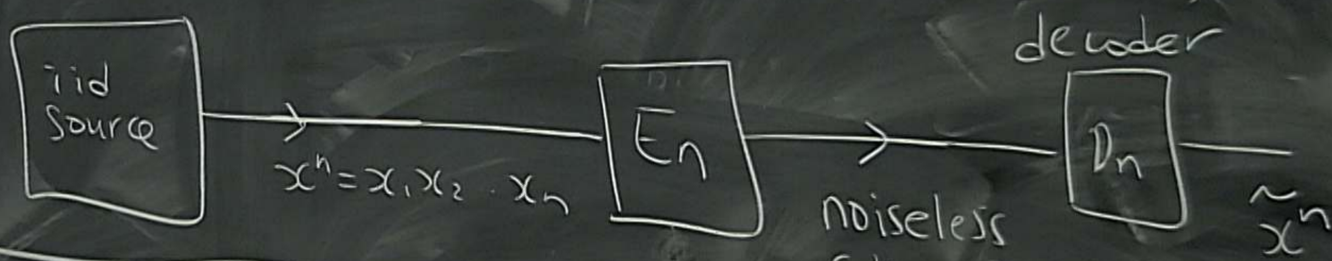
Title: PSI 2016/2017 Quantum Information (Review) - Lecture 4 (Debbie Leung)

Date: Feb 23, 2017 03:15 PM

URL: http://pirsa.org/17020115

Abstract:

App: data compression of iid sources
(Shannon's noiseless coding theorem)



Q_n: how large R should be to suppress $\Pr(x_n \neq \tilde{x}_n)$?

Shannon's noiseless coding theorem:

Let $X_1, X_2, \ldots X_n$ be iid source

① $\forall \varepsilon > 0 \quad \forall R > H(X)$

$\exists n_0$ s.t $\forall n \geq n_0, \; \exists E_n, D_n$

s.t. $\Pr(D_n \circ E_n(x^n) \neq x^n) < \varepsilon$

②

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② $\forall\, R < H(X)$

$\exists\, n_0$ s.t $\forall\, n \geq n_0$, $\forall\, E_n, D_n$

$$\Pr\left(D_n \circ E_n(x^n) = x^n \right) \leq \varepsilon + 2^{-n\left[\frac{H(X)-R}{2}\right]}$$

Pf ①: transmit only typical x^n's

ie $E_n : x^n \longrightarrow b(x^n)$ if $x^n \in T_{n,\delta}$

$\qquad\qquad x^n \longrightarrow \mathrm{ERR}$ otherwise

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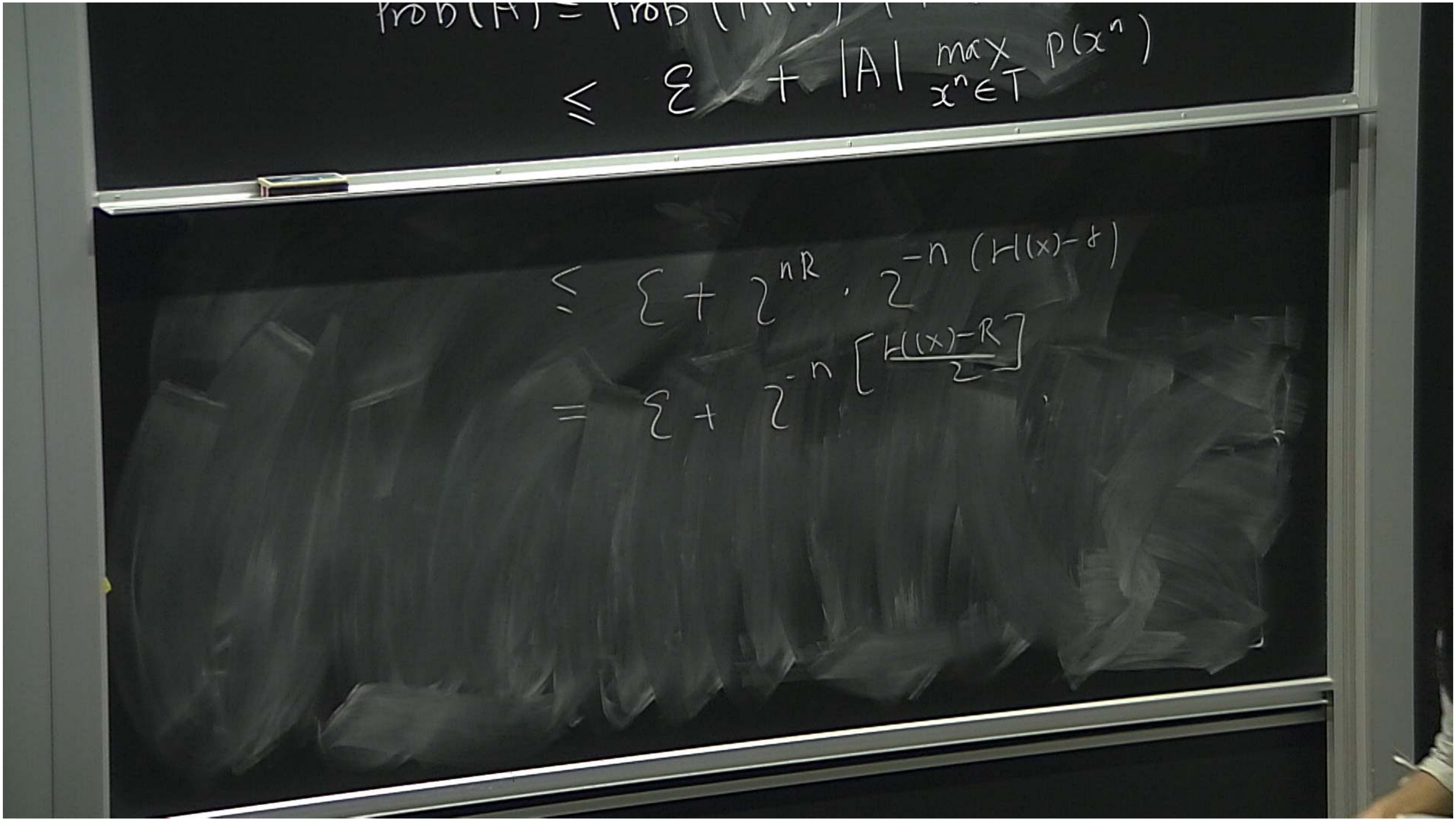
$\Pr(\tilde{x}_n \neq x_n) < \varepsilon$


$\leq \varepsilon + |A| \max_{x^n \in T} p(x^n)$
$\leq \varepsilon + 2^{nR} \cdot 2^{-n(H(X)-\delta)}$
$= \varepsilon + 2^{-n\left[\frac{H(X)-R}{2}\right]}$

$$\mathrm{Prob}(A) = \mathrm{Prob}\,(\ldots$$

$$\leq \varepsilon + |A| \max_{x^n \in T} p(x^n)$$

$$\leq \varepsilon + 2^{nR} \cdot 2^{-n(H(X)-\delta)}$$

$$= \varepsilon + 2^{-n\left[\frac{H(X)-R}{2}\right]}$$

Def [von Neumann entropy]

Let $\rho \geq 0$, $\mathrm{tr}\,\rho = 1$, $\rho \in B(\mathbb{C}^d)$

Then $S(\rho) := -\mathrm{tr}\,\rho \log \rho = -\sum_{i=1}^{d} \lambda_i \log \lambda_i$

where $\lambda_1, \ldots, \lambda_d$ are eigenvalues of ρ.

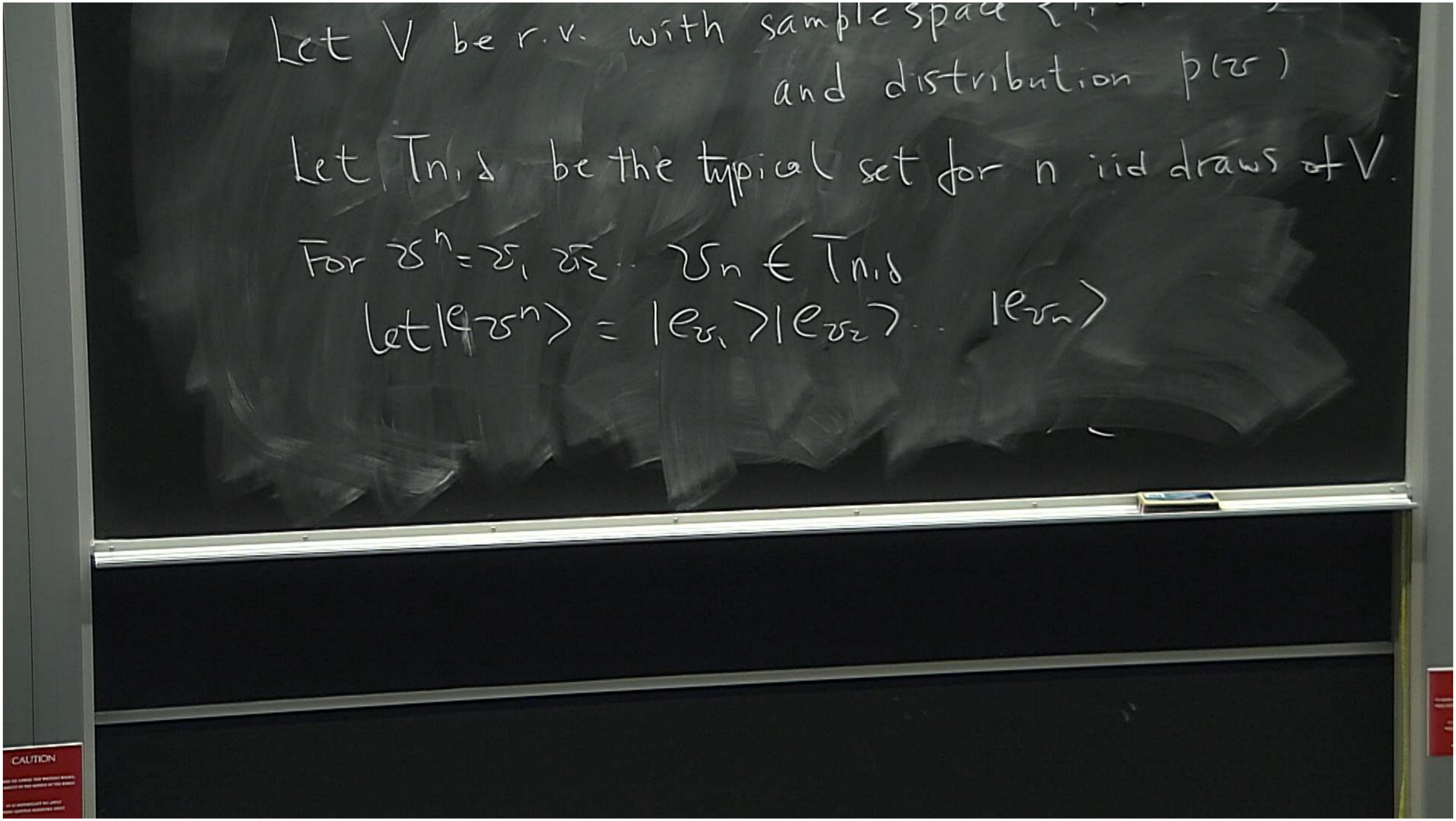
eg. B92, $\rho_0 = |0\rangle\langle 0|$, $q(0) = \frac{1}{2}$

$\rho_1 = |+\rangle\langle +|$, $q(1) = \frac{1}{2}$

<u>Def</u> Let $\rho = \sum_{v=1}^{d} p(v)\, |e_v\rangle\langle e_v|$

Let V be r.v. with samplespace $\{1, 2, \ldots d\}$

and distribution $p(v)$


Let V be r.v. with sample space $\{$
and distribution $p(v)$
Let $T_{n,\delta}$ be the typical set for n iid draws of V.
For $v^n = v_1 v_2 \ldots v_n \in T_{n,\delta}$
let $|e_{v^n}\rangle = |e_{v_1}\rangle |e_{v_2}\rangle \ldots |e_{v_n}\rangle$
CAUTION

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Let $S = \mathrm{Span}\{ |e_{v^n}\rangle : v^n \in T_{n,\delta} \} = \delta$-typical space of $\rho^{\otimes n}$

Let $\Pi_S = \sum_{v^n \in T_{n,\delta}} |e_{v^n}\rangle\langle e_{v^n}|$

CAUTION

Facts:

① $\dim S = |T_{n,\delta}| \leq 2^{n(H(V)+\delta)} = 2^{n(S(\rho)+\delta)}$

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1. $\dim S = |T_{n,\delta}| \leq 2^{n(H(V)+\delta)} = 2^{n(S(\rho)+\delta)}$

2. $\mathrm{Tr}(\rho^{\otimes n}\,\Pi_S) = \sum_{x^n \in T_{n,\delta}} p(x^n) \geq 1-\varepsilon$ if $n \geq n_0 = \ldots$

Let M_S be the following measurement:

It has POVM $= \{ \Pi_S, \ I - \Pi_S \}$

Define "transmission of S":

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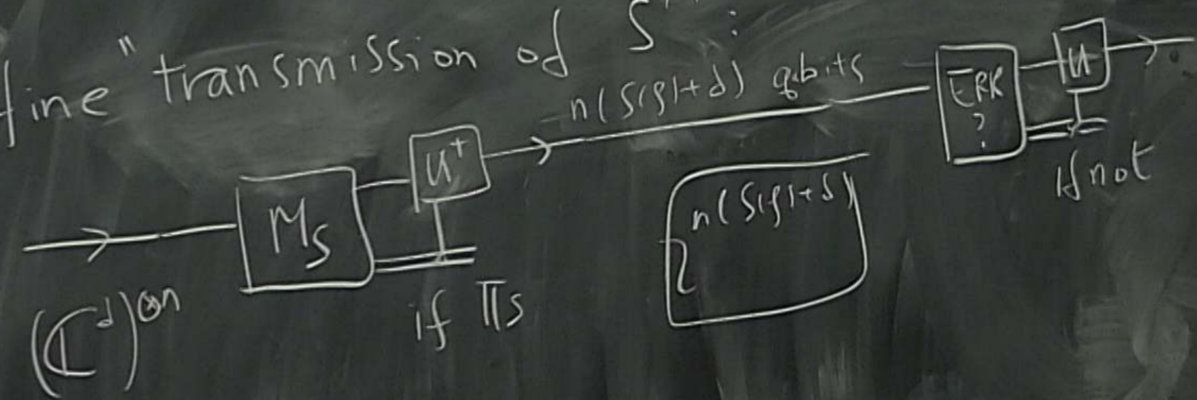
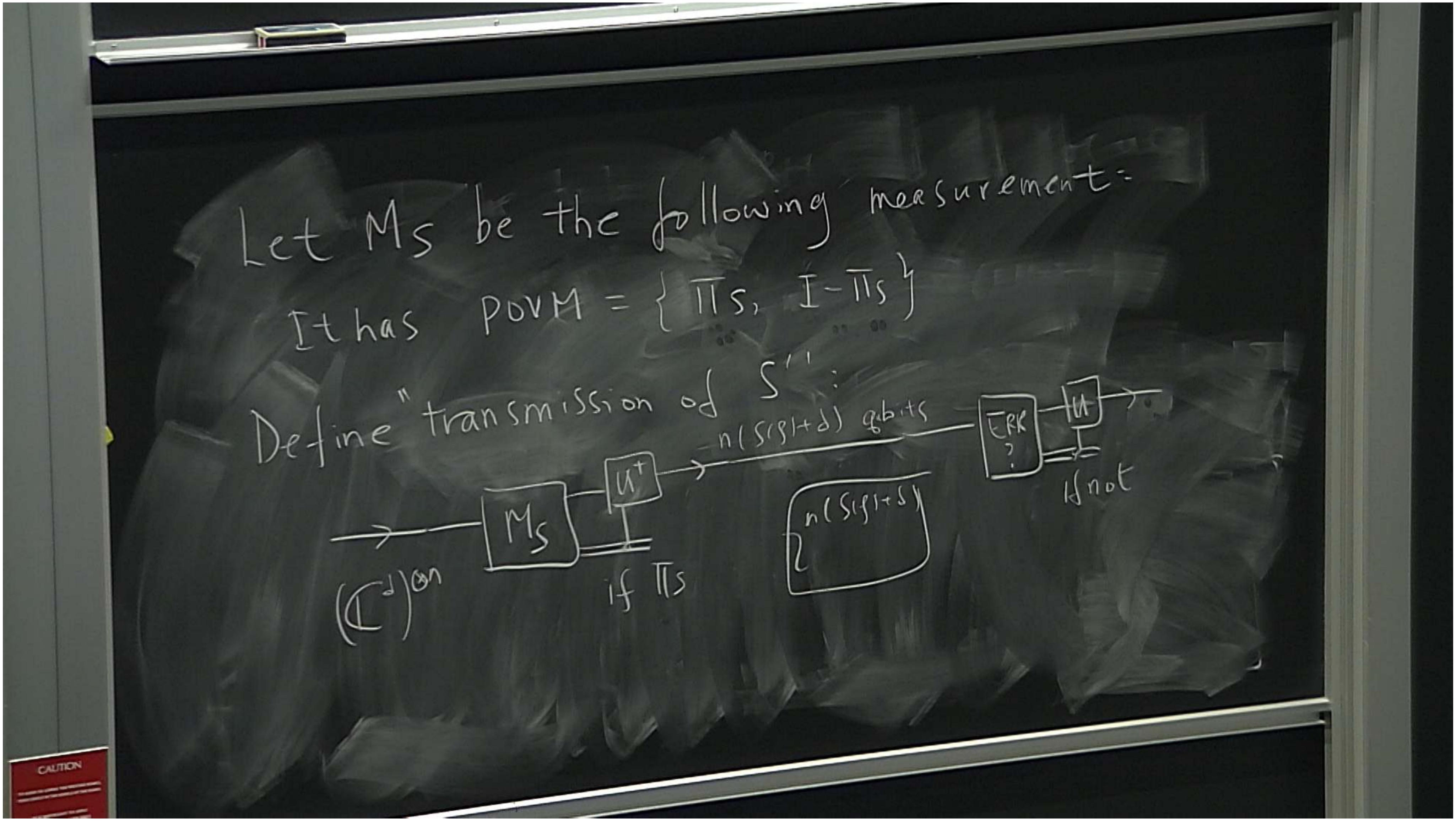
Define "transmission of S":




Let Ms be the following measurement:
It has POVM = { Πs, I − Πs }
Define "transmission of S":
n(S(ρ)+δ) qubits
Ms
U†
if Πs
(ℂ^d)^⊗n
n(S(ρ)+δ)
ERR?
U
if not
CAUTION

Q. Source. X, Ω as before, $pr(X=x) = q(x)$

Process: ① Sample X, obtain x,

② prepare ρ_x

$$\Lambda = \sum_{x \in \Omega} q(x) \, |x\rangle\langle x|_R \otimes \rho_{x\,Q}$$

Average state is $\rho = \sum_x q(x) \, \rho_x = \mathrm{tr}_R \Lambda$

Take n draws, wp $q(x^n) = q(x_1)\cdots q(x_n)$

prep state $\rho_{x_1}\otimes\rho_{x_2}\otimes\cdots\otimes\rho_{x_n}$

Resulting state:

$$\Lambda^{\otimes n} = \sum_{x^n\in\Omega^n} q(x^n)\, \underbrace{|x^n\rangle\langle x^n|}_{R^n = R_1\cdots R_n} \otimes \underbrace{\rho_{x^n}}_{\text{in } Q^n = Q_1\cdots Q_n}$$

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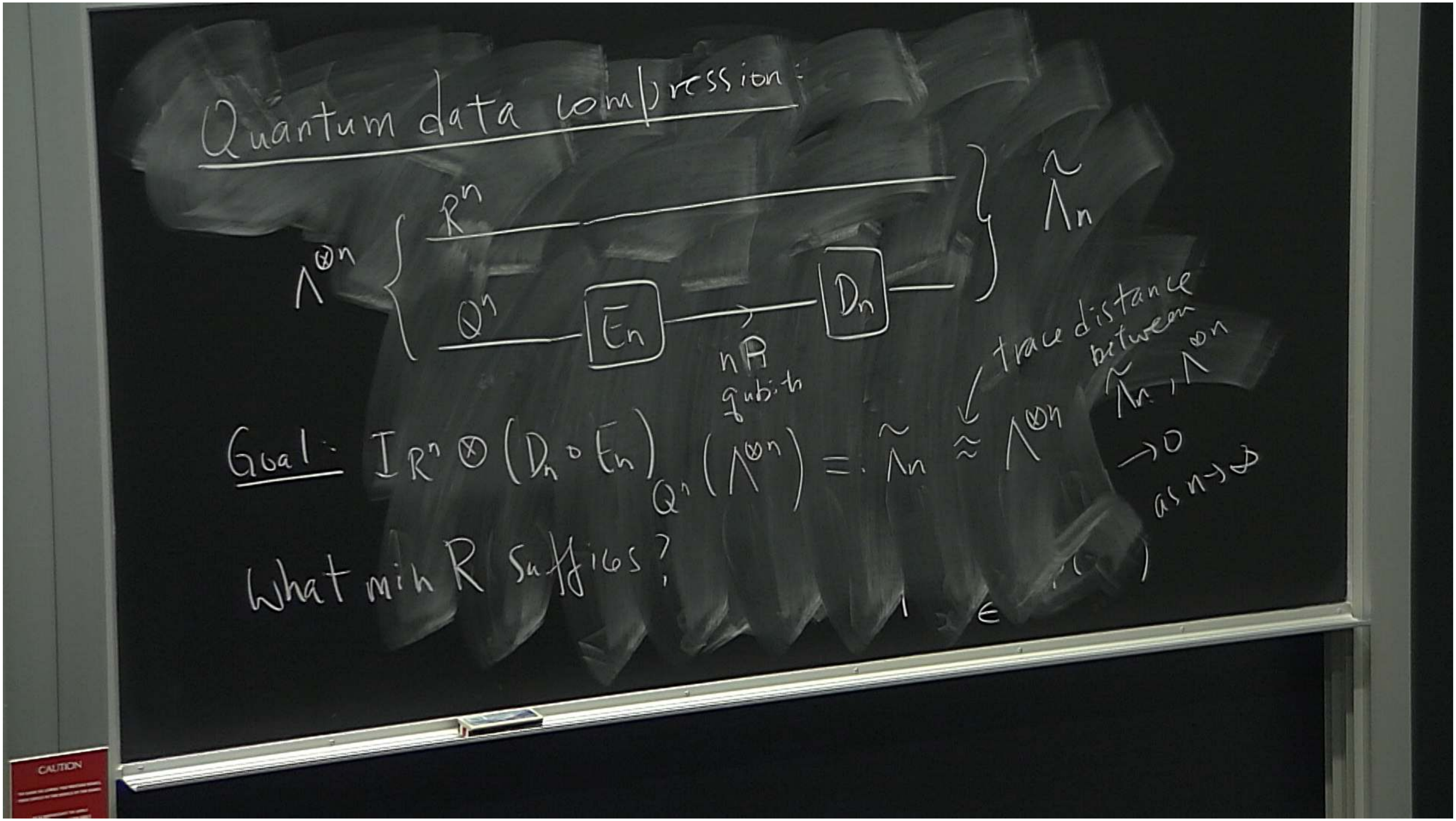
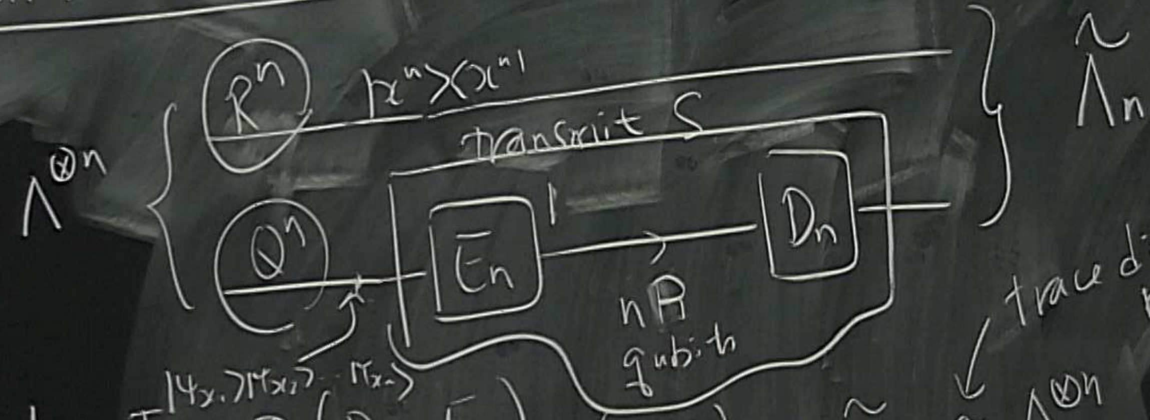
ρ_{x^n} in $Q^n = Q_1 \cdots Q_n$


Quantum data compression:
R^n
$\Lambda^{\otimes n}$
Q^n
E_n
D_n
$\tilde{\Lambda}_n$
nR qubits
trace distance between $\tilde{\Lambda}_n, \Lambda^{\otimes n}$ $\to 0$ as $n \to \infty$
Goal: $I_{R^n} \otimes (D_n \circ E_n)_{Q^n}(\Lambda^{\otimes n}) =: \tilde{\Lambda}_n \approx \Lambda^{\otimes n}$
What min R suffices?

Quantum data compression:



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trace distance between $\tilde{\Lambda}_n, \Lambda^{\otimes n} \to 0$ as $n \to \infty$

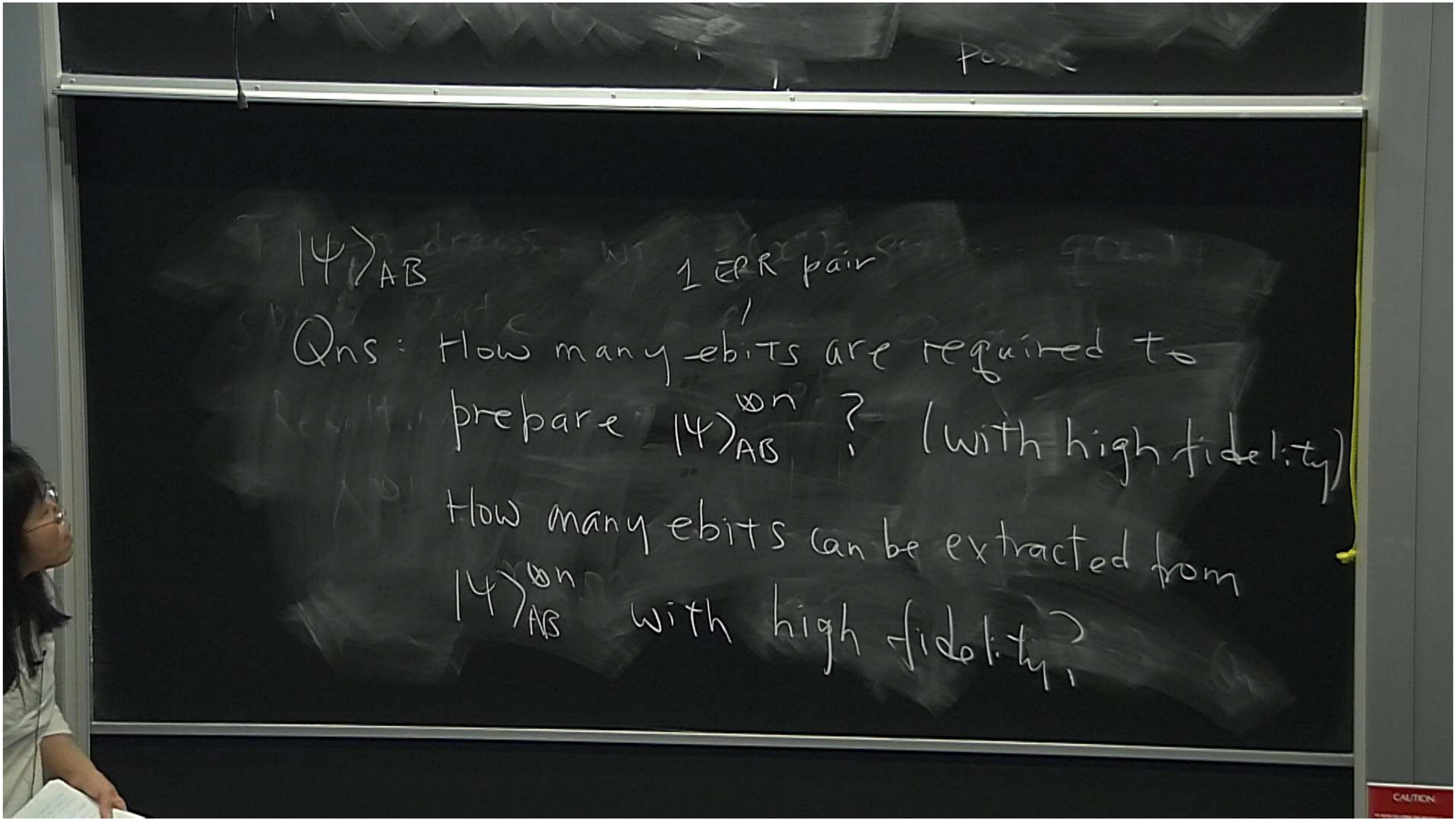
What min R suffices?
$= S(\rho)$

Intuition:

$$\text{Average fidelity} = \sum_{x^n} q(x^n) \, \Big| \underbrace{\langle \psi_{x^n} |}_{\text{input}} \underbrace{\Pi_S | \psi_{x^n} \rangle}_{\text{output}} \Big|$$

$$\geq \sum_{x^n} q(x^n) \langle \psi_{x^n} | \Pi_S | \psi_{x^n} \rangle$$

$$= \mathrm{tr}\Big(\underbrace{\sum_{x^n} q(x^n) | \psi_{x^n} \rangle\langle \psi_{x^n} |}_{\rho^{\otimes n}} \Pi_S \Big) \geq 1 - \varepsilon$$



|ψ⟩AB
1 EPR pair
Qns: How many ebits are required to
prepare |ψ⟩AB ⊗n ? (with high fidelity)
How many ebits can be extracted from
|ψ⟩AB ⊗n with high fidelity?

$|\psi\rangle_{AB} = \sum_x p(x) |x\rangle|x\rangle$

1 EPR pair

Qns: How many ebits are required to prepare $|\psi\rangle_{AB}^{\otimes n}$? (with high fidelity)

$\approx n S(\underbrace{\mathrm{tr}_B |\psi\rangle\langle\psi|}_{\rho_A})$

How many ebits can be extracted from $|\psi\rangle_{AB}^{\otimes n}$ with high fidelity?