

Title: PSI 2016/2017 Quantum Information (Review) - Lecture 3 (Debbie Leung)

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URL: <http://pirsa.org/17020114>

Abstract:

PS I Quantum info review Feb 23, 2017

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Typicality, AEP, entropy, data compression, entanglement

Goal: quantify info & redundancy

Refs: www.math.uwaterloo.ca/~wcleung/c0781-f2016-lectures6-7 (3 files) [html](http://www.math.uwaterloo.ca/~wcleung/c0781-f2016.html)

Nielsen & Chuang Sec 12.2, Cover & Thomas Cp3

Preskill Sec 10.1.1, 10.3-10.4

X : random variable

Ω : sample space

p : prob distribution of X

$$p: \Omega \rightarrow [0, 1]$$

$$x \mapsto p(x)$$

$$\sum_{x \in \Omega} p(x) = 1$$

eg. X = biased coin

$$\Omega = \{0, 1\}$$

$$p(0) = 0.1$$

$$p(1) = 0.7$$

Discrete information source: a sequence of r.v.s
 $X_1, X_2, \dots$, with a common sample space Ω .

Typicality & the asymptotic equipartition thm (AEP)

Idea: consider $X^n = X_1 X_2 \dots X_n$ ($X_i \sim X$ iid)

For large n , $\exists$ large probability set $S_n \subseteq \Sigma^n$

with low cardinality. Furthermore,

elements of S_n have "similar" probabilities

Def: Shannon entropy

$$H(X) \text{ or } H(p) := - \sum_{x \in \Omega} p(x) \log p(x)$$

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eg for the biased coin

$$H(X) = -0.1 \log 0.1 - 0.9 \log 0.9 = 0.469$$

Def [typical sequence]:

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Def [typical set]

$$T_{n,\delta} = \{x^n : x^n \text{ is } \delta\text{-typical}\}$$

Def [typical sequence]:

$$x^n \text{ is } \delta\text{-typical if } \left| -\frac{1}{n} \log p(x^n) - H(X) \right| \leq \delta$$

eg for biased coin, $n=100$, $d=0.1$

If there are t 0's & $n-t$ 1's in x^n

$$\text{then } -\frac{1}{n} \log p(x^n) = -\frac{t}{n} \log 0.1 - \left(\frac{n-t}{n}\right) \log 0.9$$

is in $[0.369, 0.569]$ for $7 \leq t \leq 13$

$$P_{0.1}(T_{100,0.1}) = 75.897\%$$

$$P_{0.1}(T_{100,0.1}) = 8.3 \times 10^5 < \sum_{i=1}^{100} \approx 1.3 \times 10^{30}$$

AEP:

$\forall \epsilon > 0 \exists \delta > 0 \exists n_0$ s.t. $\forall n \geq n_0$

$$(1) \quad p(T_{n,\delta}) \geq 1 - \epsilon$$

$$(2) \quad (1 - \epsilon) 2^{n(H(x) - \delta)} \leq |T_{n,\delta}| \leq 2^{n(H(x) + \delta)}$$

$$(3) \quad \forall A \subseteq \Sigma^n, p(A) \geq 1 - \epsilon \Rightarrow |A| \geq (1 - 2\epsilon) 2^{n(H(x) - \delta)}$$

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Ω^n

Pf ① We apply LLN on $Y = \log p(X)$ of r.v.s
 $X_1, X_2, \dots$ with a common $p(x) = \log p(x)$ is $p(x)$

only so $\mathbb{E} Y = -H(X)$

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$\because X^n$ is iid, $Y^n = Y_1 \dots Y_n$ is iid

Let $x^n = x_1 \dots x_n \in \Omega^n$, $y_i = \log p(x_i)$

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$X_1, X_2, \dots$ with $p(y) = \log p(x)$ is $p(x)$

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Let $x^n = x_1 \dots x_n \in \Omega^n$, $y_i = \log p(x_i)$

Then $x^n \notin T_{n,\delta} \Leftrightarrow \left| \frac{1}{n} \sum_{i=1}^n y_i - \mathbb{E} Y \right| > \delta$ (*)

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To bound the prob of RHS of $(*)$, we use

Def: Shannon's ineq for r.v Z :

$$\Pr(|Z - \mathbb{E}Z| > k \sqrt{\text{Var} Z}) \leq \frac{1}{k^2}$$

Choose $Z = \frac{1}{n} \sum_{i=1}^n Y_i$ ($Z = \sum_{i=1}^n y_i$), $\text{Var} Z = \frac{1}{n} \text{Var} Y$

$$k \text{ s.t. } k \sqrt{\text{Var} Z} = \delta$$

Chebyshev's

$$\Pr(|Z - \mathbb{E}Z| > k\sqrt{\text{Var}Z}) \leq \left(\frac{1}{k^2}\right)$$

Choose $Z = \frac{1}{n} \sum_{i=1}^n Y_i$ ($Z = \sum_{i=1}^n y_i$), $\text{Var}Z = \frac{1}{n} \text{Var}Y$

k s.t. $k\sqrt{\text{Var}Z} = \delta$

n s.t. $\frac{1}{k^2} \leq \epsilon$, $\frac{\text{Var}Z}{\delta^2} \leq \epsilon$, $\boxed{\frac{\text{Var}Y}{n\delta^2} \leq \epsilon}$

$$n_0 = \left\lceil \frac{\text{Var}Y}{\delta^2 \epsilon} \right\rceil$$

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$n_0 = \left\lceil \frac{\text{Var}Y}{\delta^2 \epsilon} \right\rceil$

$$\underline{Pf(2)} \quad 1 - \varepsilon \leq p(T_{n,\delta}) \leq 1$$

$$\sum_{|T_{n,\delta}|} 2^{-n(H(x) + \delta)} \leq \sum_{x^n \in T_{n,\delta}} p(x^n) \leq \sum_{|T_{n,\delta}|} 2^{-n(H(x) - \delta)}$$

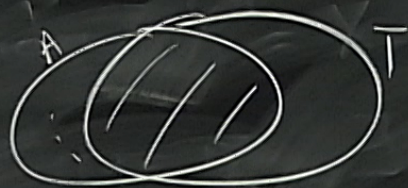
$$(1 - \varepsilon) 2^{n(H(x) - \delta)} \leq |T_{n,\delta}|$$

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$$\sum_{|T_{n,\delta}|}^{-n(H(x)+\delta)} p(x^n) \leq \sum_{x^n \in T_{n,\delta}} p(x^n) \leq \sum_{|T_{n,\delta}|}^{-n(H(x)-\delta)} p(x^n)$$

$$(1-\varepsilon) 2^{n(H(x)-\delta)} \leq |T_{n,\delta}| \leq 2^{n(H(x)+\delta)}$$

Pf ③: Let $A \subseteq \Omega^n$ have $P(A) > 1 - \epsilon$, $T = T_{n, \delta}$



$$|A| \geq |A \cap T|$$

$$\geq \frac{P(A \cap T)}{\max_{a \in A \cap T} P(a)}$$

$$\geq \frac{P(A \cap T)}{2^{-n(H(x) - \delta)}} \geq (1 - 2\epsilon) 2^{n(H(x) - \delta)}$$

$$P(A \cap T) = P(A) - P(A \setminus T)$$

$$\geq P(A) - P(\Omega^n \setminus T)$$

$$\geq 1 - \epsilon - \epsilon = 1 - 2\epsilon$$