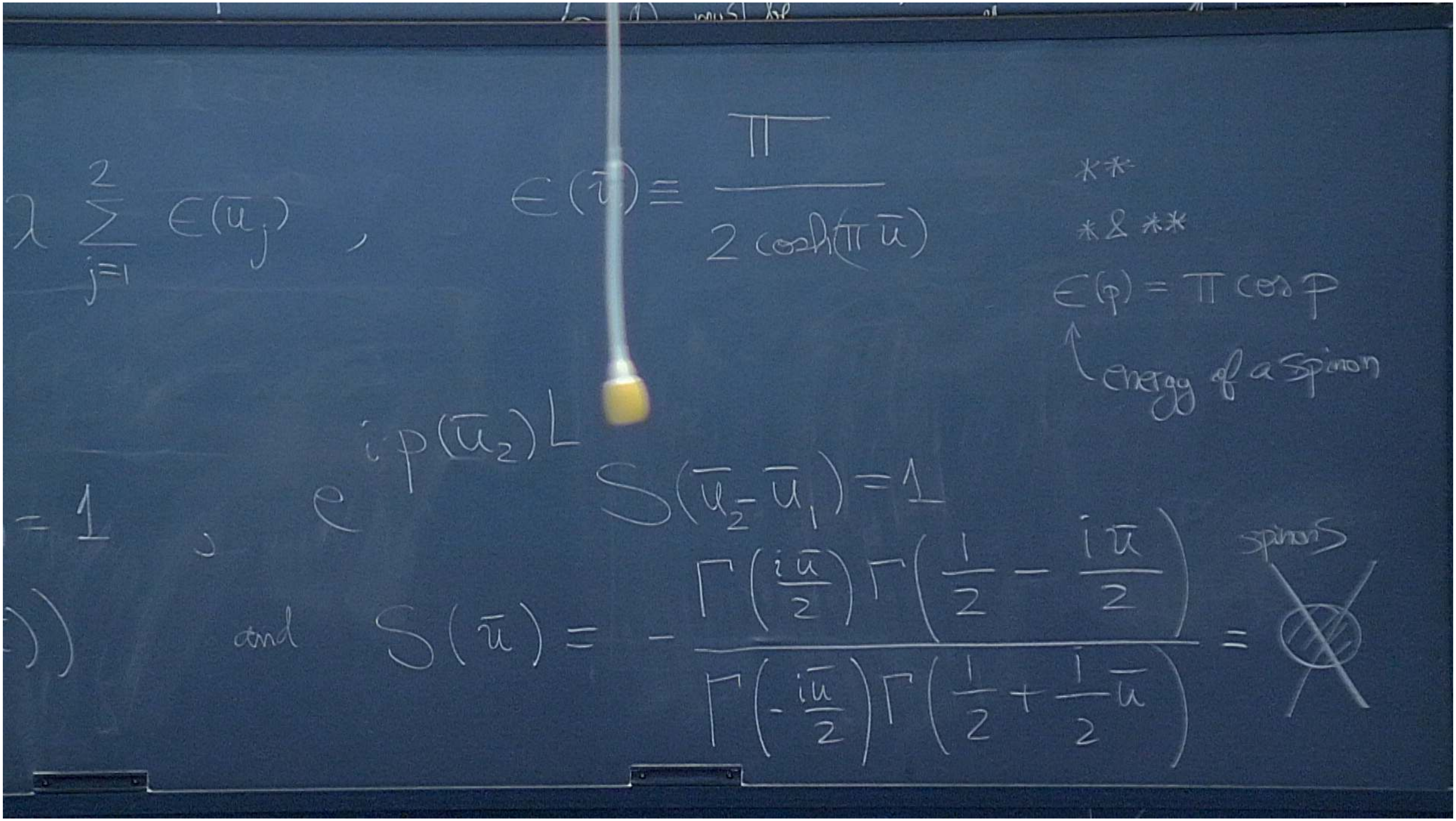
$$\rho(u) = \frac{L}{2\cosh(\pi u)}$$

$$E = \lambda\, 2 \log 2 - \lambda \sum_{j=1}^{2} \epsilon(\bar{u}_j) \,, \qquad \epsilon($$

<u>Where</u>

$$e^{ip(\bar{u}_1)L}\, S(\bar{u}_1 - \bar{u}_2) = 1 \,, \qquad e^{ip(\bar{u}_2)L}$$

$$* \quad p(\bar{u}) = \mathrm{Arctg}\left(\mathrm{Sinh}(\pi \bar{u})\right) \quad \text{and} \quad S(\bar{u}) =$$


$\lambda \sum_{j=1}^{2} \epsilon(\bar{u}_j)$,
$\epsilon(\bar{u}) = \frac{\pi}{2\cosh(\pi \bar{u})}$
**
*& **
$\epsilon(p) = \pi \cos p$
energy of a spinon
$e^{ip(\bar{u}_2)L} S(\bar{u}_2 - \bar{u}_1) = 1$
$= 1$
and
$S(\bar{u}) = -\frac{\Gamma\left(\frac{i\bar{u}}{2}\right)\Gamma\left(\frac{1}{2} - \frac{i\bar{u}}{2}\right)}{\Gamma\left(-\frac{i\bar{u}}{2}\right)\Gamma\left(\frac{1}{2} + \frac{i}{2}\bar{u}\right)} =$
spinons