Title: Perfect Embezzlement of Entangled States

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Abstract: <p>Hayden and Van Dam showed that starting with a separable state in Alice and Bobâ€™s state space and a shared entangled state in a common bipartite resource space, then using local unitary operations, it is possible to produce an entangled pair in the state space while at the same time only perturbing the shared entangled state by a small amount, which can be made arbitrarily small as the dimension of the resource space grows. They referred to this as â€œembezzling entanglementâ€• since numerically it â€œappears" that the resource state was returned exactly.</p>

<p>It is natural to wonder if using an infinite dimensional resource space and local operations, one can return the resource state exactly while producing an entangled state in their state space. Whether or not you can achieve this phenomenon of â€œperfect embezzlement of an entangled stateâ€• depends on which mathematical model one uses to describe â€œlocalâ€•.</p>

<p>We prove that perfect embezzlement is impossible in the tensor model but is possible in the commuting model. We then relate this to current work on the conjectures of Connes and Tsirelson about different models for quantum conditional probabilities.</p>

<p>This talk is based on joint work with R. Cleve, L. Liu and S. Harris.</p>

Perfect Embezzlement

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February 14, 2017

Based on joint work with
R. Cleve,
L. Liu,
S. Harris

Outline

- Van Dam and Hayden Approximate Embezzlement
- Impossibility of Perfect Embezzlement in Tensor Framework
- Commuting Framework
- The C*-algebra of Non-commuting Unitaries
- Perfect Embezzlement
- New Versions of Tsirelson, Connes, and Kirchberg
- The Coherent Embezzlement Game

Approximate Embezzlement of A Bell State

It is well-known that entangled states cannot be produced from separable states by local operations. But Van Dam and Hayden showed a method that, in a certain sense, *appears* to produce entanglement by local methods. Hence, their term *embezzlement*.

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$$|0\rangle_A |0\rangle_B \otimes \psi \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \otimes \psi_\epsilon$$

where $\|\psi - \psi_\epsilon\| < \epsilon$ for any $\epsilon > 0$.

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More precisely, given $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$, there are finite dimensional spaces $\mathcal{R}_A, \mathcal{R}_B$ and unitaries, U_A on $\mathcal{H}_A \otimes \mathcal{R}_A$, U_B on $\mathcal{R}_B \otimes \mathcal{H}_B$ such that on $(\mathcal{H}_A \otimes \mathcal{R}_A) \otimes (\mathcal{R}_B \otimes \mathcal{H}_B)$,

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We now show why perfect embezzlement is impossible, in this tensor product framework.

Proposition (CLP)

Perfect embezzlement is impossible in the above tensor product framework.

Proof: Write a Schmidt decomposition

$$|0\rangle \otimes \psi \otimes |0\rangle = \sum_j t_j(|0\rangle \otimes u_j) \otimes (v_j \otimes |0\rangle),$$

with $u_j \in \mathcal{R}_A$ orthonormal and $v_j \in \mathcal{R}_B$ orthonormal.

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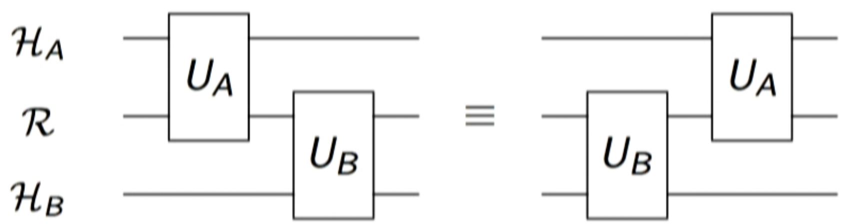

Given a commuting operator framework, we say that $\psi \in \mathcal{R}$ *is a catalyst vector for perfect embezzlement of a Bell state* provided that

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In the rest of this talk, I want to outline the proof and show why the fact that perfect embezzlement is possible in this commuting framework but not possible in a tensor product framework is closely related to the Tsirelson conjectures and to Connes' embedding conjecture.

Suppose that $\mathcal{H}_A = \mathbb{C}^n$ and identify $\mathbb{C}^n \otimes \mathcal{R} = \mathcal{R} \oplus \cdots \oplus \mathcal{R}$(n times). Using this identification, we can write an operator on $\mathbb{C}^n \otimes \mathcal{R}$ as $U_A = (U_{i,j})$ where $U_{i,j} \in B(\mathcal{R})$, $0 \le i, j \le n-1$.

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Lemma

$(U_A \otimes id_B)$ commutes with $(id_A \otimes U_B)$ if and only if $U_{i,j}V_{k,l} = V_{k,l}U_{i,j}$ and $U_{i,j}^*V_{k,l} = V_{k,l}U_{i,j}^*$ for all i, j, k, l.

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This last condition is called **-commuting*.
Thus, we see that having commuting operator frameworks as above is exactly the same as having operator matrices $U_A = (U_{i,j})$ and $U_B = (V_{k,l})$ that yield unitaries and whose entries pairwise *-commute.

The C*-algebra $U_{nc}(n)$

L. Brown introduced a C*-algebra denoted $U_{nc}(n)$. It has n^2 generators $u_{i,j}$ and the "universal" property that whenever there are n^2 operators $U_{i,j}$ on a Hilbert space $\mathcal{R}$ such that $(U_{i,j})$ defines a unitary operator on $\mathbb{C}^n \otimes \mathcal{R}$ then there is a *-homomorphism $\pi : U_{nc}(n) \to B(\mathcal{R})$ with $\pi(u_{i,j}) = U_{i,j}$.
Thus, a representation of $U_{nc}(n) \otimes_{max} U_{nc}(m)$ corresponds to operators $U_{i,j}, V_{k,l}$ where the $U_{i,j}$'s *-commute with the $V_{k,l}$'s such that $(U_{i,j})$ and $(V_{k,l})$ are unitary operator matrices.

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Recall that a *state* on a C*-algebra is just a positive linear functional s with $s(1) = 1$.

Theorem (CLP)

Perfect embezzlement of a Bell state is possible in a commuting operator framework if and only if there is a state s on $U_{nc}(2) \otimes_{max} U_{nc}(2)$ *satisfying* $s(u_{00} \otimes v_{00}) = s(u_{10} \otimes v_{10}) = 1/\sqrt{2}$ *and* $s(u_{00} \otimes v_{10}) = s(u_{10} \otimes v_{00}) = 0$.

To prove, take the GNS representation of any such state then $\psi = [1]$ is a catalyst vector for perfect embezzlement of a state in a commuting operator framework.

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Corollary

The van Dam–Hayden approximate embezzlement results imply that there exists a state on $U_{nc}(2) \otimes_{min} U_{nc}(2)$ *satisfying the above equations.*

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The representation of $U_{nc}(2) \otimes_{min} U_{nc}(2)$ given by the Corollary can not decompose as a spatial tensor product of a representation of each factor or else we would contradict the fact that perfect embezzlement is impossible in a tensor product framework!

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We now want to draw an analogy with quantum correlation matrices.

Tsirelson, Connes and all that

Suppose that Alice and Bob each have n quantum experiments and each experiment has m outcomes. We let $p(a, b|x, y)$ denote the conditional probability that Alice gets outcome a and Bob gets outcome b given that they perform experiments x and y, respectively. There are several possible models for describing the set of all such tuples.

One model is that Alice and Bob have finite dimensional state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. For each experiment x, Alice has projections $\{E_{x,a}, 1 \le a \le m\}$ such that $\sum_a E_{x,a} = I_A$. Similarly, for each y, Bob has projections $\{F_{y,b} : 1 \le b \le m\}$ such that $\sum_b F_{y,b} = I_B$. They share an entangled state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ and

$$p(a, b|x, y) = \langle \psi | E_{x,a} \otimes F_{y,b} | \psi \rangle.$$

We let $C_q(n,m) = \{p(a,b|x,y) : \text{ obtained as above }\} \subseteq \mathbb{R}^{n^2m^2}$.
We let $C_{qs}(n,m)$ denote the possibly larger set that we could obtain if we allowed the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ to also be infinite dimensional.
We let $C_{qc}(n,m)$ denote the possibly larger set that we could obtain if instead of requiring the common state space to be a tensor product, we just required one common state space, and demanded that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all a, b, x, y, i.e., a commuting model.
Tsirelson was the first to examine these sets and study the relations between them. In fact, he wondered if they could all be equal.
Here are some of the things that we know/don't know about these sets.

- $C_q(n,m) \subseteq C_{qs}(n,m) \subseteq C_{qc}(n,m)$.
- We don't know if the sets $C_q(n,m)$ and $C_{qs}(n,m)$ are closed, but $C_{qc}(n,m)$ is closed.
- $C_q(n,m)^- = C_{qs}(n,m)^-$ and this can be identified with the states on a minimal tensor product.
- Werner-Scholz speculated that $C_{qs}(n,m) = C_q(n,m)^-$.
- (JNPPSW + Ozawa)$C_q(n,m)^- = C_{qc}(n,m), \forall n, m$ iff Connes' Embedding conjecture has an affirmative answer.
- (Slofstra, April 2016) there exists an n, m(very large) such that $C_{qs}(n,m) \neq C_{qc}(n,m)$.

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Unitary Tensors

Theorem (Harris)

The following are equivalent.

1. *Connes' Embedding conjecture is true.*
2. $U_{nc}(n) \otimes_{min} U_{nc}(m) = U_{nc}(n) \otimes_{max} U_{nc}(m), \forall n, m.$

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3. $U_{nc}(2) \otimes_{min} U_{nc}(2) = U_{nc}(2) \otimes_{max} U_{nc}(2).$
4. *The unitary correlation sets(defined later) satisfy* $UC_q(n, m)^- = UC_{qc}(n, m), \forall n, m.$

The equivalence of the first three, is the analogue of Kirchberg's theorem relating Connes to tensor products of free group C*-algebras. The equivalence of the first and last is the analogue of the result of [Junge $\cdots$ Ozawa].

Unitary Correlation Sets

We set

$$UC_q(n,m) = \{\langle\psi|U_{ij}\otimes V_{kl}|\psi\rangle : (U_{i,j}),(V_{k,l}) \text{ are unitary}, \\ U_{i,j}\in M_p, V_{k,l}\in M_q, \exists p,q, ||\psi|| = 1\} \subset M_n \otimes M_m.$$

so these are $(n^2)(m^2)$-tuples.
For the set $UC_{qs}(n,m)$ we drop the requirement that each $U_{i,j}$ and $V_{k,l}$ act on finite dimensional spaces.
For the set $UC_{qc}(n,m)$ we replace the tensor product of two spaces by a single space and instead demand that the $U_{i,j}$'s *-commute with the $V_{k,l}$'s.

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Here are some of the things that we know/don't know about these sets.

- $UC_q(n,m) \subseteq UC_{qs}(n,m) \subseteq UC_{qc}(n,m)$.
- For each n, m, $UC_q(n,m)$ and $UC_{qs}(n,m)$ are not closed.
- $UC_{qc}(n,m)$ is closed.
- $UC_q(n,m)^- = UC_{qs}(n,m)^- = \{(s(x \otimes y)) :$
  $s : U_{nc}(n) \otimes_{min} U_{nc}(m) \to \mathbb{C}$ is a state, x, y as above $\}$.
- $UC_{qs}(2,2) \neq UC_{qc}(2,2)$, a consequence of the embezzlement results.
- (Harris) $UC_q(n,m)^- = UC_{qc}(n,m), \forall n, m \iff$ Connes Embedding is true.

The Coherent Embezzlement Game

This game was introduced by Regev and Vidick, also known as the T_2 game.
The Referee prepares one of two states, $\phi_0, \phi_1 \in \mathcal{H}_A \otimes \mathcal{H}_B$ where

$$\phi_c = \frac{1}{\sqrt{2}}|00\rangle \otimes |00\rangle + \frac{1}{\sqrt{2}}(-1)^c\left(\frac{1}{\sqrt{2}}|10\rangle \otimes |01\rangle + \frac{1}{\sqrt{2}}|11\rangle \otimes |11\rangle\right),$$

$c \in \{0, 1\}$.
Alice and Bob each output a classical bit a, b.
They win if input $\phi_0 \implies a + b = 0$, and input $\phi_1 \implies a + b = 1$.

Assume that they are allowed to share a state $\psi \in \mathcal{R}$ and act with unitaries on $\mathcal{H}_A \otimes \mathcal{R}$ and $\mathcal{R} \otimes \mathcal{H}_B$, respectively, where necessarily these unitaries commute.

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Theorem (CLP)

There is a perfect strategy for the coherent embezzlement game in the commuting framework. But there is no perfect strategy if we require that $\mathcal{R} = \mathcal{R}_A \otimes \mathcal{R}_B$ and that their unitaries act locally, even when we allow $\mathcal{R}_A$ and $\mathcal{R}_B$ to be infinite dimensional.

Idea of proof: 1)This game is embezzlement in reverse!

The Coherent Embezzlement Game

This game was introduced by Regev and Vidick, also known as the T_2 game.
The Referee prepares one of two states, $\phi_0, \phi_1 \in \mathcal{H}_A \otimes \mathcal{H}_B$ where

$$\phi_c = \frac{1}{\sqrt{2}}|00\rangle \otimes |00\rangle + \frac{1}{\sqrt{2}}(-1)^c\Big(\frac{1}{\sqrt{2}}|10\rangle \otimes |01\rangle + \frac{1}{\sqrt{2}}|11\rangle \otimes |11\rangle\Big),$$

$c \in \{0, 1\}$.
Alice and Bob each output a classical bit a, b.

Assume that they are allowed to share a state $\psi \in \mathcal{R}$ and act with unitaries on $\mathcal{H}_A \otimes \mathcal{R}$ and $\mathcal{R} \otimes \mathcal{H}_B$, respectively, where necessarily these unitaries commute.

Theorem (CLP)

There is a perfect strategy for the coherent embezzlement game in the commuting framework. But there is no perfect strategy if we require that $\mathcal{R} = \mathcal{R}_A \otimes \mathcal{R}_B$ and that their unitaries act locally, even when we allow $\mathcal{R}_A$ and $\mathcal{R}_B$ to be infinite dimensional.

Idea of proof: 1)This game is embezzlement in reverse!
2) Unitaries are reversible, i.e., invertible.

Summary: Problems of Connes and Tsirelson are closely tied to unitary correlations.

Techniques from embezzlement can be used to produce some elements of these unitary correlation sets.

Continuing to study geometry of these unitary correlation sets.

Unitary correlation sets should determine if perfect strategies exist for games with finite quantum inputs–finite classical outputs in the same way that probabilistic correlations are used in finite input-output non-local games.

Thanks!