

Title: PSI 2016/2017 Quantum Field Theory III - Lecture 4

Date: Feb 02, 2017 11:30 AM

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Abstract:

Operators in a CFT are either.

1- Primary. (annihilated by k_μ). Transform as tensors

$$\delta O^A(x) = -\xi^\mu \partial_\mu O^A(x) + \frac{i}{2} \Omega_{\mu\nu}(x) (M^{\mu\nu})^A_B O_B(x) - \Delta w(x) O_A(x)$$

$$\Omega_{\mu\nu} = w_{\mu\nu} - 2(x_\mu b_\nu - x_\nu b_\mu)$$

$$w(x) = \lambda - 2x \cdot b.$$

2- Descendants: Derivatives of primaries $\partial_{\mu_1} \dots \partial_{\mu_n} O_A(x)$

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3rd CFT

$$\begin{cases} \Omega_{\mu\nu} = w_{\mu\nu} - 2(x_\mu b_\nu - x_\nu b_\mu) \\ w(x) = \lambda - 2x \cdot b. \end{cases}$$

- reps of Lorentz

- Δ : scaling dimension.

2. Descendants: Derivatives of primaries $\partial_{\mu_1} \dots \partial_{\mu_n} O_A(x)$

Σ Descendants : Derivatives of primaries $\partial_{p_1} \dots \partial_{p_n} \mathcal{O}(x)$

Lie bracket of CKL's

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$\nearrow$ 3rd CFT

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$\nearrow$ $\mathfrak{so}(1,3)$ CFT

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2- Descendants: Derivatives of primaries $\partial_{\mu_1} \dots \partial_{\mu_n} O_A(x)$

Correlators constrained by conformal symmetry (Ward identities)
 $x'' \rightarrow x'(x)$ a conformal transformation.

$$\langle \hat{O}_{A_1}(x_1) \dots \hat{O}_{A_n}(x_n) \rangle = \langle O_{A_1}(x_1) \dots O_{A_n}(x_n) \rangle$$

local version

$$\delta \langle O_{A_1}(x_1) \dots O_{A_n}(x_n) \rangle = 0$$

$$\delta(O_1 O_2) = \delta O_1 \cdot O_2 + O_1 \delta O_2$$

1pt function

$$\langle \theta(x) \rangle = 0 \text{ unless } 1$$

2pt function

$$\langle O_I(x) O_J(y) \rangle = \frac{\delta_{\Delta_I, \Delta_J}}{(x-y)^{2\Delta_I}} \rightarrow \{ \Delta_I \} \text{ scaling dimensions}$$

3pt function

$$\langle O_I(x) O_J(y) O_K(z) \rangle = C_{IJK}$$

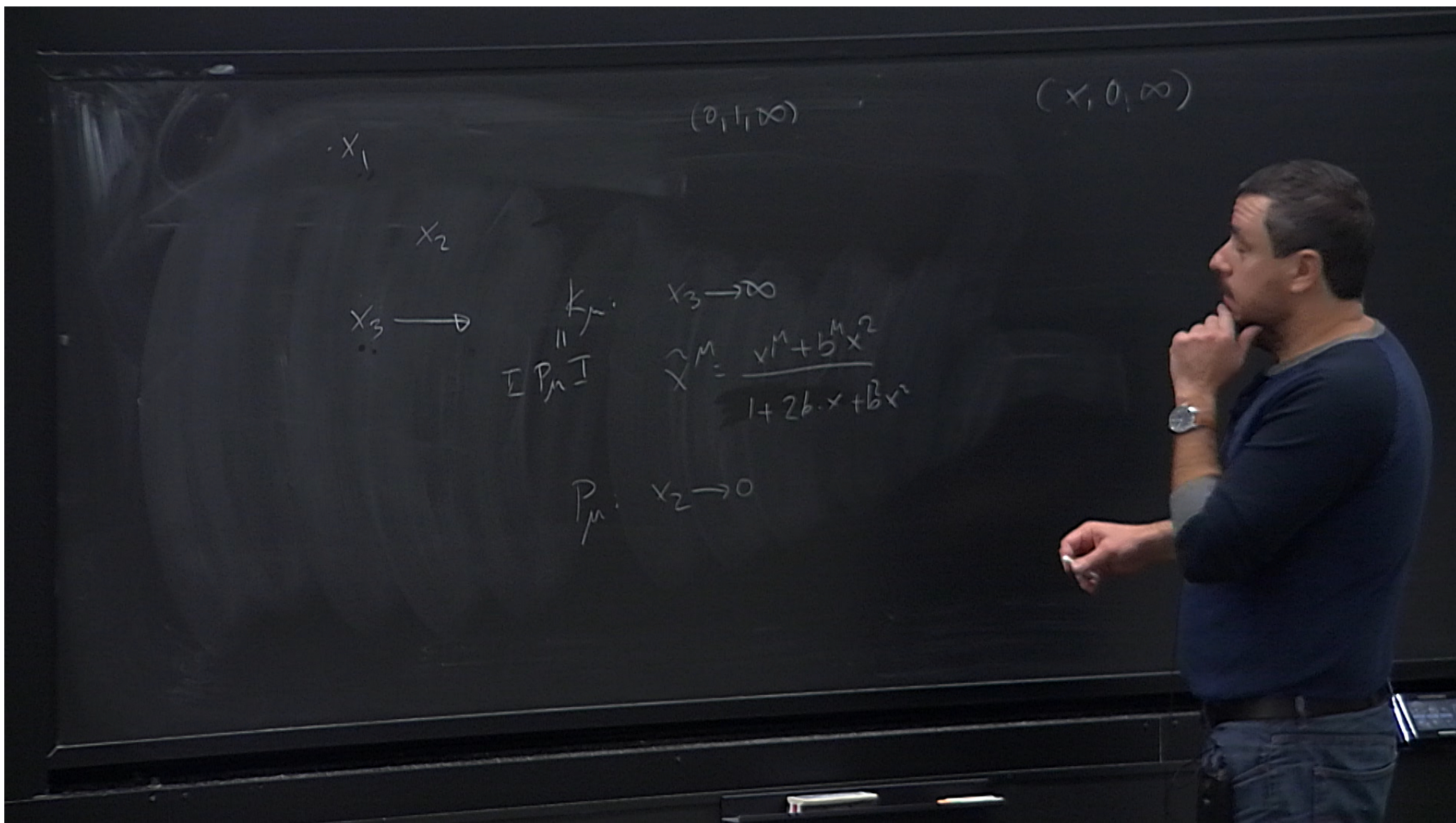
$\rangle = 0$ unless Δ

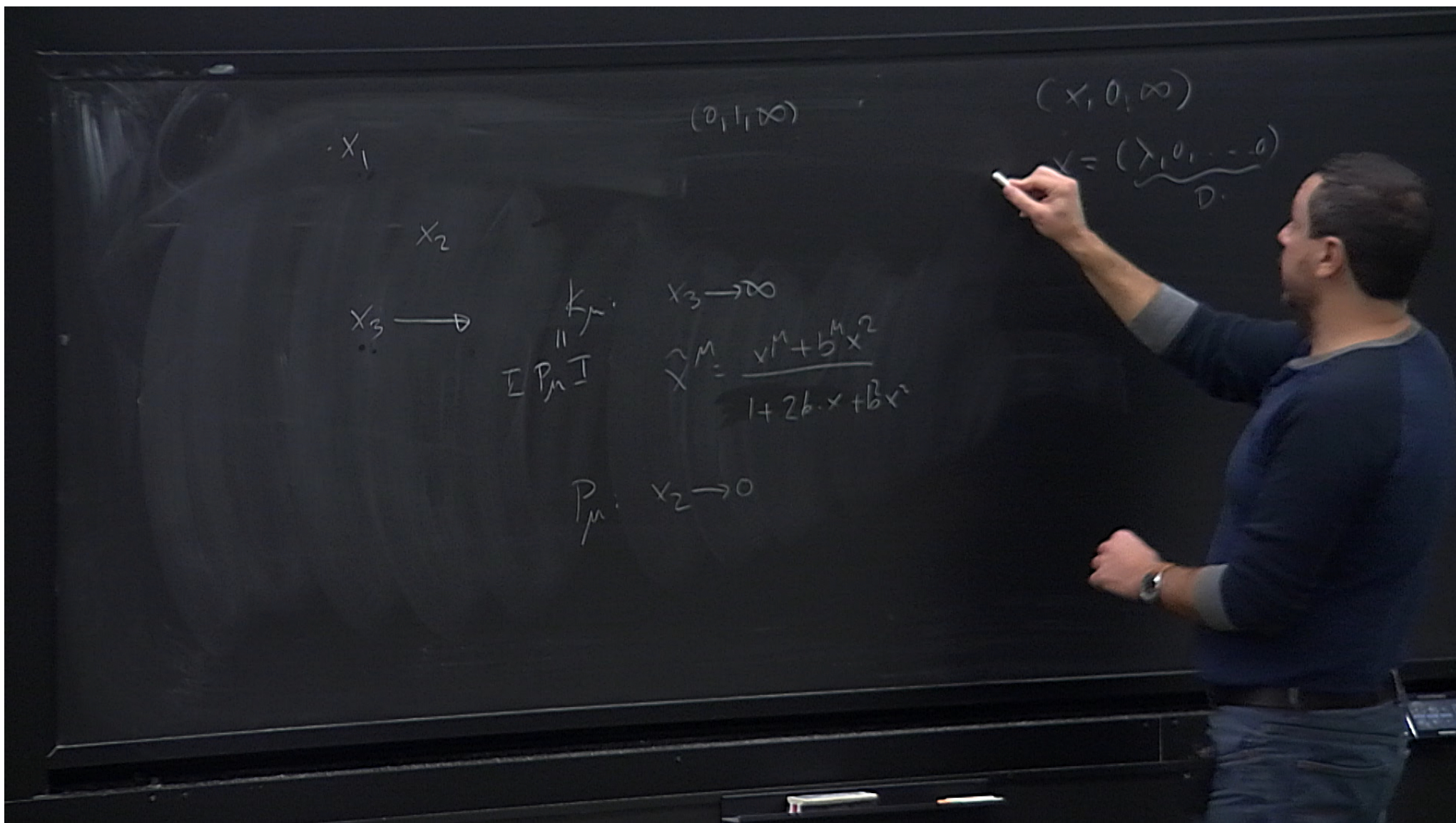
$$\langle y \rangle = \frac{\delta \Delta_I, \Delta_J}{(x-y)^{2\Delta_I}} \rightarrow \{ \Delta_I \} \text{ scaling dimensions.}$$

$$\langle y \rangle O_k(z) = \frac{C_{IJK}}{(x-y)^{2[\frac{\Delta_I + \Delta_J - \Delta_K}{2}]} (y-z)^{2[\frac{\Delta_J + \Delta_K - \Delta_I}{2}]} (x-z)^{2[\frac{\Delta_I + \Delta_K - \Delta_J}{2}]} \Rightarrow \{$$

$$\frac{\Gamma, \Delta_j}{(x-y)^{2\Delta_I}} \rightarrow \{ \Delta_I \} \text{ scaling dimensions.}$$

$$= \frac{C_{Ijk}}{(x-y)^2 [\frac{\Delta_I + \Delta_j - \Delta_k}{2}] (y-z)^2 [\Delta_j + \Delta_k - \Delta_I] (x-z)^2 [\Delta_I + \Delta_k - \Delta_j]} \Rightarrow \{ C_{Ijk} \}.$$





$$(0, 1, \infty)$$

$$(x, 0, \infty)$$

$$M_{\mu\nu}: x = (\underbrace{\lambda_1, 0, \dots, 0}_{D.})$$

$$D: x = (1, 0, \dots, 0)$$

$$-x_1$$

$$x_2$$

$$x_3 \rightarrow \infty$$

$$\boxed{k_{\mu}} \\ \parallel \\ \mathbb{I} P_{\mu} \mathbb{I}$$

$$x_3 \rightarrow \infty$$

$$\tilde{x}^{\mu} = \frac{x^{\mu} + b^{\mu} x^2}{1 + 2b \cdot x + b^2 x^2}$$

$$P_{\mu}: x_2 \rightarrow 0$$

1pt function, $\langle \theta(x) \rangle = 0$ unless 1

2pt function, $\langle O_I(x) O_J(y) \rangle = \frac{\delta_{IJ}}{(x-y)^{2\Delta_I}} \rightarrow \{\Delta_I\} \text{ scaling dimensions}$

3pt function, $\langle O_I(x) O_J(y) O_K(z) \rangle = \frac{C_{IJK}}{(x-y)^{2[\frac{\Delta_I + \Delta_J - \Delta_K}{2}]} (y-z)^{2[\frac{\Delta_J + \Delta_K - \Delta_I}{2}]}}$

if we imposed (Poincaré + D)

$(x, 0, \infty)$

$$x = (\lambda, 0, \dots, 0)$$

$$x = (1, \dots, 1)$$

4 pt function : of identical operators

$$\langle O(x_1) O(x_2) O(x_3) O(x_4) \rangle = \frac{F(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}$$

Cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

4 pt function, of identical operators

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cross ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z \bar{z}$$

$$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

OPE in CFT conformal symmetry

F

4 pt function, of identical operators

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OPE in CFT conformal symmetry
F(u,v) determined by the

CFT data

$$\{\Delta_I, C_{IJK}\}$$

This data has an important consistency condition

$\Rightarrow$ measure critical exponents of 3D Ising CFT

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Any CFT has $T_{\mu\nu}$, want to 2,3pt Functions of $T_{\mu\nu}$

CFT has a global symmetry: $\partial^\mu j_\mu = 0$

$$\langle j_\mu(x) j_\nu(y) \rangle = \frac{-I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}}$$

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$T_{\mu\nu}$ is related to 4-point function prod

we want to 2,3pt Functions of $T_{\mu\nu}$

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$$\langle j_\nu(y) \rangle = \frac{-I_{\mu\nu}(x-y)}{(x-y)^2 \Delta}$$

$I_{\mu\nu}$ is related to the Jacobian produced by inverse

$$\tilde{x}^\mu = \frac{x^\mu}{x^2}$$

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \frac{1}{(x^2)} \left[\delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2} \right]$$

$I_{\mu\nu}(x)$

$$I^T \eta I = \eta.$$

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$I_{\mu\nu}(x)$

$I^\top$

$$I_{\mu\nu}(\bar{x}-\bar{y}) = I_{\mu\rho}(x) \boxed{I_{\rho\sigma}(x-y)} I_{\sigma\nu}(y)$$

↓
transforms as a connection
if

$$I^T \eta I = \eta.$$

$\dot{x}$

induced by inverse

$$\left[\frac{x_1}{x_2} \right]$$

Any CFT has $T_{\mu\nu}$, want to 2,3pt Functions of $T_{\mu\nu}$

CFT has a global symmetry: $\partial^\mu j_\mu = 0$

$$\langle j_\mu(x) j_\nu(y) \rangle = \frac{T_{\mu\nu}(x-y)}{(x-y)^{2\Delta}}$$

$$\hat{j}_\mu(\bar{x}) \hat{j}_\nu(\bar{y}) = \langle j_\mu(\bar{x}) j_\nu(\bar{y}) \rangle,$$

$T_{\mu\nu}$ is related to the Jacobian p

$$\tilde{x}^\mu = \frac{x^\mu}{x^2}$$

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \frac{1}{(x^2)} \left[\delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2} \right]$$

$$T_{\mu\nu}(x)$$

$$\frac{\sum \Delta_I, \Delta_J}{(x-y)^{2\Delta_I}} \rightarrow \{ \Delta_I \} \text{ scaling dimensions}$$

$$(y) O_k(z) = \frac{C_{IJK}}{(x-y)^2 [\frac{\Delta_I + \Delta_J - \Delta_K}{2}] (y-z)^2 [\Delta_J + \Delta_K - \Delta_I] (x-z)^2 [\Delta_I + \Delta_K - \Delta_J]} \Rightarrow$$

$$ae' + D)$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle = \frac{c}{(x-y)^{2D}} I_{\mu\nu\rho\sigma}(x-y)$$

$$I_{\mu\nu\rho\sigma}(x) = \frac{1}{2} \left(I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x) - \frac{1}{D} \delta_{\mu\nu} \delta_{\rho\sigma} \right)$$

- Specially in $D=2$ captures lot's of physical system

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle = \frac{c}{(x-y)^{2D}} I_{\mu\nu\rho\sigma}(x-y)$$

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- Specially in $D=2$ captures lot's of physical system

c. flat space correlator or by coupling $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$

c is interpreted as a conformal anomaly

$$T_{\mu}^{\mu} = 0 \quad \text{in a CFT in flat space}$$

in a curved
background

$$T_{\mu}^{\mu} \neq 0 \propto c.$$

$$\partial^{\mu} j_{\mu}^D = T_{\mu}^{\mu}$$

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5)

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$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(q) \rangle = \frac{\quad}{\quad} \delta(p+q)$$

↑
most general

- Lorentz invariance

- Bose symmetry $\mu \leftrightarrow \nu$

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(q) \rangle = \underbrace{\hspace{10em}}_{\substack{\text{5 different structures} \\ \uparrow \\ \text{most general}}} \delta(p+q)$$

- Lorentz invariance
- Bose symmetry $\mu\nu \leftrightarrow \rho\sigma$

$\eta_{\mu\nu}, p_\mu$

Impose conservation

conservation

$\rightarrow = 0$: 2 independent structures

$$p) T_{\mu\sigma}(-p) = \left[\pi_{\mu\rho} \pi_{\nu\sigma} + \pi_{\mu\sigma} \pi_{\nu\rho} - \frac{2}{D-1} \pi_{\mu\nu} \pi_{\rho\sigma} \right] F(p^2) + \pi_{\mu\nu} \pi_{\rho\sigma} G(p^2)$$

trace

Symmetric tensors

$$\pi_{\mu\nu} = (P_\mu P_\nu - p^2 \eta_{\mu\nu})$$

$$P^\mu \pi_{\mu\nu} = 0$$

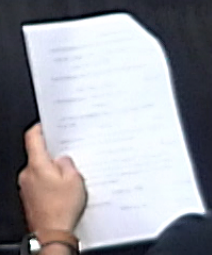
the first term just vanishes

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p) \rangle = \pi_{\mu\nu} \pi_{\rho\sigma} G(p^2)$$

$$\pi_{\mu\nu} \propto \hat{P}_\mu \tilde{P}_\nu$$

In 2D $\hat{P}_\mu = \epsilon_{\mu\lambda} P^\lambda$

$$\epsilon_{\mu\lambda} \epsilon_{\nu\rho} = \eta_{\mu\nu} \eta_{\lambda\rho} - \eta_{\mu\rho} \eta_{\nu\lambda}$$



conservation

$$p \cdot \gamma = 0$$

2 independent structures

$$p) T_{\mu\nu}(-p) = \left[\pi_{\mu\rho} \pi_{\nu\sigma} + \pi_{\mu\sigma} \pi_{\nu\rho} - \frac{2}{D-1} \pi_{\mu\nu} \pi_{\rho\sigma} \right] F(p) + \pi_{\mu\nu} \pi_{\rho\sigma} G(p^2)$$

trace

Symmetric tensor

$$\pi_{\mu\nu} = (P_\mu P_\nu - p^2 \eta_{\mu\nu})$$

$$p^\mu \pi_{\mu\nu} = 0$$

the first term just vanishes

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p) \rangle = \pi_{\mu\nu} \pi_{\rho\sigma}$$

In 2D $\tilde{P}_\mu = \epsilon_{\mu\lambda} p^\lambda$

$$\pi_{\mu\nu} \propto \tilde{P}_\mu \tilde{P}_\nu$$

$$\epsilon_{\mu\alpha} \epsilon_{\nu\beta} = \eta_{\mu\nu} \eta_{\alpha\beta} - \eta_{\mu\beta} \eta_{\alpha\nu}$$

In $D=2$

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p) \rangle = c \underbrace{\tilde{P}_\mu \tilde{P}_\nu \tilde{P}_\rho \tilde{P}_\sigma}_4 \frac{1}{p^2}$$

- conformal symmetry

$$0 = 4 - 2 + [F]$$

$$[F] = -2$$

In $D=2$

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$$p^\mu \tilde{P}_\mu = 0$$

- conformal symmetry

$$\langle T^\mu_\mu(p) T_{\rho\sigma}(-p) \rangle =$$

$$\eta^{\mu\nu} \tilde{P}_\mu \tilde{P}_\nu = p^2$$

In $D=2$

$$\langle T_{\mu\nu}(p) T_{\rho\sigma}(-p) \rangle = c \underbrace{\tilde{P}_\mu \hat{P}_\nu \tilde{P}_\rho \tilde{P}_\sigma}_4 \frac{1}{p^2}$$

$$p^\mu \tilde{P}_\mu = 0$$

- conformal symmetry

$$\langle T_\mu^\mu(p) T_{\rho\sigma}(-p) \rangle = c \tilde{P}_\rho \tilde{P}_\sigma \iff$$

$$\langle T_\mu^\mu(x) T_{\rho\sigma}(0) \rangle = c (\delta_{\mu\rho} \delta_{\mu\sigma} - \delta_{\mu\sigma} \delta_{\mu\rho}) \delta^2(x)$$

$$\eta^{\mu\nu} \hat{P}_\mu \hat{P}_\nu = p^2$$



$\Rightarrow$ conformal anomaly

$$\langle T_M^M(0) \rangle_{h_{\mu\nu}} = \langle T_M^M(0) e^{\int h^{\mu\nu} T_{\mu\nu}} \rangle = \int dx \langle T_M^M(0) T_{\mu\nu}(x) \rangle h^{\mu\nu}(x)$$

$\Rightarrow$ conformal anomaly

$$\langle T_{\mu}^{\mu}(0) \rangle_{h_{\mu\nu}} = \langle T_{\mu}^{\mu}(0) e^{\int h^{\mu\sigma} T_{\mu\sigma}} \rangle = \int dx \langle T_{\mu}^{\mu}(0) T_{\mu\sigma}(x) \rangle h^{\mu\sigma}(x)$$

$$= c (\partial_{\rho} \partial_{\sigma} - \square \eta_{\rho\sigma}) h^{\rho\sigma}(x)$$

$$T_{\mu}^{\mu} = c (\partial_{\rho} \partial_{\sigma} - \square \eta_{\rho\sigma}) h^{\rho\sigma}$$

$$= c R |_{\eta_{\mu\nu} + h_{\mu\nu}}$$

$$\Rightarrow \boxed{T_{\mu}^{\mu} = c R}$$

conformal anomalies in 2d and 4d.

(not in odd-dimensions)

CFT_{UV}

$$C_{UV} > C_{IR}$$

CFT_{IR}

conformal anomalies in 2d and 4d.

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conformal anomalies in 2d and 4d.

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CFT_{IR}

$Z_{\text{CFT}}[S^D]$ measures the monotonic quantity