Title: PSI 2016/2017 Quantum Field Theory III - Lecture 15

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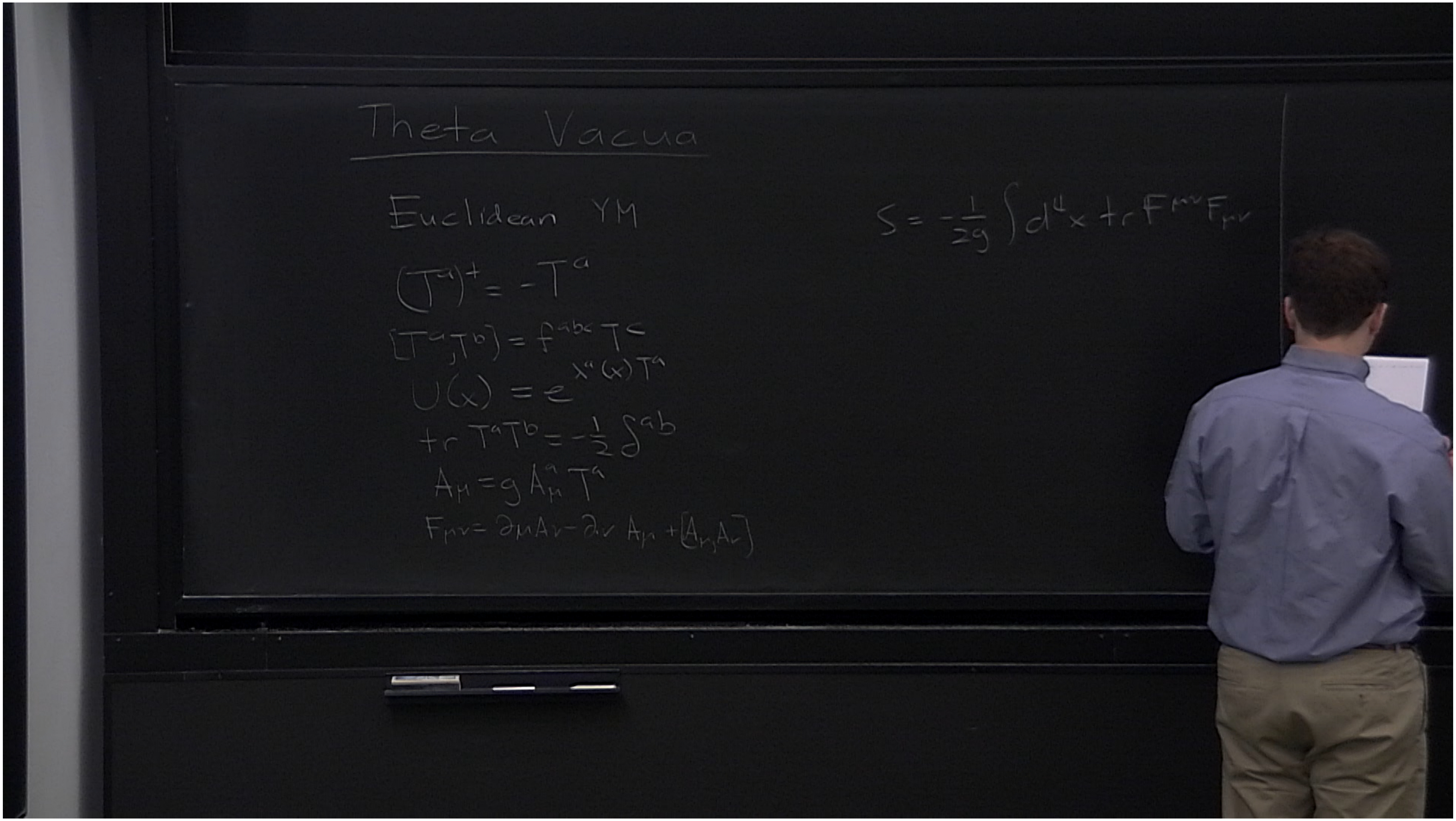
Abstract:



Theta Vacua

Euclidean YM

$(T^a)^\dagger = -T^a$

$[T^a, T^b] = f^{abc} T^c$

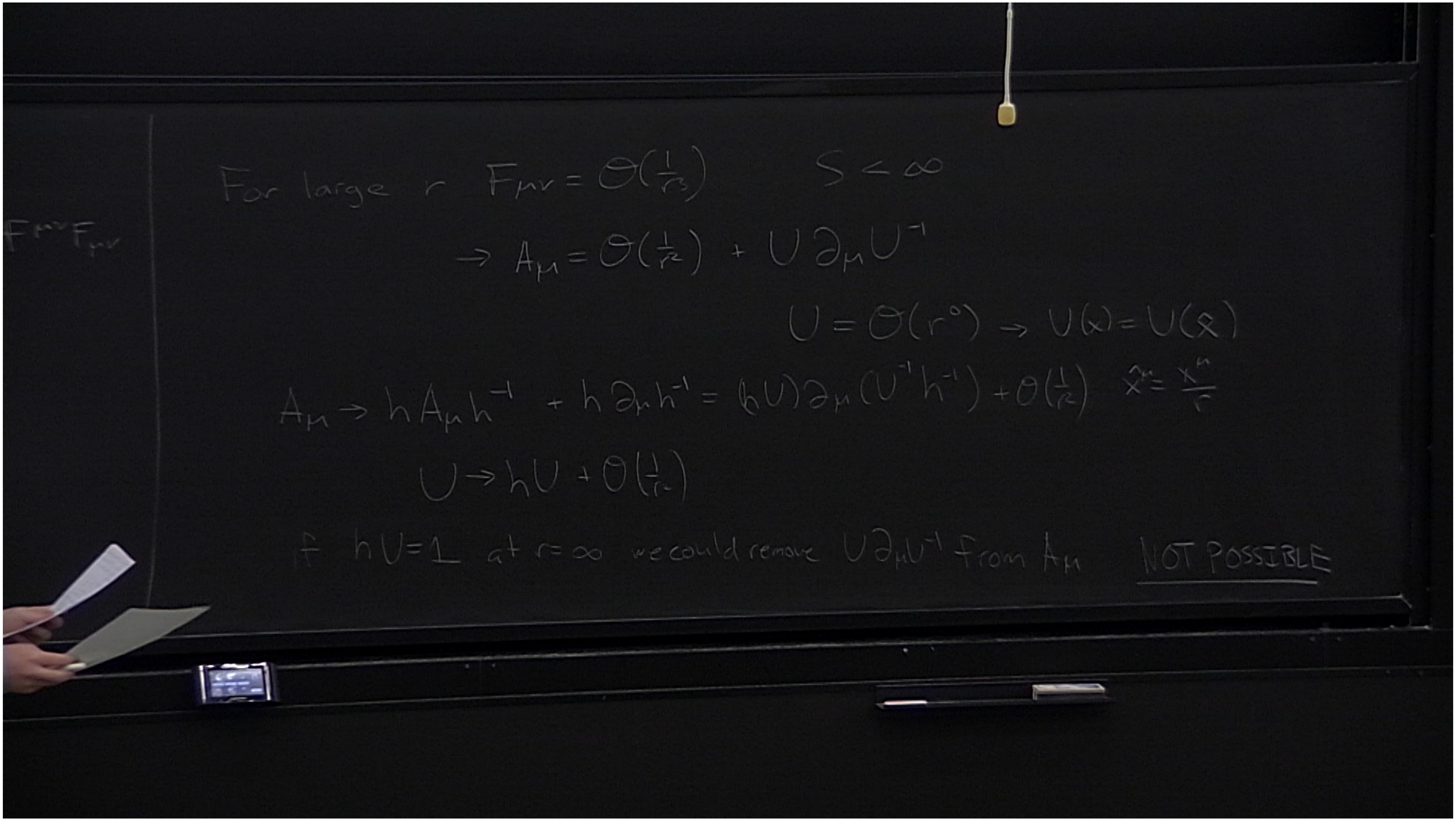
$U(x) = e^{\lambda^a(x) T^a}$

$\mathrm{tr}\, T^a T^b = -\frac{1}{2} \delta^{ab}$

$A_\mu = g A_\mu^a T^a$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

$S = -\frac{1}{2g} \int d^4x \, \mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}$



$F^{\mu\nu}F_{\mu\nu}$

For large r $F_{\mu\nu} = \mathcal{O}(\frac{1}{r^3})$ $S < \infty$

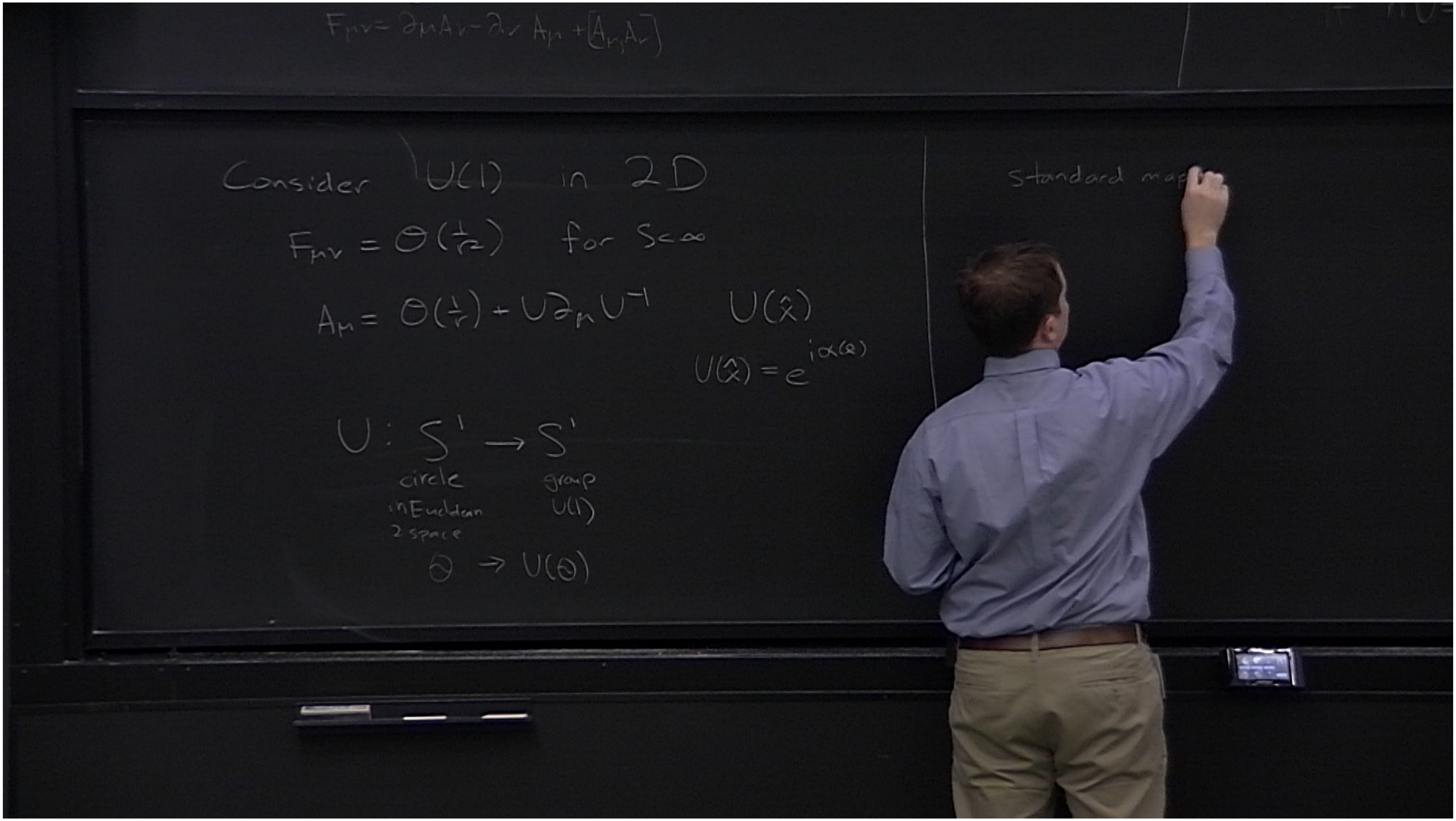
$\rightarrow A_\mu = \mathcal{O}(\frac{1}{r^2}) + U\partial_\mu U^{-1}$

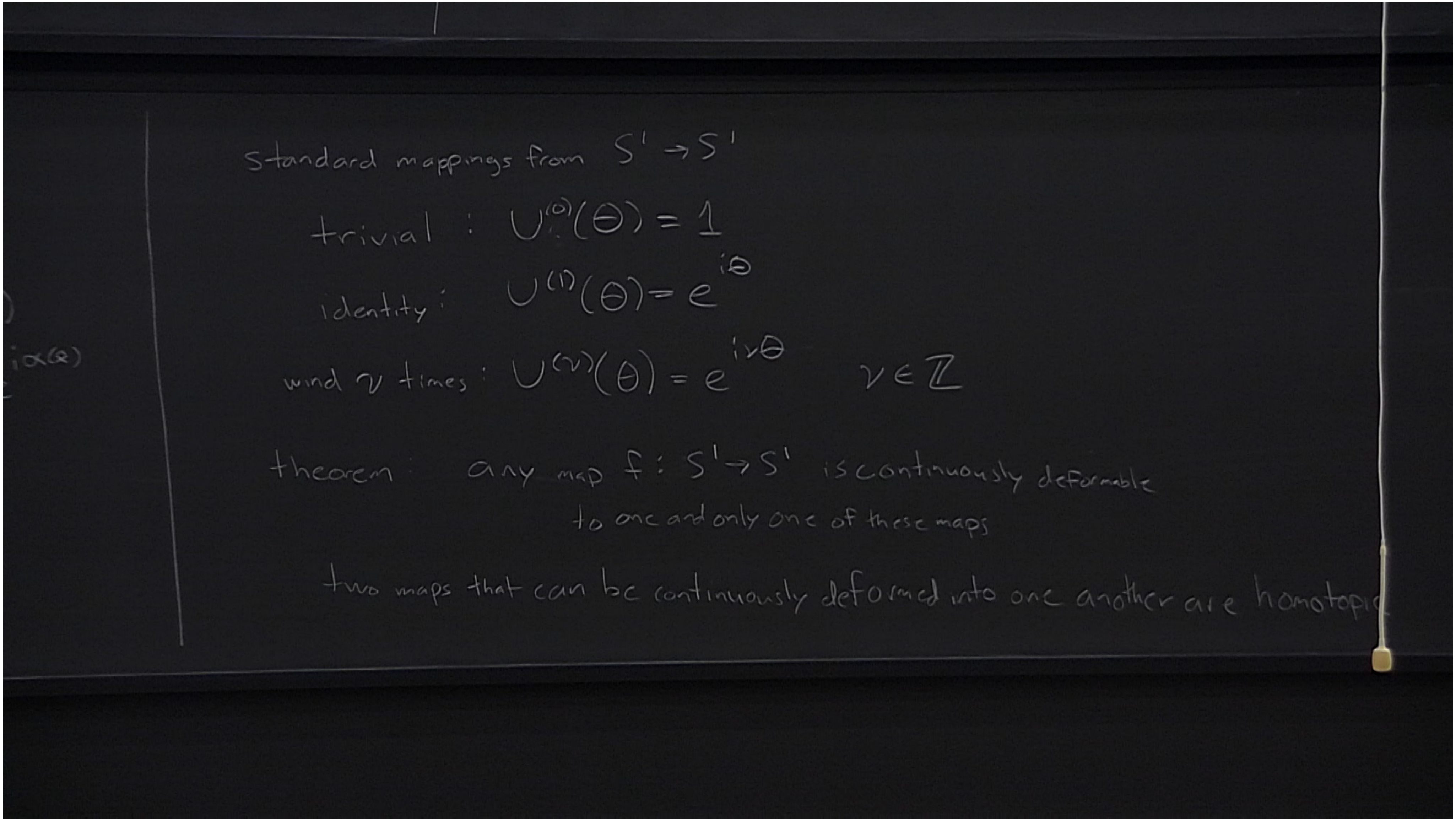
$U = \mathcal{O}(r^0) \rightarrow U(x) = U(\hat{x})$

$A_\mu \rightarrow h A_\mu h^{-1} + h\partial_\mu h^{-1} = (hU)\partial_\mu(U^{-1}h^{-1}) + \mathcal{O}(\frac{1}{r^2})$ $\hat{x}^\mu = \frac{x^\mu}{r}$

$U \rightarrow hU + \mathcal{O}(\frac{1}{r^2})$

If $hU = 1$ at $r = \infty$ we could remove $U\partial_\mu U^{-1}$ from A_μ $\underline{\text{NOT POSSIBLE}}$





standard mappings from $S^1 \to S^1$

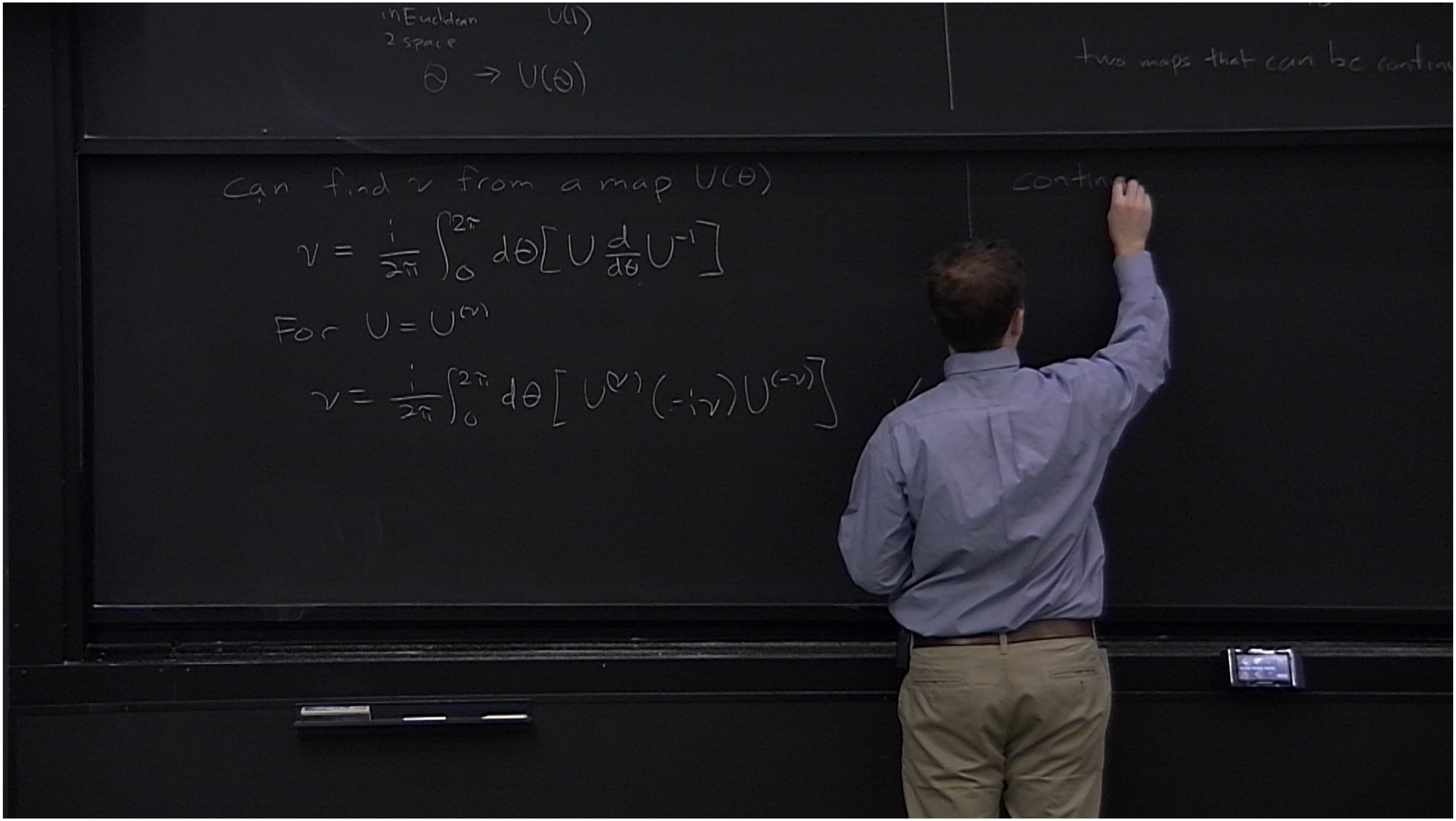
trivial : $U^{(0)}(\theta) = 1$

identity: $U^{(1)}(\theta) = e^{i\theta}$

wind ν times: $U^{(\nu)}(\theta) = e^{i\nu\theta}$ $\nu \in \mathbb{Z}$

theorem: any map $f : S^1 \to S^1$ is continuously deformable to one and only one of these maps

two maps that can be continuously deformed into one another are homotopic



in Euclidean 2 space $U(1)$

$\theta \to U(\theta)$

two maps that can be contin

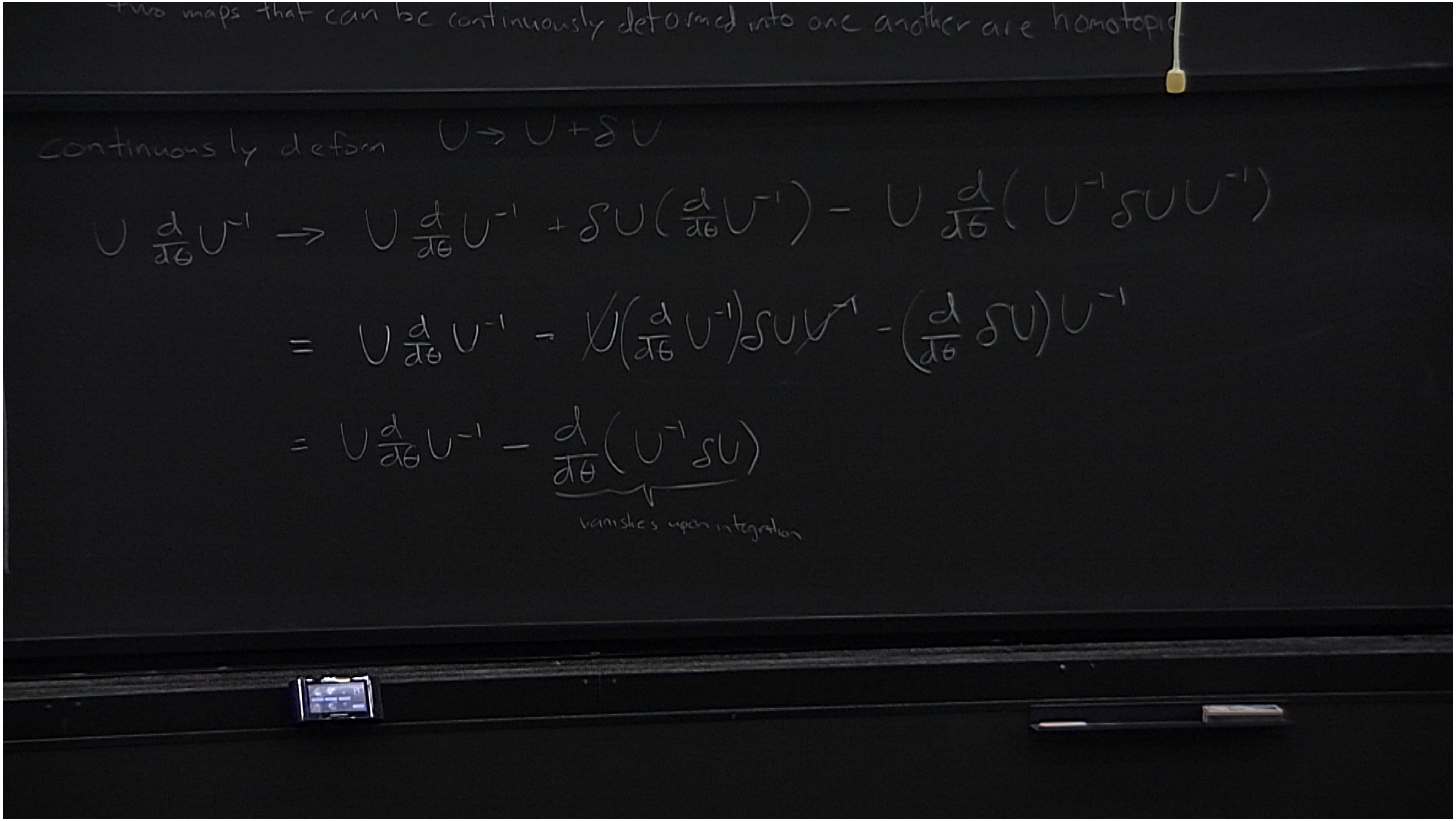
Can find ν from a map $U(\theta)$

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta \left[U \frac{d}{d\theta} U^{-1} \right]$$

For $U = U^{(\nu)}$

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta \left[U^{(\nu)} (-i\nu) U^{(-\nu)} \right]$$

contin



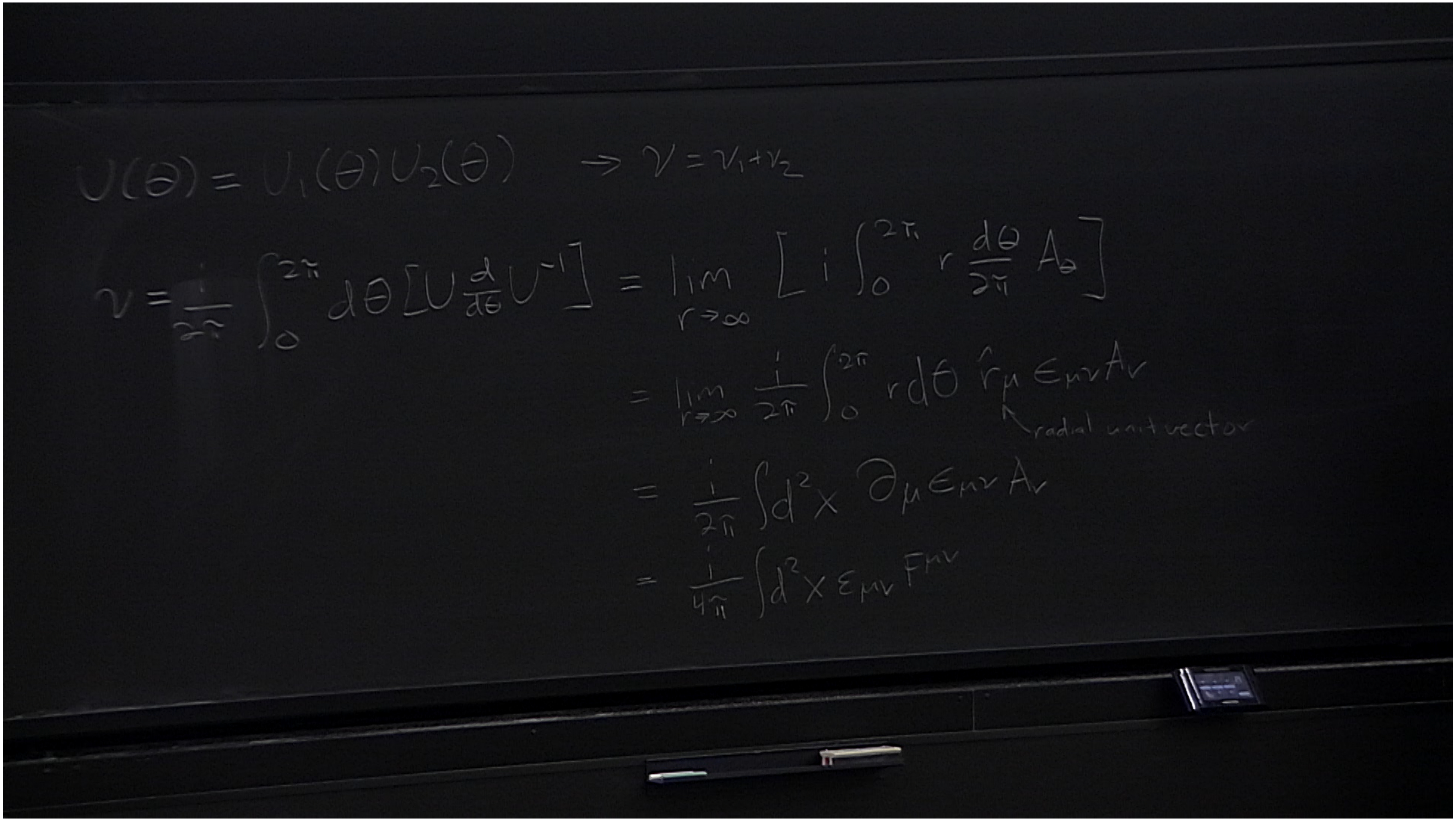
two maps that can be continuously deformed into one another are homotopic

continuously deform $U \to U + \delta U$

$$U \frac{d}{d\theta} U^{-1} \longrightarrow U \frac{d}{d\theta} U^{-1} + \delta U \left(\frac{d}{d\theta} U^{-1} \right) - U \frac{d}{d\theta} \left(U^{-1} \delta U U^{-1} \right)$$

$$= U \frac{d}{d\theta} U^{-1} - \cancel{U} \left(\frac{d}{d\theta} U^{-1} \right) \delta U \cancel{U^{-1}} - \left(\frac{d}{d\theta} \delta U \right) U^{-1}$$

$$= U \frac{d}{d\theta} U^{-1} - \underbrace{\frac{d}{d\theta} \left(U^{-1} \delta U \right)}_{\text{vanishes upon integration}}$$



$$U(\theta) = U_1(\theta)U_2(\theta) \rightarrow \nu = \nu_1 + \nu_2$$

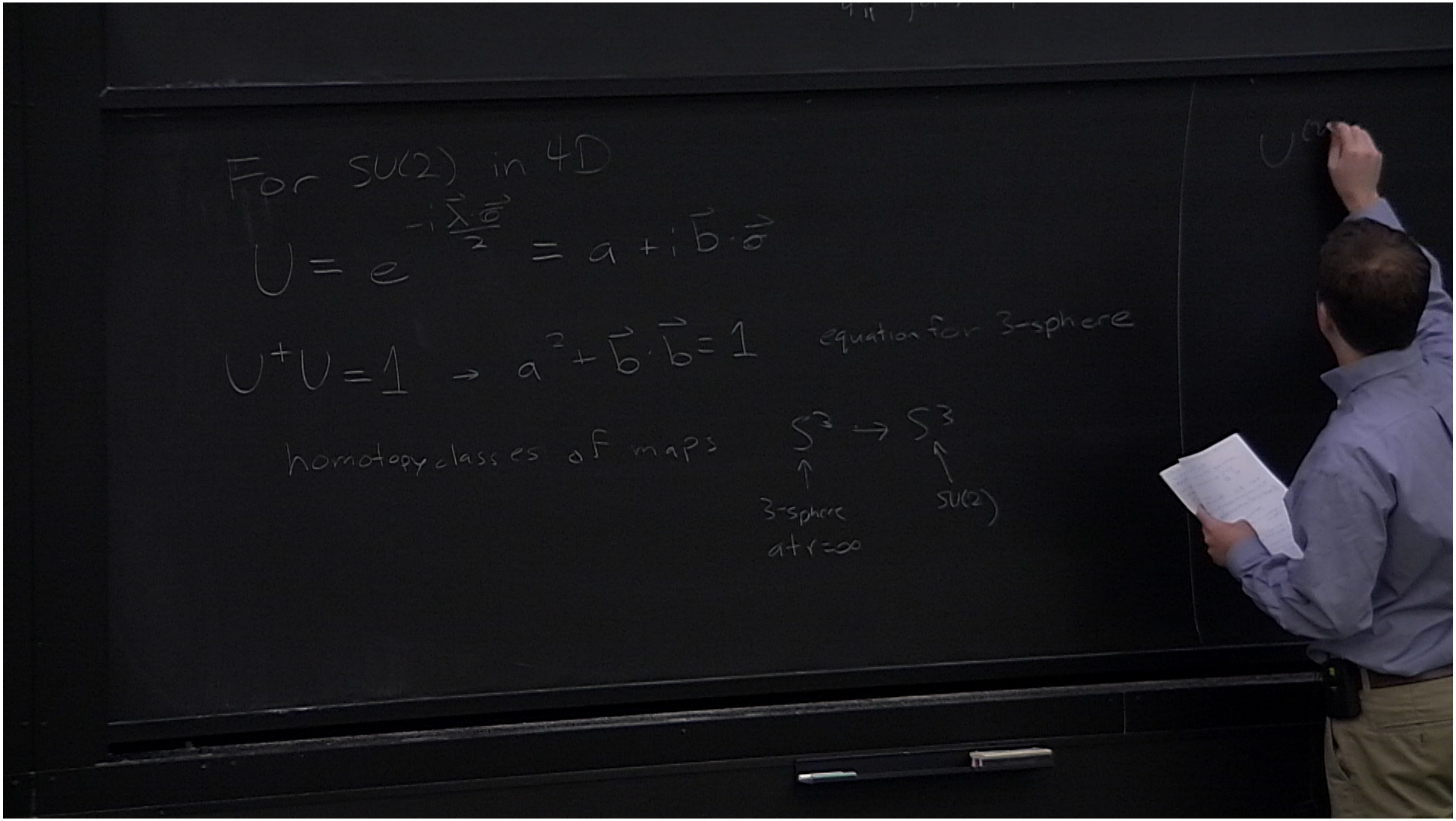
$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta \left[U \frac{d}{d\theta} U^{-1}\right] = \lim_{r\to\infty}\left[i \int_0^{2\pi} r \frac{d\theta}{2\pi} A_\theta \right]$$

$$= \lim_{r\to\infty} \frac{i}{2\pi}\int_0^{2\pi} r d\theta \, \hat{r}_\mu \epsilon_{\mu\nu} A_\nu$$

radial unit vector

$$= \frac{i}{2\pi}\int d^2x \, \partial_\mu \epsilon_{\mu\nu} A_\nu$$

$$= \frac{i}{4\pi}\int d^2x \, \varepsilon_{\mu\nu} F^{\mu\nu}$$



For SU(2) in 4D

$$U = e^{-i\frac{\vec{X}\cdot\vec{\sigma}}{2}} = a + i\vec{b}\cdot\vec{\sigma}$$

$$U^{+}U = 1 \rightarrow a^2 + \vec{b}\cdot\vec{b} = 1$$

equation for 3-sphere

homotopy classes of maps $\underset{\substack{\uparrow \\ \text{3-sphere} \\ \text{at } r=\infty}}{S^3} \rightarrow \underset{\substack{\uparrow \\ SU(2)}}{S^3}$

U

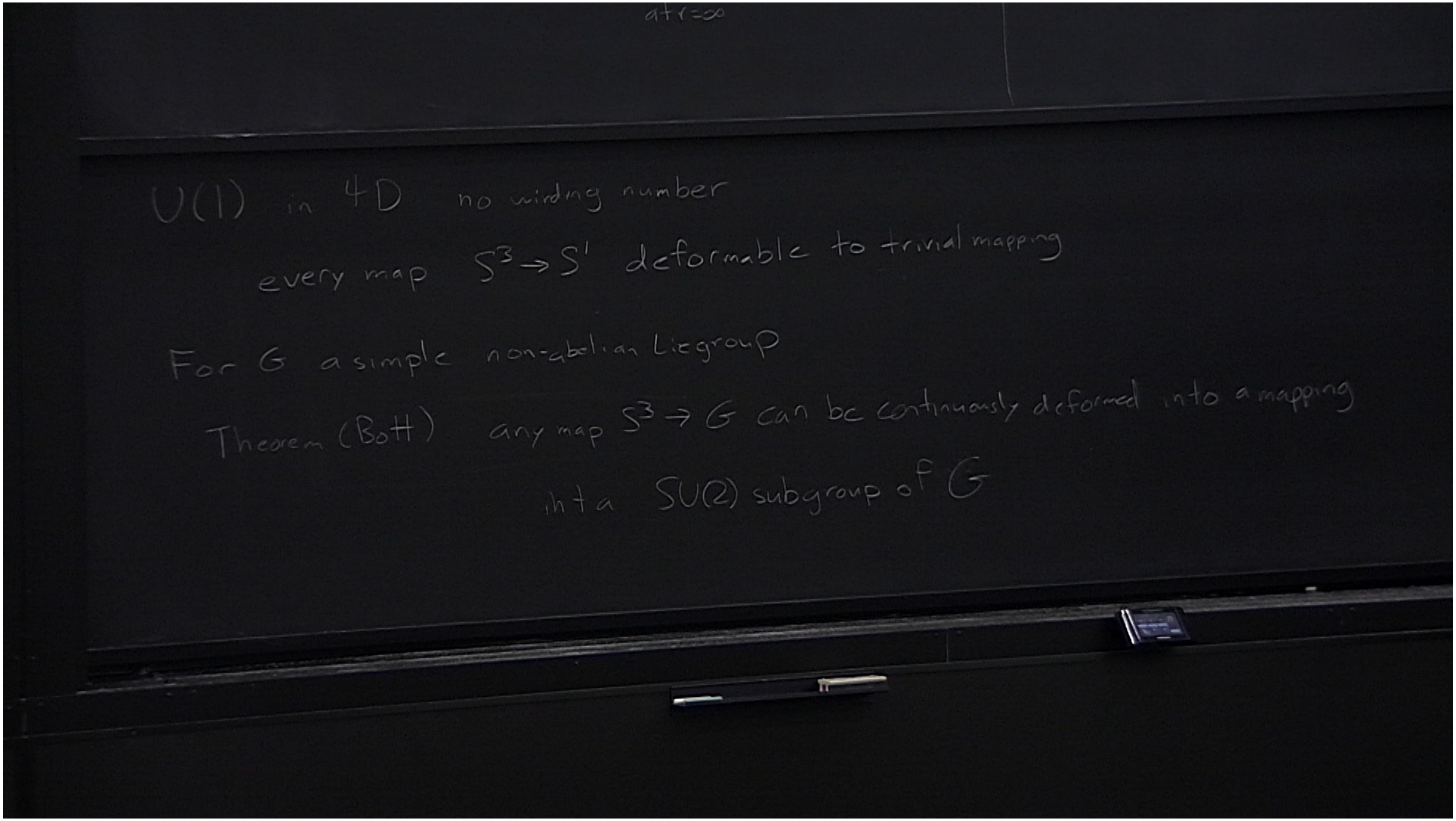
re

$$U^{(\nu)}(\hat{x}) = [\hat{x}_4 + i\hat{\vec{x}}\cdot\vec{\sigma}]^{\nu} \qquad \hat{x}^{\mu} = \frac{x^{\mu}}{r} \qquad r^2 = x_4^2 + \vec{x}^2$$

any $f : S^3 \to S^3$ homotopic to one and only one $U^{(\nu)}$

$$\nu = -\frac{1}{24\pi^2}\int d\Omega \, \mathrm{Tr}\left[\epsilon^{ijk}(U\partial_i U^{-1})(U\partial_j U^{-1})(U\partial_k U^{-1})\right]$$

$$= \frac{1}{16\pi^2}\int d^4x \, \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} \, \mathrm{tr} \, F^{\mu\nu}F^{\rho\sigma}$$



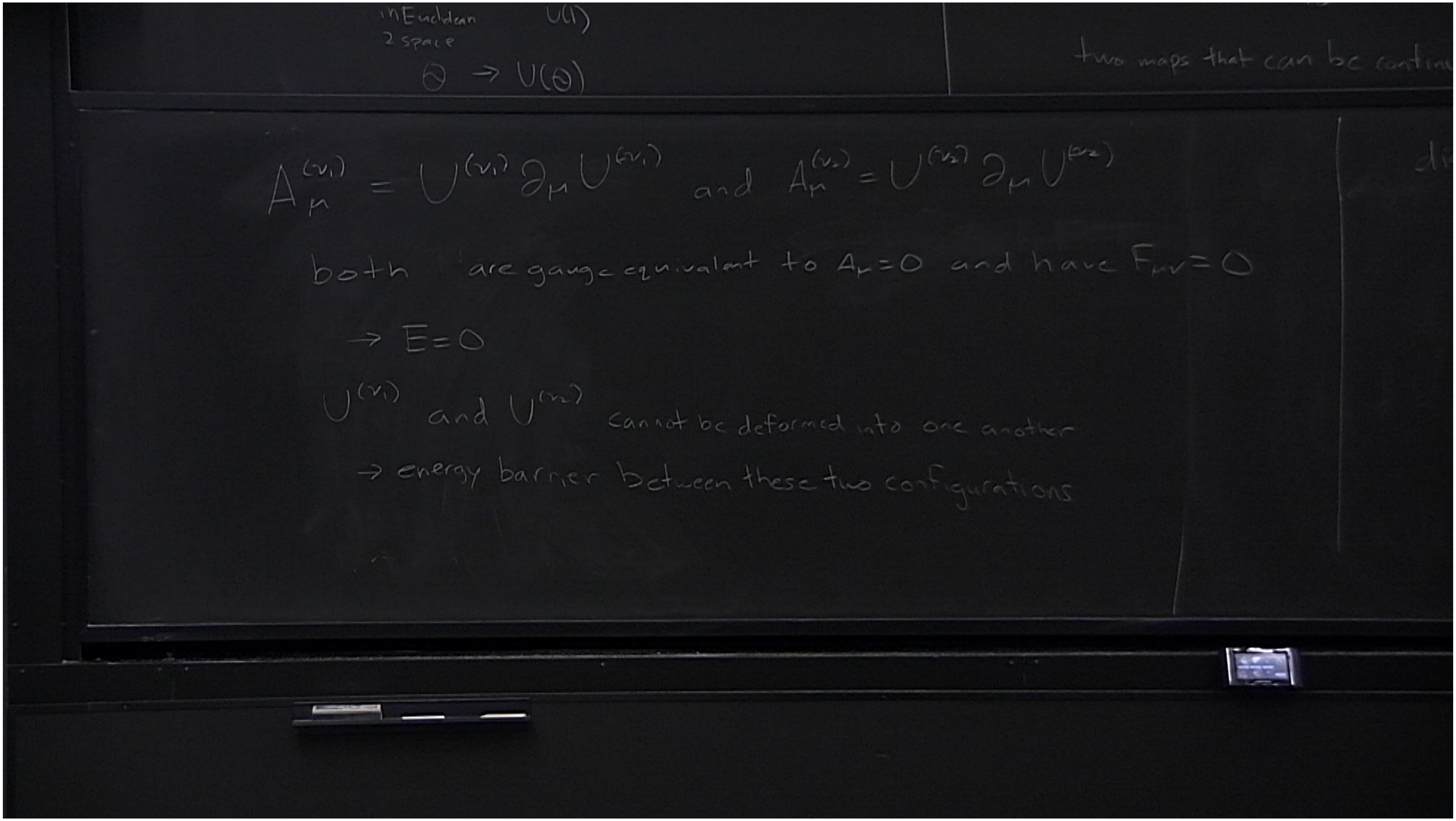
at $r=\infty$

U(1) in 4D no winding number

every map $S^3 \to S^1$ deformable to trivial mapping

For G a simple non-abelian Lie group

Theorem (Bott) any map $S^3 \to G$ can be continuously deformed into a mapping into a SU(2) subgroup of G



in Euclidean 2 space

U(1)

$\theta \to U(\theta)$

two maps that can be contin

$$A_\mu^{(\nu_1)} = U^{(\nu_1)} \partial_\mu U^{(\nu_1)} \quad \text{and} \quad A_\mu^{(\nu_2)} = U^{(\nu_2)} \partial_\mu U^{(\nu_2)}$$

both are gauge equivalent to $A_\mu = 0$ and have $F_{\mu\nu} = 0$

→ $E = 0$

$U^{(\nu_1)}$ and $U^{(\nu_2)}$ cannot be deformed into one another

→ energy barrier between these two configurations

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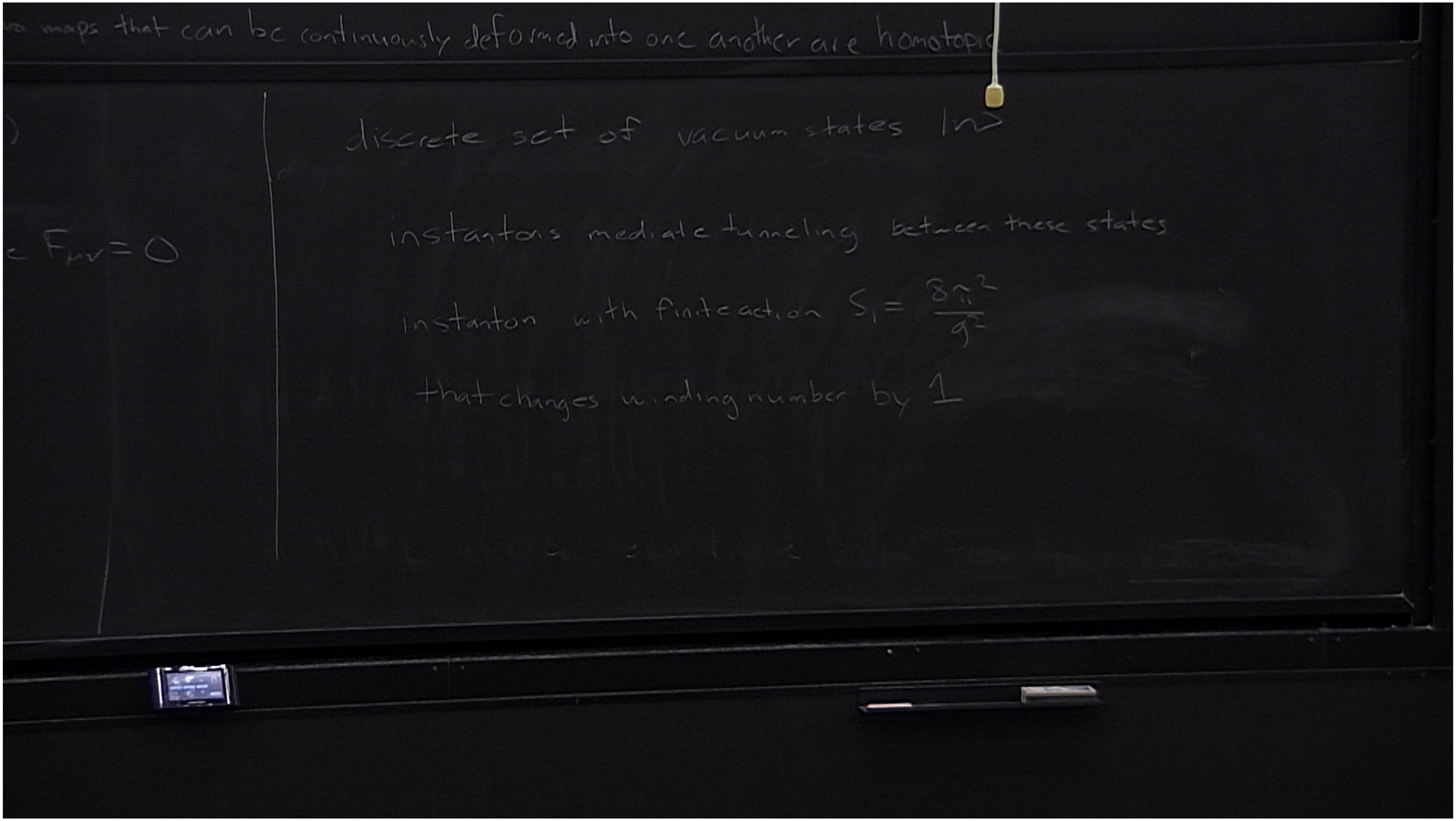


o maps that can be continuously deformed into one another are homotopic

)

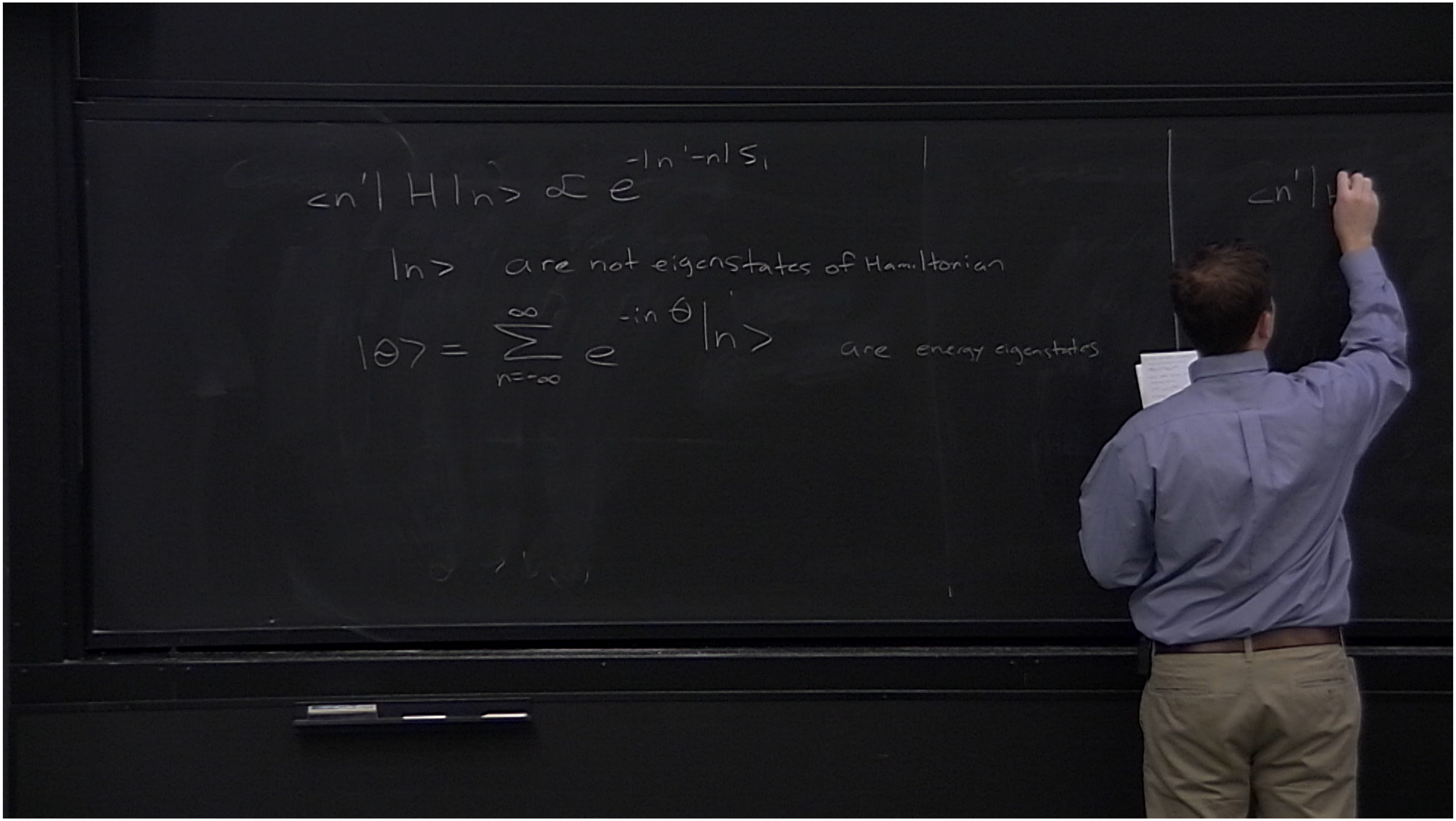
e $F_{\mu\nu}=0$

discrete set of vacuum states $|n\rangle$

instantons mediate tunneling between these states

instanton with finite action $S_1 = \frac{8\pi^2}{g^2}$

that changes winding number by 1

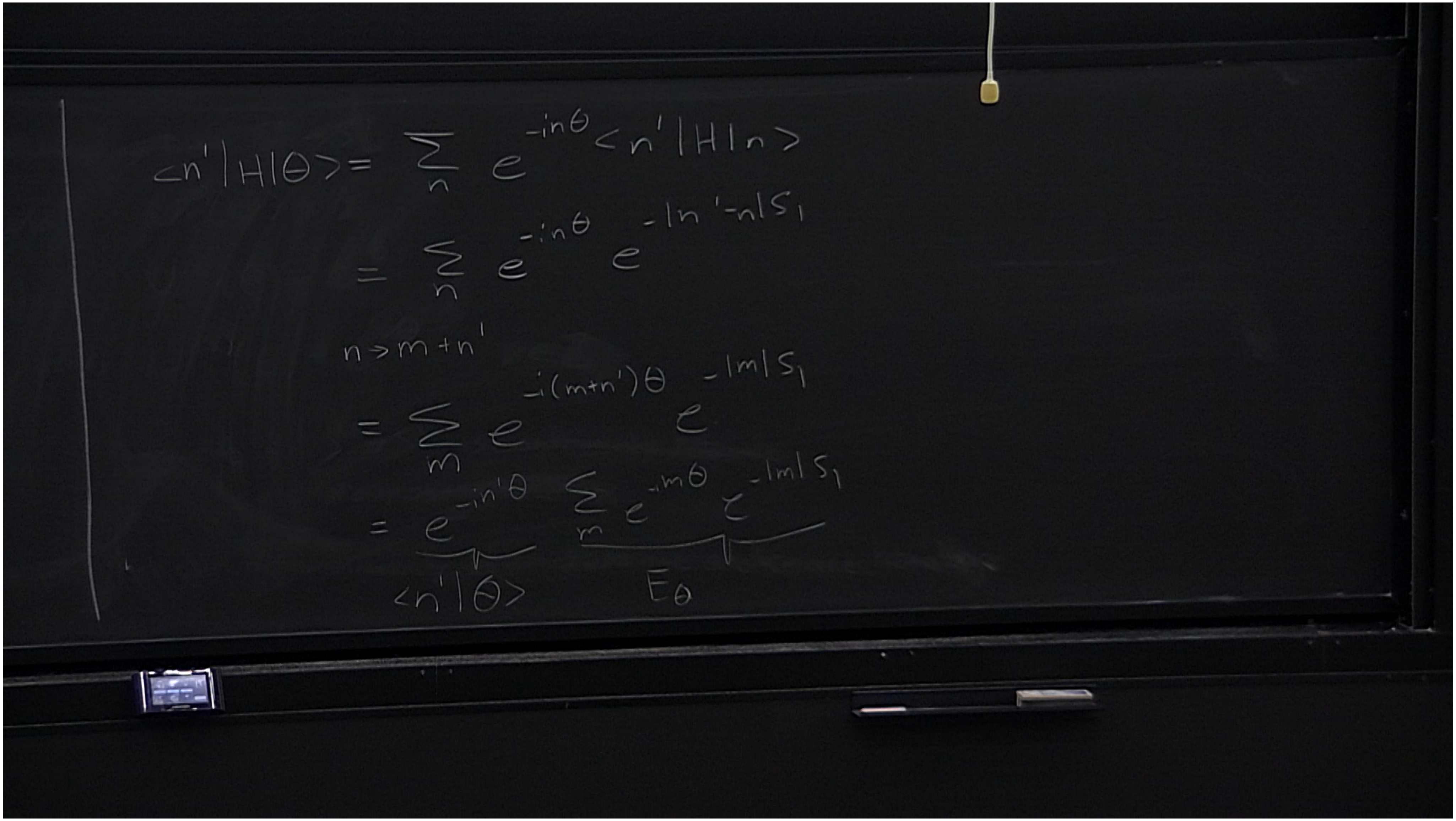


$$\langle n'| H |n\rangle \propto e^{-|n'-n| S_1}$$

$|n\rangle$ are not eigenstates of Hamiltonian

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle$$ are energy eigenstates

$\langle n'|$



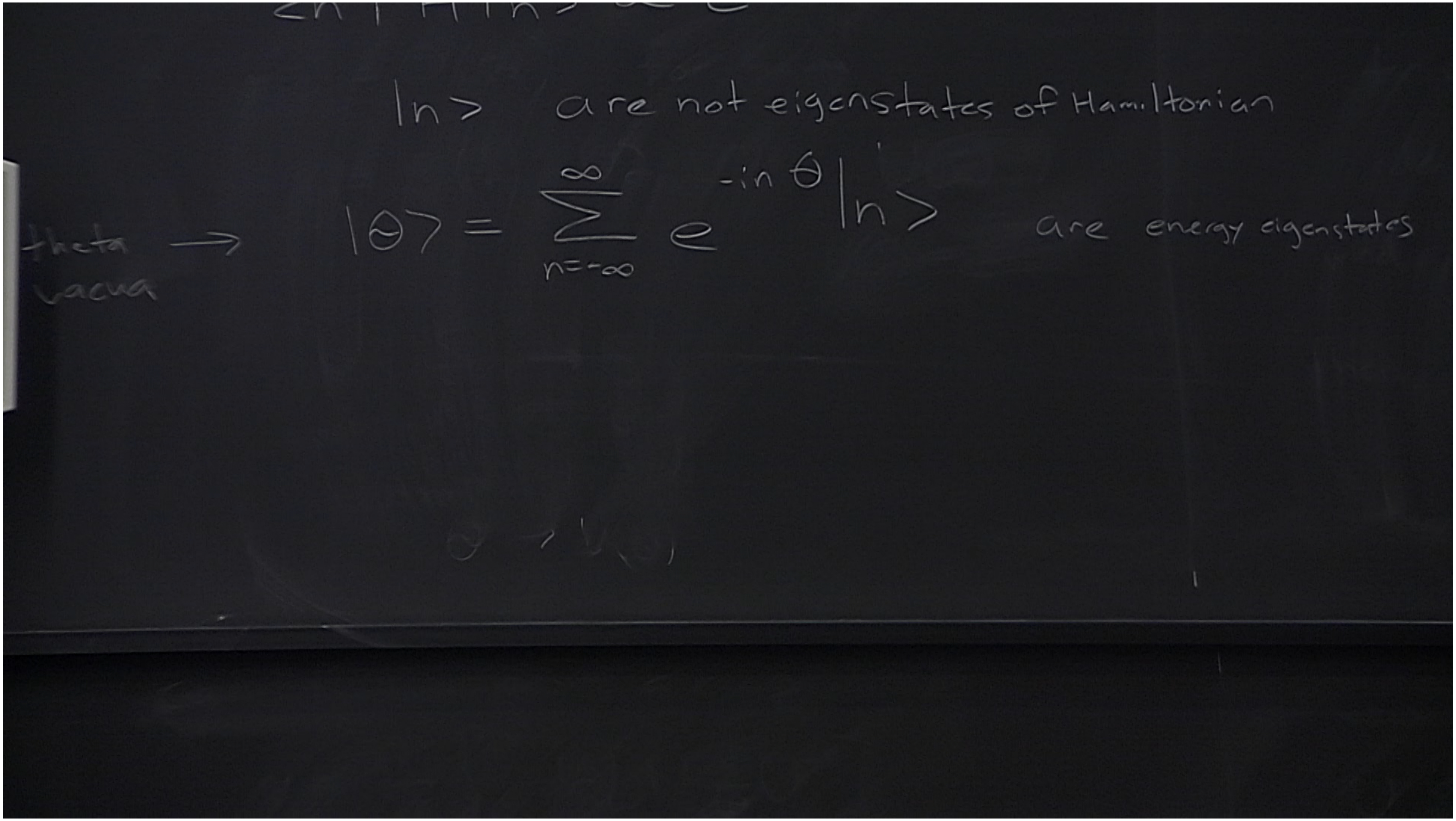
$$\langle n'|H|\theta\rangle = \sum_n e^{-in\theta}\langle n'|H|n\rangle$$

$$= \sum_n e^{-in\theta} e^{-|n'-n|S_1}$$

$n \to m + n'$

$$= \sum_m e^{-i(m+n')\theta} e^{-|m|S_1}$$

$$= \underbrace{e^{-in'\theta}}_{\langle n'|\theta\rangle} \underbrace{\sum_m e^{-im\theta} e^{-|m|S_1}}_{E_\theta}$$



$\langle n | H | n \rangle \propto e$

$|n\rangle$ are not eigenstates of Hamiltonian

theta vacua $\longrightarrow$ $$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle$$ are energy eigenstates

$\langle n'|\theta\rangle$ E_θ

$$\langle\theta'|e^{-HT}|\theta\rangle = \sum_{n,m}\langle n|e^{-HT}|m\rangle e^{i(n\theta'-m\theta)}$$

$\nu = n-m$

$$= \sum_n e^{in(\theta'-\theta)}\left(\sum_\nu \int[\mathcal{D}A]_\nu\, e^{i\nu\theta}\, e^{-S[A]}\right)$$

↑ configuration with winding number ν

$$= \delta(\theta'-\theta)\int \mathcal{D}A\, e^{-S+\frac{i\theta}{16\pi^2}\mathrm{Tr}\,\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}}$$

↑ violates P an CP

$|\theta_{QCD}| \lesssim 10^{-10}$