

Title: PSI 2016/2017 Cosmology (Review) - Lecture 14

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Abstract:

QUANTIZING TEST SCALAR FIELD IN FRW: ($k=0$)

$$\begin{aligned} S &= \int d^4x \left(-\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) \\ &\stackrel{d\eta = dt/a}{=} \int dt d^3x \frac{1}{2} a^3 \left(\dot{\phi}^2 - \frac{1}{a^2} |\nabla \phi|^2 \right) \\ &= \int d\eta d^3x \frac{1}{2} a^2 \left(\phi'^2 - |\nabla \phi|^2 \right) \end{aligned}$$

MUKHANOV'S VARIABLE

$$\boxed{V = a^2 \phi}$$

RW: ($k=0$)

$$S = \int d\eta d^3x \frac{1}{2} \left(\dot{v}^2 - |\nabla v|^2 + \frac{a''}{a} v^2 \right)$$

• EOM:
$$v'' - (\nabla^2 + \frac{a''}{a}) v = 0$$

• WANT

$$V(x, \eta) = \int d^3k \left(a_k u_k(x, \eta) + a_k^\dagger u_k^*(x, \eta) \right)$$

• SCALAR PRODUCT $(v_1, v_2) = i \int d^3x \left(v_1^* \partial_\eta v_2 - (\partial_\eta v_1^*) v_2 \right)$

- $(u_k, u_{k'}) = \delta^3(k - k')$

- $u_k(x, \eta) = \frac{1}{(2\pi)^{3/2}} f_k(\eta) e^{i\vec{k} \cdot \vec{x}}$
(BY SYMMETRY)

PLUG TO (EOM) $\Rightarrow$

MUKHANOV'S EQ:

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0$$

NUMBER OF TIME-DEP HARM. OSCILLATORS (1 FOR EACH Q)

$$\cdot (f_k, f_k)_{\text{OLD}} = 1, \dots$$

OLD SCALAR PRODUCT,

• QUANTIZE: $\pi = \dot{v}$ CAN. MOMENTUM

$$[v(\eta, x), \pi(\eta, x')] = i \delta^3(x - x')$$

$$\Leftrightarrow [a_k, a_{k'}^\dagger] = \delta^3(k - k')$$

$$a_k |0\rangle \quad \forall \underline{k}$$

ACH Q)

TIME-DEP HARM. OSCILLATORS (1 FOR EACH Q)

• POWER SPECTRUM

$$\begin{aligned}
 \langle \phi_k \phi_{k'} \rangle &= \int d^3x d^3x' e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'} \langle \phi(x, \eta) \phi(x', \eta) \rangle & \phi(x, \eta) &= \frac{1}{a} \int \frac{d^3k}{(2\pi)^{3/2}} (a_k f_k(\eta) e^{i\vec{k}\cdot\vec{x}} + \dots) \\
 &\stackrel{\text{TRANS. INV.}}{=} (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \underbrace{P_\phi(|\vec{k}|, \eta)}_{\text{POWER SPECTRUM}} f(x-x') \\
 &= \frac{1}{a^2} \int d^3x \frac{d^3x'}{(2\pi)^3} \int d^3k d^3k' \langle a_k a_{k'}^\dagger f_k f_{k'}^* e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \rangle e^{i\vec{k}\cdot\vec{x}} e^{i\vec{k}'\cdot\vec{x}'}
 \end{aligned}$$

NUMBER OF TIME-DEP HARM. OSCILLATORS (1 FOR EACH Q)

• POWER SPECTRUM

$$\begin{aligned}
 \langle \phi_Q \phi_{Q'} \rangle &= \int d^3x d^3x' e^{i\vec{k} \cdot \vec{x}} e^{i\vec{k}' \cdot \vec{x}'} \langle \phi(x, \eta) \phi(x', \eta) \rangle \quad \phi(x, \eta) = \frac{1}{a} \int \frac{d^3k}{(2\pi)^3} \\
 &\stackrel{\text{TRANSL. INV.}}{=} (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \underbrace{P_Q(|\vec{k}|, \eta)}_{\text{POWER SPECTRUM}} f(x-x') \\
 &= \frac{1}{a^2} \int d^3x d^3x' \frac{d^3k d^3k'}{(2\pi)^3} \langle a_k a_{k'}^\dagger f_k f_{k'}^* e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k}' \cdot \vec{x}'} \rangle e^{i\vec{k} \cdot \vec{x}} \\
 &= \frac{1}{a^2} (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') |f_k|^2
 \end{aligned}$$

ORS (1 FOR EACH Q)

$$\vec{q} \cdot \vec{x} \quad \langle \phi(x, \eta) \phi(x', \eta) \rangle \quad \phi(x, \eta) = \frac{1}{a} \int \frac{d^3 k}{(2\pi)^{3/2}} (a_k f_k(\eta) e^{i\vec{k} \cdot \vec{x}} + \dots)$$

$$P_\phi(|\vec{k}|, \eta) \quad f(x-x')$$

POWER SPECTRUM

$$P_\phi(k) = \frac{1}{a^2} |f_k|^2$$

$$\langle a_k a_k^\dagger f_k f_k^* e^{i\vec{k} \cdot \vec{x}} e^{-i\vec{k} \cdot \vec{x}'} \rangle = |f_k|^2$$

GIVEN BY A SOL. OF
MUKHANOV'S EQ.
PROVIDED SOME INITIAL
COND.

CHOISE OF INITIAL COND

$$\boxed{\frac{a''}{a} \rightarrow 0}$$

AT EARLY TIMES

$$(\omega(\eta) \rightarrow k = \text{CONST})$$

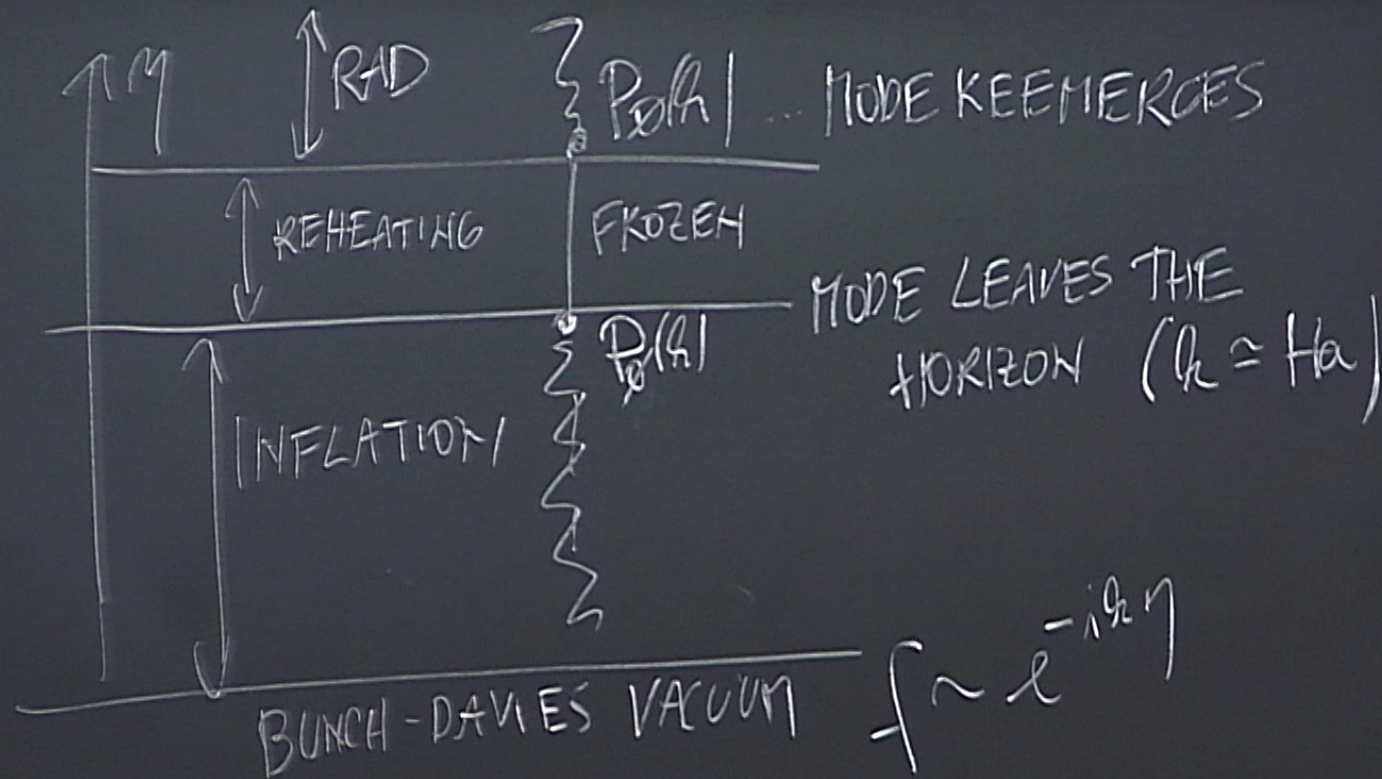
$$\Rightarrow \boxed{f_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}} \quad (f_{IN})$$

$\Rightarrow$ USING THIS $f_k \rightarrow$ VACUUM

$$a_k |0\rangle$$

BUNCH-DAVIES VACUUM





• CONCRETELY: DS EXPANSION

$$a(t) = l^{Ht} \quad | \quad H = \text{CONST}$$

$$\eta = \int_{-\infty}^t \frac{dt}{a(t)} = -\frac{1}{H l^{-Ht}} = -\frac{1}{Ha} \quad (\eta \in (-\infty, 0])$$

$\eta = -\infty$... B.B. ... BUNCH-DAVIES

$\eta = 0$... "END OF INFL"

MUKHANOV'S EQ:

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = f_k'' + \underbrace{\left(k^2 - \frac{k^2}{\gamma^2}\right)}_{\omega^2 \rightarrow k^2} f_k$$

AT EARLY TIMES.

(-90)

INIT. COND $f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta}$ $\eta \rightarrow -\infty$

→ SOLUTION

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{1}{k\eta}\right)$$

= CONST

$$\frac{1}{H} \frac{d}{dt} = -\frac{1}{Ha} \quad (\eta \in (-\infty, 0])$$

B.R. ... BUNCH-DAVIES

"END OF INF" "

$$P_{\delta k} = \frac{|f_k|^2}{a^2} = H^2 \eta^2 \frac{1}{2k} \left(1 + \frac{1}{k^2 \eta^2}\right) \xrightarrow{\eta \rightarrow 0} \boxed{\frac{H^2}{2k^3} = P_{\delta k}}$$

→ SOLUTION

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{1}{k\eta}\right)$$

$$\text{INIT. COND } f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad \eta \rightarrow -$$

$$f_k'' + \left(k^2 - \frac{a}{a}\right) f_k = f_k'' + \dots$$

FAMOUS "SCALE INVARIANT" SPECTRUM

WHY? $\phi(\vec{x}) \xrightarrow{\text{SCALING TRANSF.}} \phi(\lambda \vec{x})$

$\lambda = 1 + \varepsilon$
 $\Rightarrow \boxed{\delta\phi = \vec{x} \cdot \frac{\partial\phi}{\partial\vec{x}}}$

2-POINT FUNCTION IS INVARIANT UNDER SCALING TRANSF.

$$0 = \vec{x} \cdot \frac{\partial}{\partial\vec{x}} \langle \phi(x) \phi(0) \rangle$$

$$0 = \vec{x} \cdot \frac{\partial}{\partial \vec{x}} \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 q'}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{x}} \underbrace{\langle \phi_{\vec{q}} \phi_{\vec{q}'} \rangle}_{(2\pi)^3 \delta(\vec{q} + \vec{q}') P_D(q)}$$

$$= \vec{x} \cdot \frac{\partial}{\partial \vec{x}} \int \frac{d^3 q}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{x}} P(q)$$

$$= \int \frac{d^3 q}{(2\pi)^3} \left(-\frac{\partial}{\partial \vec{q}} (e^{-i\vec{q} \cdot \vec{x}}) \cdot \vec{q} P(q) \right) \quad \text{BY PARTS}$$

$$= \int \frac{d^3 q}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{x}} \underbrace{\left(3 + \vec{q} \cdot \frac{\partial}{\partial \vec{q}} \right) P(q)}_{\emptyset}$$

$$\Rightarrow P(q) \propto \frac{1}{q^3}$$

SCALE
INVARIANT
SPECTRUM

MUKHANOV'S EQ:

$$f_k'' + (k^2 - \frac{a''}{a})f_k = 0$$

NUMBER OF TIME-DEP HARM. OSCILLATORS (1 FOR EACH Q)

• SPECTRAL INDEX

$$n_s = 1 + k \frac{d}{dk} \log(k^3 P(k))$$

$$P(k) =$$

• $n_s = 1 \dots$ SCALE INVARIANT

$n_s < 1 \dots$ RED TILTED

$n_s > 1 \dots$ BLUE TILTED

• PROVIDED $n_s = \text{CONST}$

$$P(k) \propto k^{-3 + (n_s - 1)}$$

$$P(k) = A_0 \left(\frac{k}{k_0} \right)^{-3+m_s-1}$$

↑ AMPLITUDE AT $k=k_0$

2 PARAMS DESCRIBING PRIM. POWER SPECTRUM $\{A_0, m_s\}$.

• $m_s \neq \text{CONST}$... "RUNNING SPECTRAL INDEX"

PLANCK DATA: $m_s = \text{CONST.}$

$$m_s = 0.96 \pm 0.01$$

RED TILTED SPECTRUM

PROVIDED SOME INITIAL
COND.