Title: Self-Linking for Legendrian Knots

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Abstract: <p>The Feynman diagram expansion for a Wilson loop observable in Chern-Simons gauge theory generates an infinite series of topological invariants for framed knots. In this talk, I will describe a new perturbative formalism which conjecturally generates the same invariants for Legendrian knots in the standard contact R^3. The formalism includes a `perturbative' localization principle which drastically simplifies the structure of calculations. As time permits, I will provide some examples and applications. This talk is based upon joint work with Brendan McLellan and Ruoran Zhang.</p>

"Self-Linking for Legendrian Knots" CS

- joint w/ B. McLellan and R. Zhang

egendrian

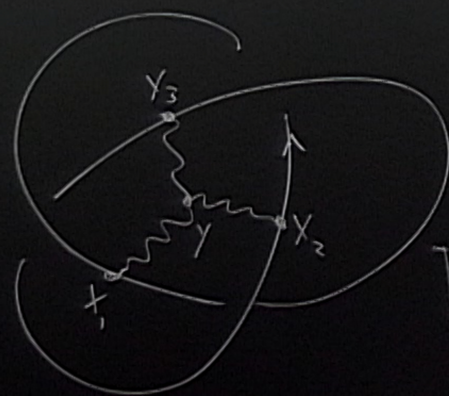
nots "

G = cpct Lie gp

$$CS(A) = \frac{1}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A\right)$$

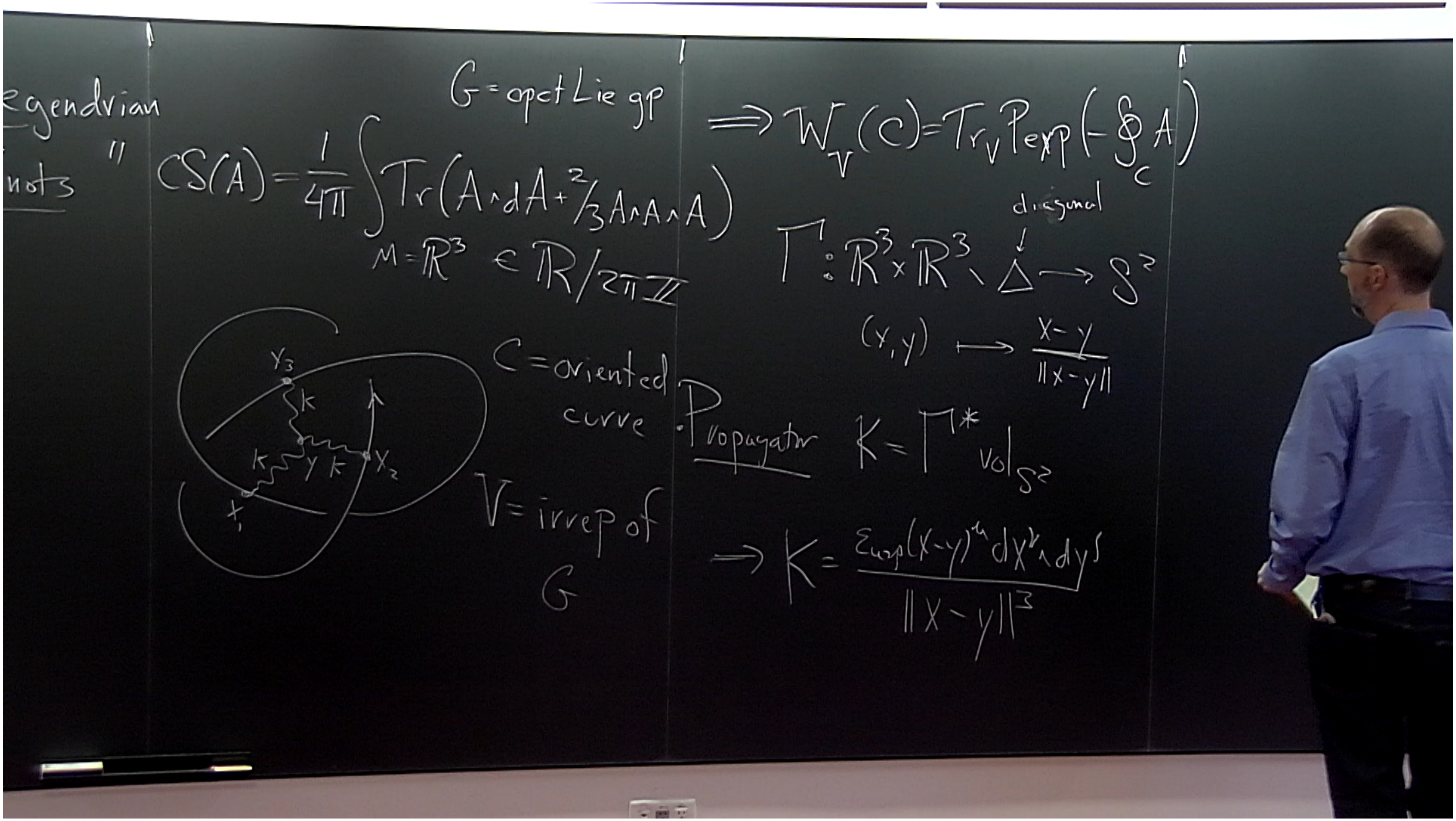
$$M = \mathbb{R}^3 \quad \in \mathbb{R}/2\pi\mathbb{Z}$$

$$\Longrightarrow W_V(C) = \mathrm{Tr}_V\, P\exp\left(-\oint_C A\right)$$



C = oriented curve

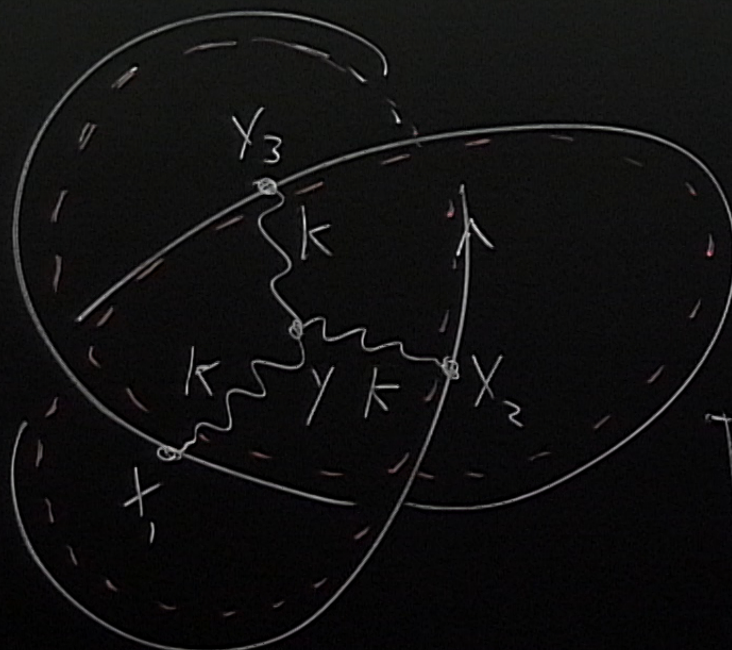
V = irrep of G


egendrian
"
nots
G = cpct Lie gp
$CS(A) = \frac{1}{4\pi} \int \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)$
$M = \mathbb{R}^3$ $\in \mathbb{R}/2\pi\mathbb{Z}$
C = oriented curve
V = irrep of G
$\Longrightarrow W_V(C) = \mathrm{Tr}_V P\exp(-\oint_C A)$
diagonal
$\Gamma: \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta \longrightarrow S^2$
$(x, y) \longmapsto \frac{x-y}{\|x-y\|}$
Propagator $K = \Gamma^* \mathrm{vol}_{S^2}$
$\longrightarrow K = \frac{\varepsilon_{\mu\nu\rho}(x-y)^\mu dx^\nu \wedge dy^\rho}{\|x-y\|^3}$
x_3
x_2
x_1
k
y

Knots

$$CS(A) = \frac{\quad}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \tfrac{2}{3}A\wedge A\wedge A\right)$$

$M = \mathbb{R}^3 \quad \in \mathbb{R}/2\pi\mathbb{Z}$



C = oriented curve

V = irrep of G

· Propagator

$\to K$

E.g. S^1  $= \lim_{\varepsilon \to 0} \int_{C \times C \setminus \Delta_2} K$

"$slk(C)$" $\in \mathbb{R}$ is well-defined

$\notin \mathbb{Z}$, not isotopy invariant

YM-CS

Action

$F_A = dA + A \wedge A$

$$\mathbb{I}(A) = \frac{\exp(-\lambda)}{8\pi e^2} \int_M \mathrm{Tr}\left(F_A \wedge *_0 F_A\right) + i\, k\, CS(A)$$

metric $\quad g = g_0 \cdot \exp(2\lambda), \ \lambda \in \mathbb{R}$

... g for Legendrian Knots

... Lellan and ... Zhang

$T_p M \simeq \mathbb{R} \times \mathbb{C} \circlearrowleft U(1)$

Reeb vector R

$\iota_R \kappa = 1, \ \iota_R d\kappa = 0$

$\Rightarrow \mathcal{L}_R \kappa = 0$

$J^2 = -\mathbb{1}$

$d\kappa|_H \neq 0$

contact condition

$H = \ker(\kappa)$

$\left\{ \begin{array}{l} \kappa \in \Omega^1_M \\ \kappa \sim f\kappa, \ f \neq 0 \end{array} \right.$

contact metric

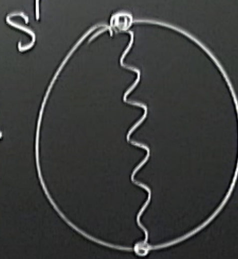
$$ds^2_M = \frac{t}{2\pi} d\kappa(\cdot, J\cdot) + t^2 \kappa \otimes \kappa$$

$r, t \in \mathbb{R}_+$

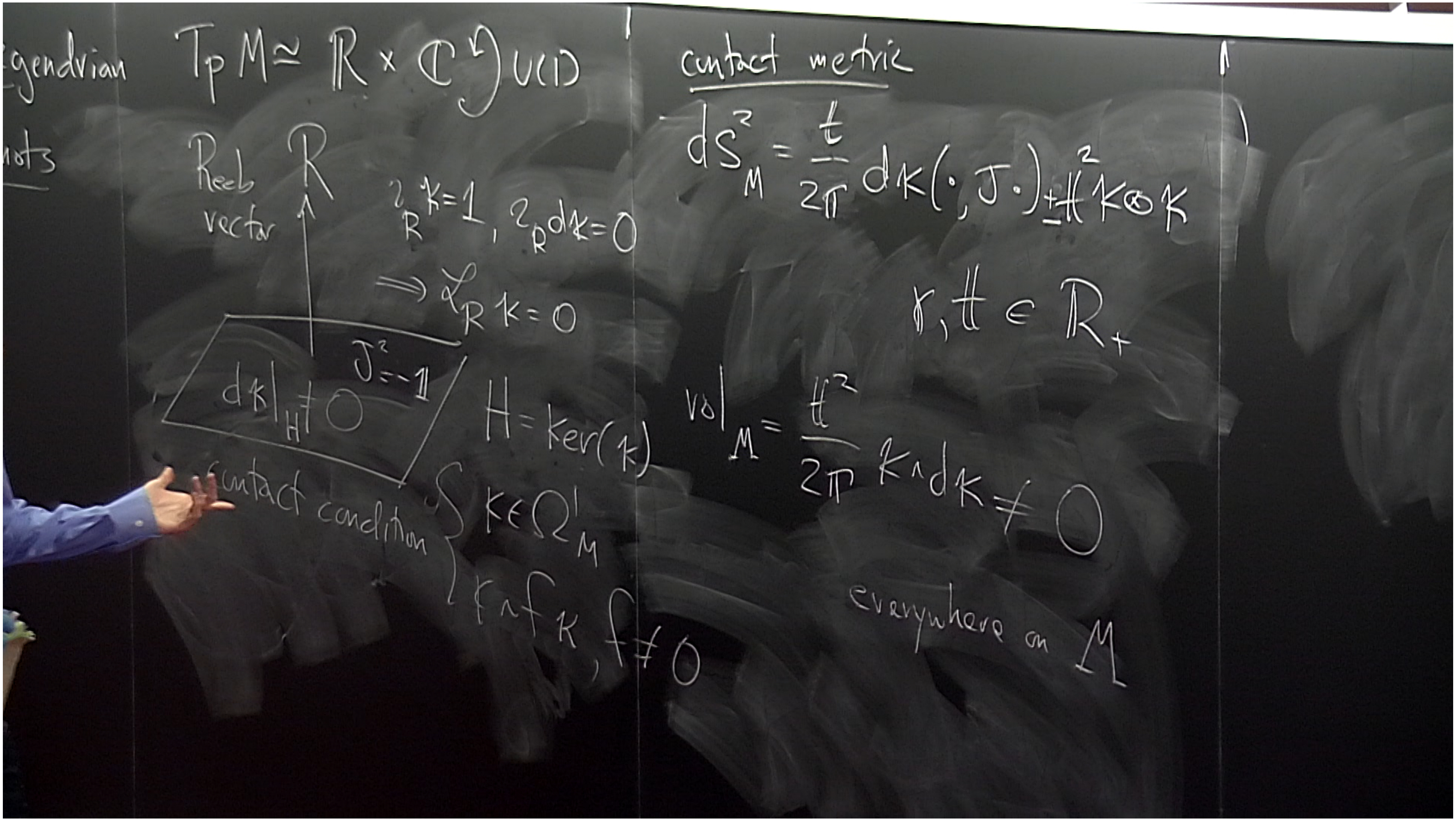
$$\mathrm{vol}_M = \frac{t^2}{2\pi} \kappa \wedge d\kappa \neq 0$$

everywhere on M

E.g. S^1



"$slk(C)$"



...gendrian
...nots
$T_p M \simeq \mathbb{R} \times \mathbb{C}^r / U(1)$
Reeb vector R
$\iota_R \kappa = 1$, $\iota_R d\kappa = 0$
$\Rightarrow \mathcal{L}_R \kappa = 0$
$J^2 = -\mathbb{1}$
$d\kappa|_H \neq 0$
$H = \ker(\kappa)$
contact condition
$\kappa \in \Omega^1_M$
$\kappa \sim f\kappa$, $f \neq 0$
contact metric
$ds^2_M = \frac{r}{2\pi} d\kappa(\cdot, J\cdot) \pm \ell^2 \kappa \otimes \kappa$
$r, \ell \in \mathbb{R}_+$
$vol_M = \frac{\ell^2}{2\pi} \kappa \wedge d\kappa \neq 0$
everywhere on M

...gendrian $\quad T_pM \simeq \mathbb{R}\times\mathbb{C} \circlearrowleft U(1)$

...nots

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$$ds^2_M = \frac{t}{2\pi} d\kappa(\cdot, J\cdot) \pm t^2 \kappa\otimes\kappa$$

$$r, t \in \mathbb{R}_+$$

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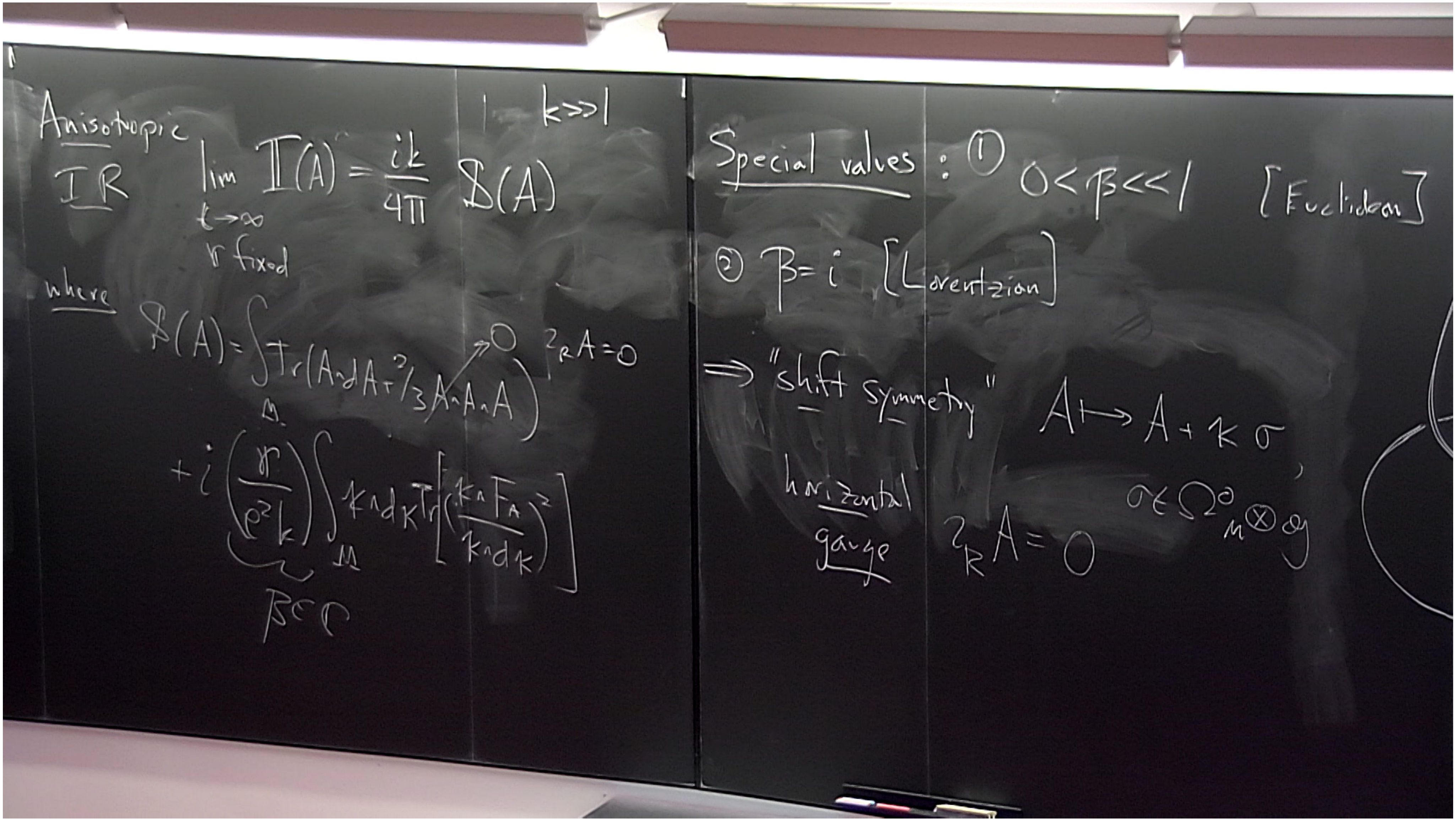
everywhere on M


Anisotropic IR
$\lim_{\substack{t\to\infty \\ r\ \text{fixed}}} \mathbb{I}(A) = \frac{ik}{4\pi} S(A)$
$k \gg 1$
where $S(A) = \int_M \mathrm{Tr}(A\wedge dA + \tfrac{2}{3} A\wedge A\wedge A)$
$\partial_R A = 0$
$+ i \left(\frac{r}{e^2 k}\right) \int_M \kappa\wedge d\kappa\, \mathrm{Tr}\left[\left(\frac{\kappa\wedge F_A}{\kappa\wedge d\kappa}\right)^2\right]$
$\beta \in \mathbb{C}$
Special values: ① $0 < \beta \ll 1$ [Euclidean]
② $\beta = i$ [Lorentzian]
$\Rightarrow$ "shift symmetry" $A \mapsto A + \kappa\sigma$,
horizontal gauge $\partial_R A = 0$
$\sigma \in \Omega^0_M \otimes \mathfrak{g}$

nking for Legendrian Knots

B. McLellan and R. Zhang

Shift-inv Wilson loop

$$W_V(C) = \mathrm{Tr}_V \, P\exp\left(-\oint_C A\right)$$

$$\delta_r W_V(C) = 0 \iff \kappa\big|_C = 0$$

$$\iff T_pC \subset H \quad \forall p \in C$$

$$\Rightarrow C \text{ is Legendrian knot}$$

contact metric

$$ds^2_M = \frac{t}{2\pi}\, d\kappa(\cdot, J\cdot) \pm t^2\, \kappa\otimes$$

$$r, t \in \mathbb{R}_+$$

$$\mathrm{vol}\big|_M = \frac{t^2}{2\pi}\, \kappa\wedge d\kappa \neq 0$$

everywhere on M

don]

$\Pi : \mathbb{R}^3_{std} \to \mathbb{C}$

$\llcorner \mathbb{C}$

$\Pi(c)$

Eg. $M = \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C} \ni (\tau, \xi, \bar{\xi})$

$\alpha_{std} = d\tau + \frac{i}{2}(\xi\, d\bar{\xi} - \bar{\xi}\, d\xi)$

$U(1)$

$R = \partial/\partial\tau$

• SUSY $\Rightarrow \langle W_v(C)\rangle$

$\oint_C A)$

independent of β

$= 0$ $\Longrightarrow$ # of H, modulo

$\forall p \in C$ framing

Legendrian knot

Perturbation theory:

$A = \cancel{A_0}^{\text{trivial}} + \mathcal{B}, \quad \partial_R \mathcal{B} = 0$

harmonic gauge on H

$$\mathcal{S}_{g.f.}(\mathcal{B}) = \int_M \mathrm{Tr}\left(\mathcal{B}\wedge \underset{\mathcal{B}}{\partial}\mathcal{B} - \frac{i\beta}{2}\kappa\wedge\mathcal{B}\wedge *_H(d_H d_H^\dagger)\mathcal{B}\right) =$$

$$+\frac{i\beta}{2}\int_M \kappa\wedge\mathrm{Tr}\left(d_H\mathcal{B}\wedge *_H[\mathcal{B},\mathcal{B}] + \frac{1}{4}[\mathcal{B},\mathcal{B}]\wedge *_H[\mathcal{B},\mathcal{B}]\right)$$

$+$ ghosts

κ

d_H

$\iota_R B = 0$

gauge on H

$\wedge *_H (d_H d_H^\dagger) B$

K

$\wedge *_H [B, B]$

Contact operators

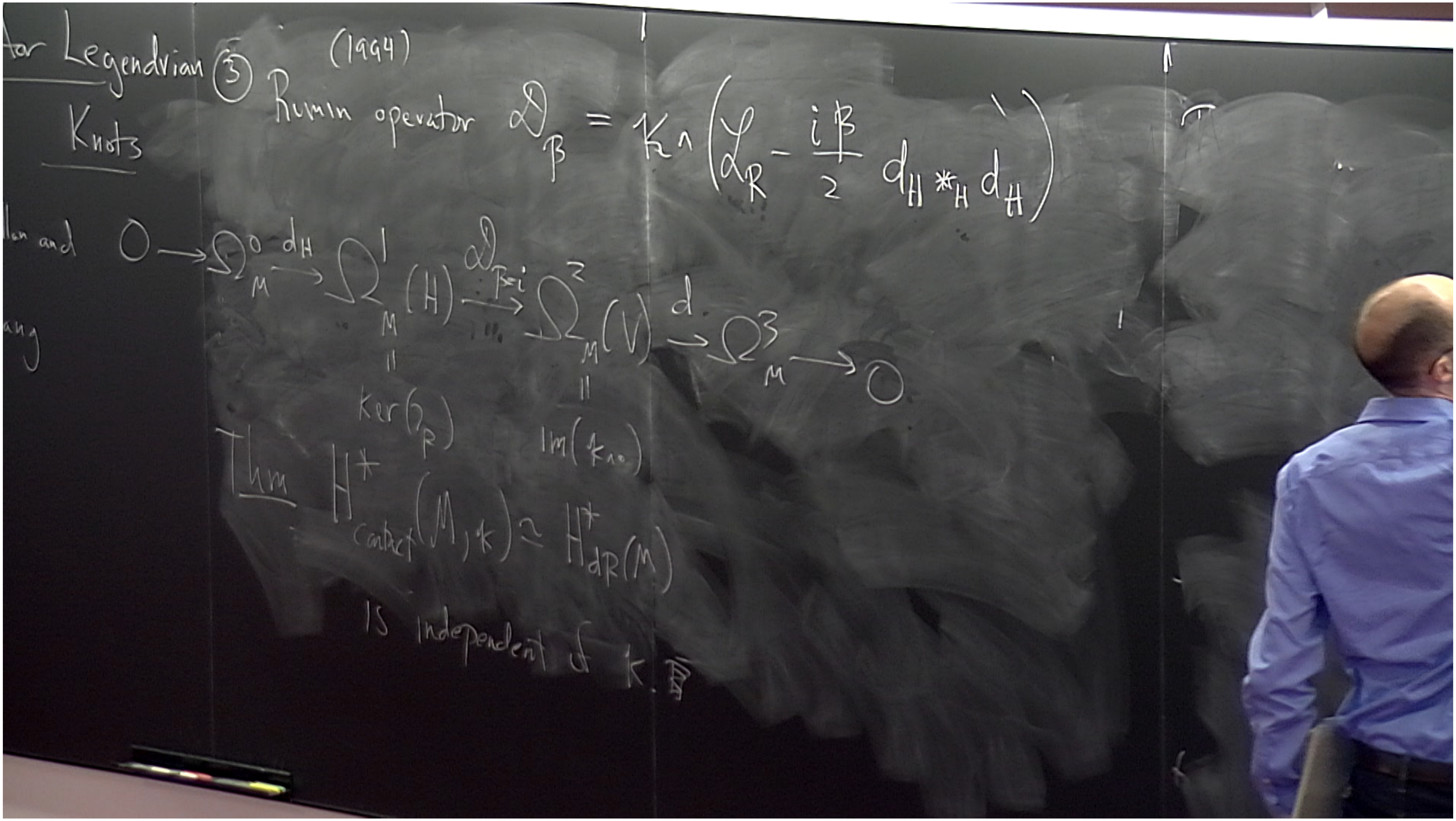
① $\Omega^0_M \xleftrightarrow{*_H} \Omega^2_M \qquad *_H 1 = \frac{1}{2} d\kappa$

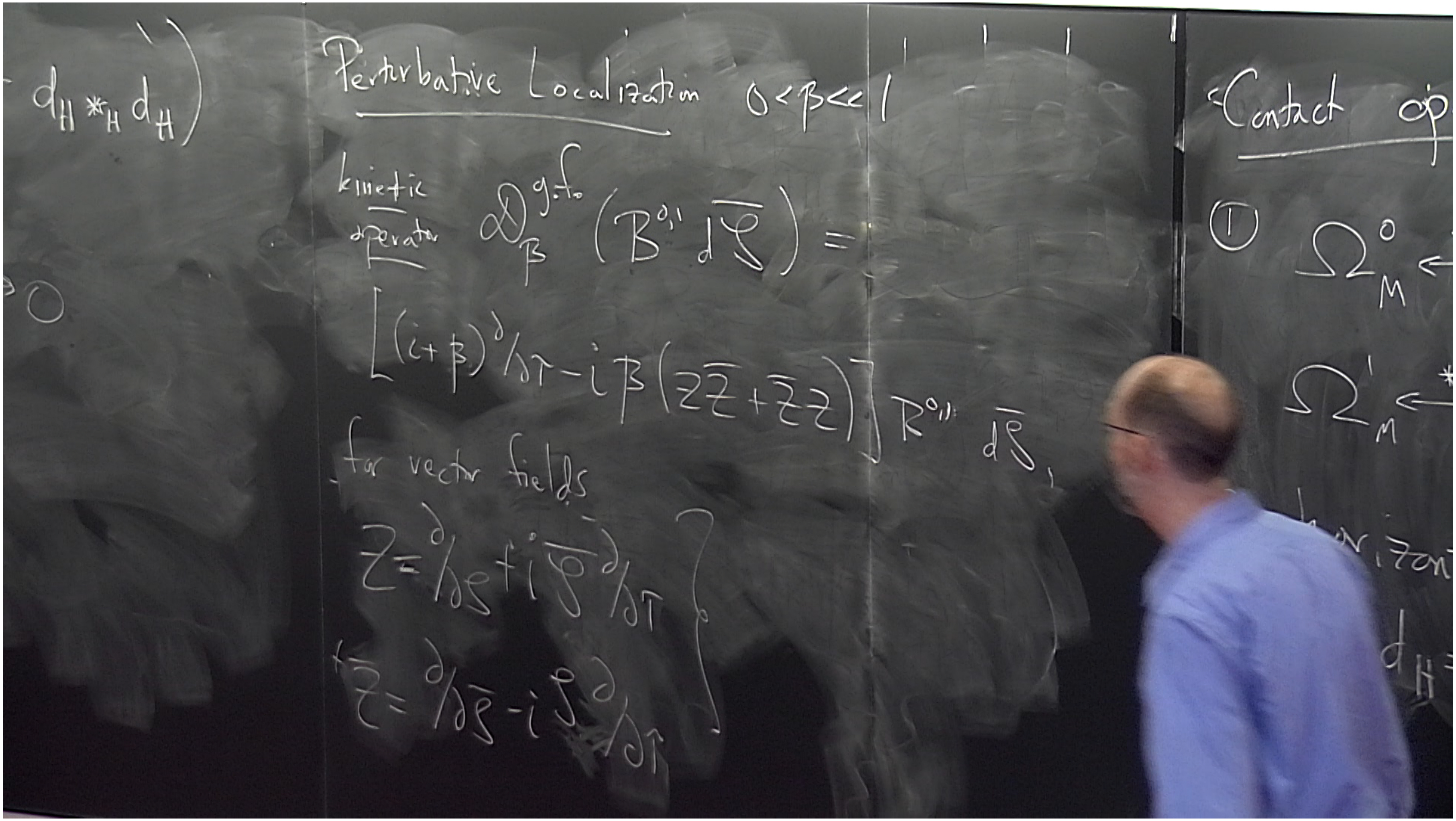
$\Omega^1_M \xleftrightarrow{*_H} \Omega^1_M \qquad *_H = *(\kappa \wedge \cdot)$

② horizontal de Rham

$$d_H = d - \kappa \wedge \iota_R \circ d$$

$\Pi : \mathbb{R}^3_{std} \to \mathbb{C}$


for Legendrian
Knots
(1994)
③ Rumin operator $\mathcal{D}_\beta = \kappa \wedge \left(\mathcal{L}_R - \frac{i\beta}{2} d_H *_H d_H\right)$
and
$0 \to \Omega^0_M \xrightarrow{d_H} \Omega^1_M(H) \xrightarrow{\mathcal{D}_{\beta=i}} \Omega^2_M(V) \xrightarrow{d} \Omega^3_M \to 0$
$\ker(\eta_R)$
$\mathrm{Im}(\kappa\wedge\cdot)$
Thm $H^*_{\text{contact}}(M,\kappa) \simeq H^*_{dR}(M)$
is independent of κ.


Perturbative Localization 0<β<<1
kinetic operator
for vector fields
Contact op

$d_H *_H d_H \big)$

$\ominus \bigcirc$

Perturbative Localization $0 < \beta << 1$

kinetic operator $\mathcal{D}_\beta^{g.f.} (B^{0,1} d\bar{\rho}) =$

$\left[(i+\beta)\partial/\partial\tau - i\beta\left(Z\bar{Z} + \bar{Z}Z\right)\right] B^{0,1}\, d\bar{\rho},$

for vector fields

$$\left.\begin{aligned} Z &= \partial/\partial\rho + i\bar{\rho}\,\partial/\partial\tau \\ \bar{Z} &= \partial/\partial\bar{\rho} - i\rho\,\partial/\partial\tau \end{aligned}\right\}$$

Contact op

① $\Omega^0_M \leftarrow$

$\Omega^1_M \leftarrow$

② horizont

d_H

$d_H *_H d_H \Big)$

$= 0$

Perturbative Localization $\quad 0<\beta<<1$

kinetic operator $\quad \mathcal{D}_{\beta}^{g.f.}\left(B^{0,1}\, d\bar{\varrho}\right) =$

$$\boxed{\left[(i+\beta)\,\partial/\partial\tau - i\beta\left(Z\bar{Z}+\bar{Z}Z\right)\right] B^{0,1}\, d\bar{\varrho}},$$

for vector fields

$$\left.\begin{aligned} Z &= \partial/\partial\varrho + i\bar{\varrho}\,\partial/\partial\tau \\ \bar{Z} &= \partial/\partial\bar{\varrho} - i\varrho\,\partial/\partial\tau \end{aligned}\right\}$$

left-inv Heisenberg $\quad \mathbb{R}^3 \simeq H$

$$[Z,\bar{Z}] = -i\,\partial/\partial\tau$$

$=$

2

$$\Rightarrow \mathcal{D}^{g.f.}_{\beta}\left(B^{0,1}\, d\bar{\xi}\right) = i\left[\partial/\partial\tau - 2\beta \frac{\partial^2}{\partial\xi\,\partial\bar{\xi}}\right] B^{0,1}\, d\bar{\xi}$$

$$+\,O(|\xi|)$$

$R^{0,1}\ d\bar{\xi}$,

Heisenberg

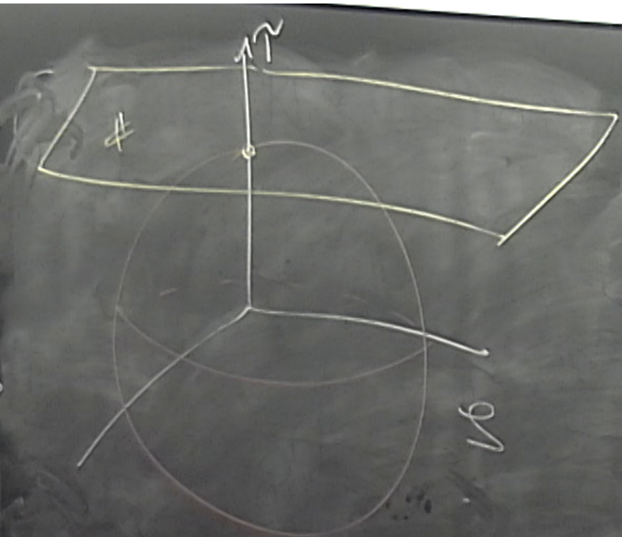
$\mathbb{R}^3 \curvearrowright H$

$= -i\,\partial/\partial\tau$

Π :

...ing for Legendrian Knots

McLellan and Zhang



• Gaussian class $\omega_\beta = \frac{i}{4\pi\beta} \exp\left[-|\xi|^2/2\beta\right] d\xi \wedge d\bar{\xi}$

• Parabolic scaling $f_+ : \mathbb{R}^3_+ \longrightarrow \mathbb{R}^2_1$

$f_+(\tau, \xi, \bar{\xi}) = \left(\frac{\xi}{\sqrt{\tau}}, \frac{\bar{\xi}}{\sqrt{\tau}}\right)$

Perturbati...

kinetic operator

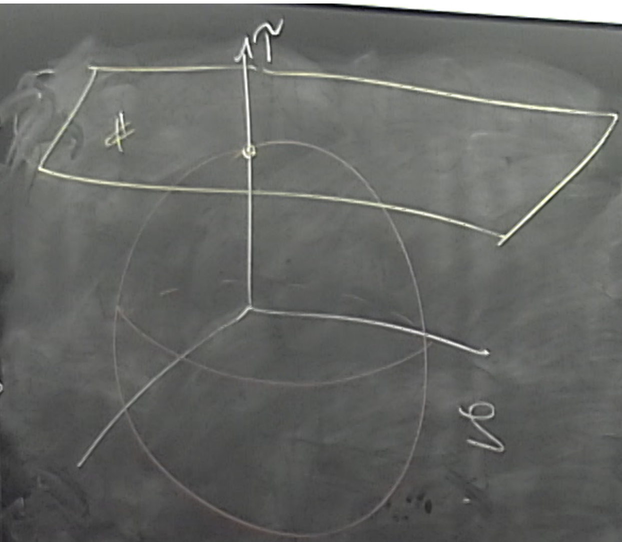
$\Big[(i + \beta)$ ...

for vector

$Z = \partial/\partial\xi$

...ing for Legendrian Knots

McLellan and Zhang

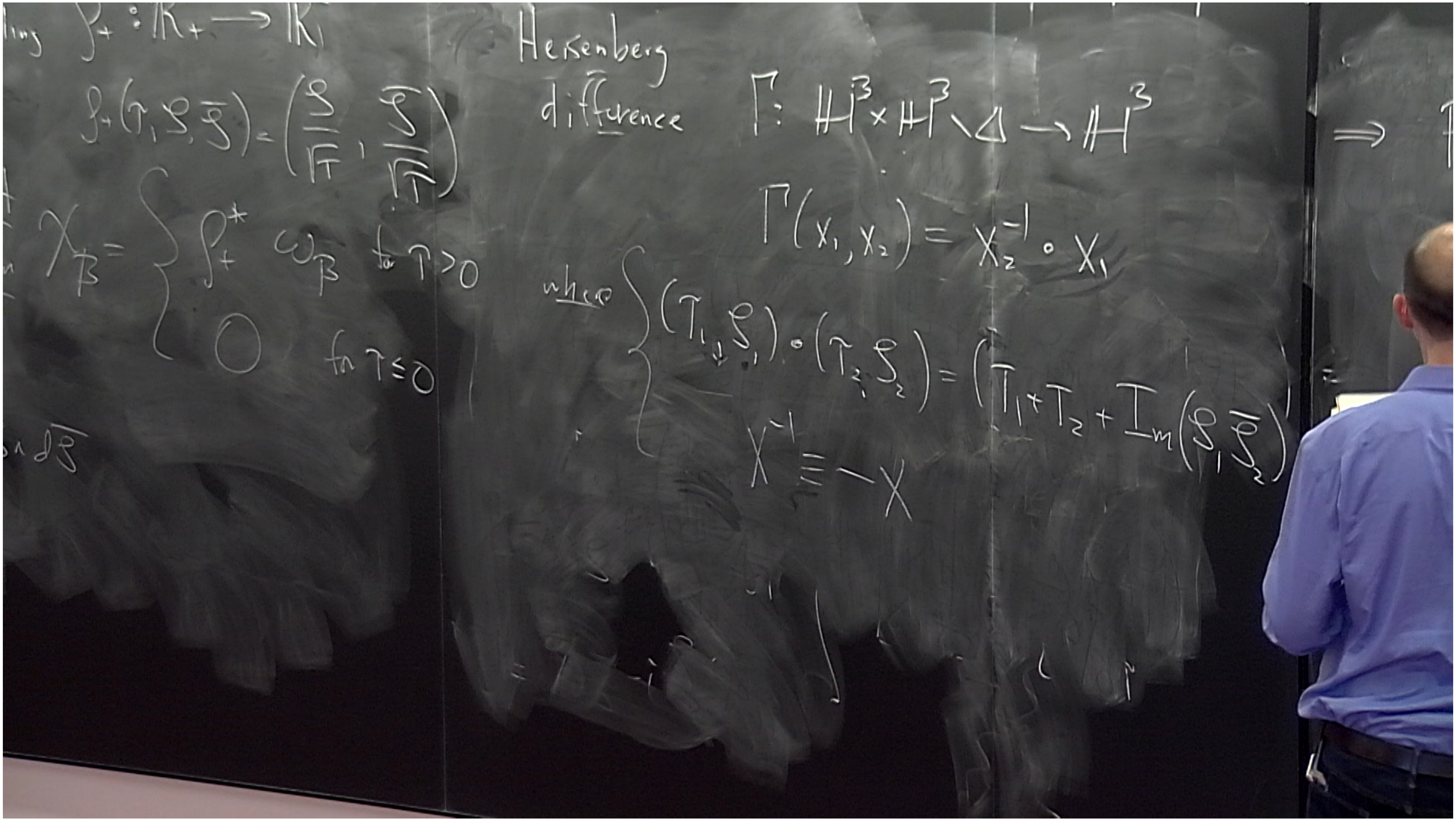


∘ Gaussian class $\omega_\beta = \frac{i}{4\pi\beta}\exp\left[-|\rho|^2/2\beta\right] d\rho \wedge d\bar{\rho}$

∘ Parabolic scaling $\rho_+ : \mathbb{R}^3_+ \longrightarrow \mathbb{R}^2_1$

$$\rho_+(\tau, \rho, \bar{\rho}) = \left(\frac{\rho}{\sqrt{\tau}}, \frac{\bar{\rho}}{\sqrt{\tau}}\right)$$

∘ Heat form $\chi_\beta = \begin{cases} \rho_+^* \, \omega_\beta & \text{for } \tau > 0 \\ 0 & \text{for } \tau \leq 0 \end{cases}$


$\rho_+ : \mathbb{R}_+ \longrightarrow \mathbb{R}$
$\rho_+(\tau, \varsigma, \bar\varsigma) = \left(\frac{\varsigma}{\sqrt{\tau}}, \frac{\bar\varsigma}{\sqrt{\tau}}\right)$
$\chi_\beta = \begin{cases} \rho_+^* \, \omega_\beta & \text{for } \tau > 0 \\ 0 & \text{for } \tau \leq 0 \end{cases}$
Heisenberg difference
$\Gamma : \mathbb{H}^3 \times \mathbb{H}^3 \setminus \Delta \longrightarrow \mathbb{H}^3$
$\Gamma(x_1, x_2) = x_2^{-1} \circ x_1$
where
$(\tau_1, \varsigma_1) \circ (\tau_2, \varsigma_2) = (\tau_1 + \tau_2 + \mathrm{Im}(\varsigma_1 \bar\varsigma_2)$
$x^{-1} \equiv -x$

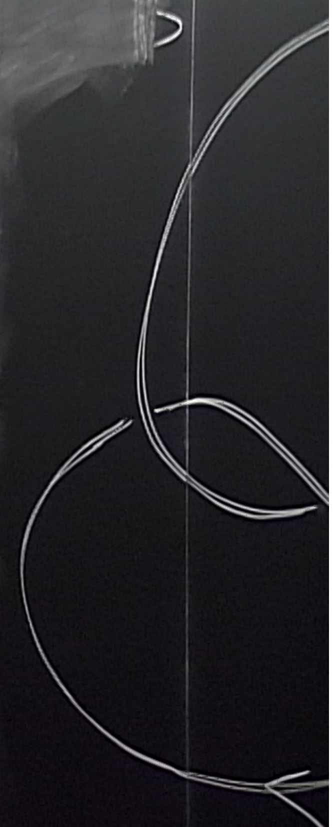
$\Delta \to \mathbb{H}^3$

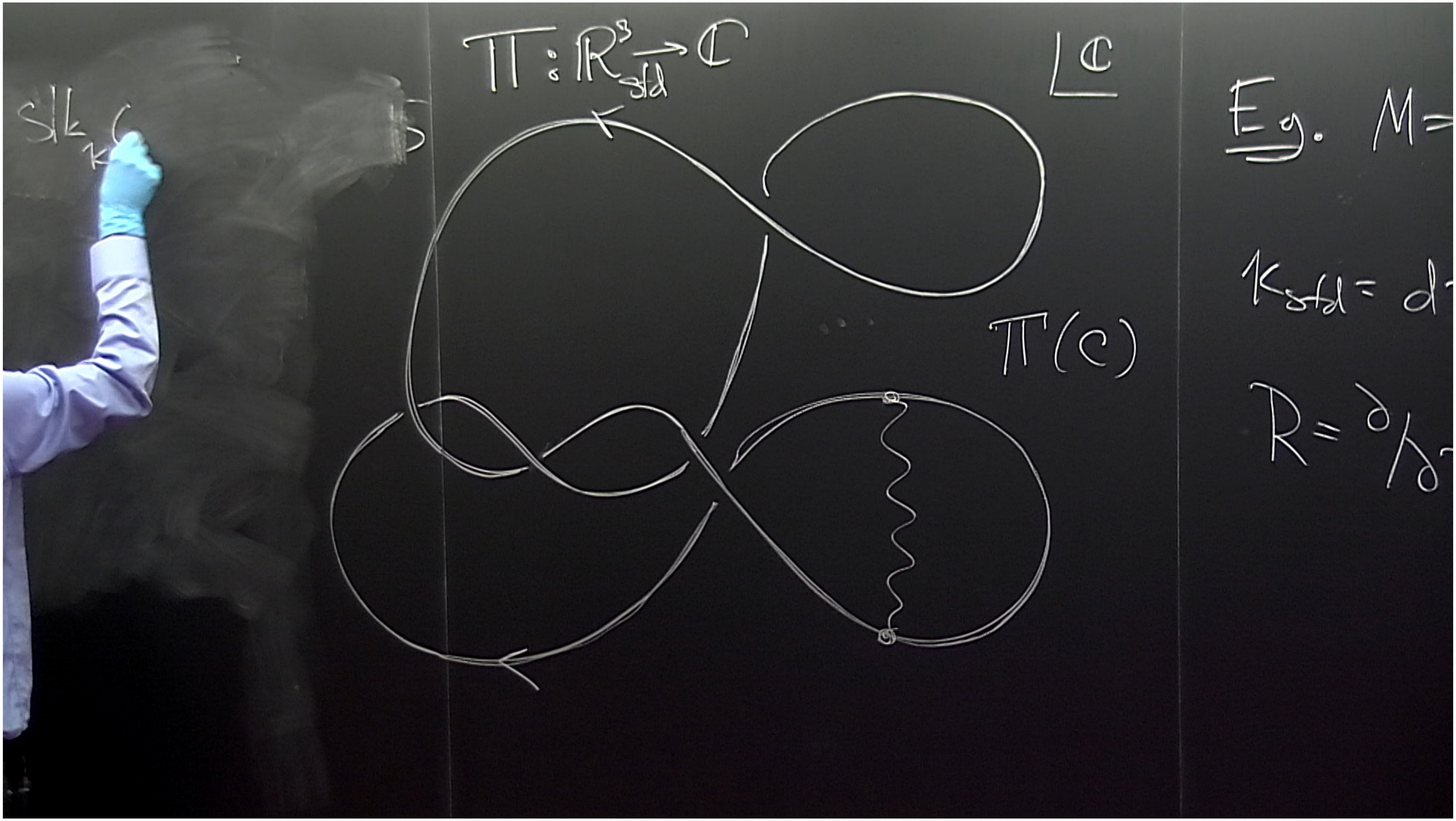
$X_2^{-1} \circ X_1$

$\left(T_1 + T_2 + I_m(\rho_1 \bar{\rho}_2)\right) \quad (\rho_1 + \rho_2)$

$\Rightarrow$ For $0 < \beta \ll 1$,

$$\boxed{K_\beta = \Gamma^* \chi_\beta}$$




Π : R³std → ℂ
L ⊂ ℂ
π(c)
E.g. M=
R =

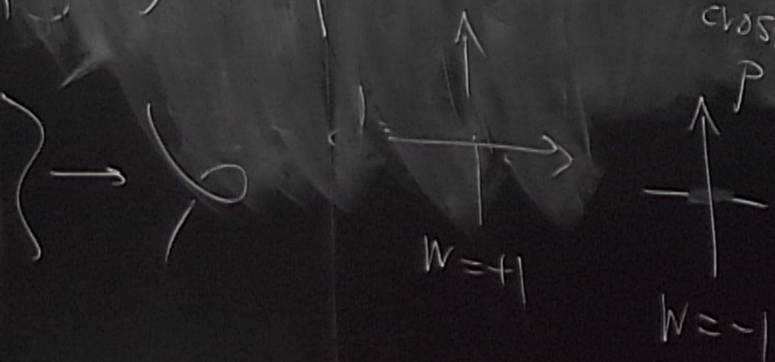
$\Rightarrow$ For $0 < \beta << 1$,

$$\boxed{K_\beta = \Gamma^* \chi_\beta}$$

(ξ_1, ξ_2) $(\xi_1 + \xi_2)$

$$slk_K(C) = \lim_{\varepsilon \to 0} \int_{C \times C \setminus \Delta_\varepsilon} K_\beta = \sum_{\substack{\text{crossing} \\ p}} w_p = tb(C)$$

"Thurston-Bennequin"



$\Pi : \mathbb{R}^3_{std} \to \mathbb{C}$