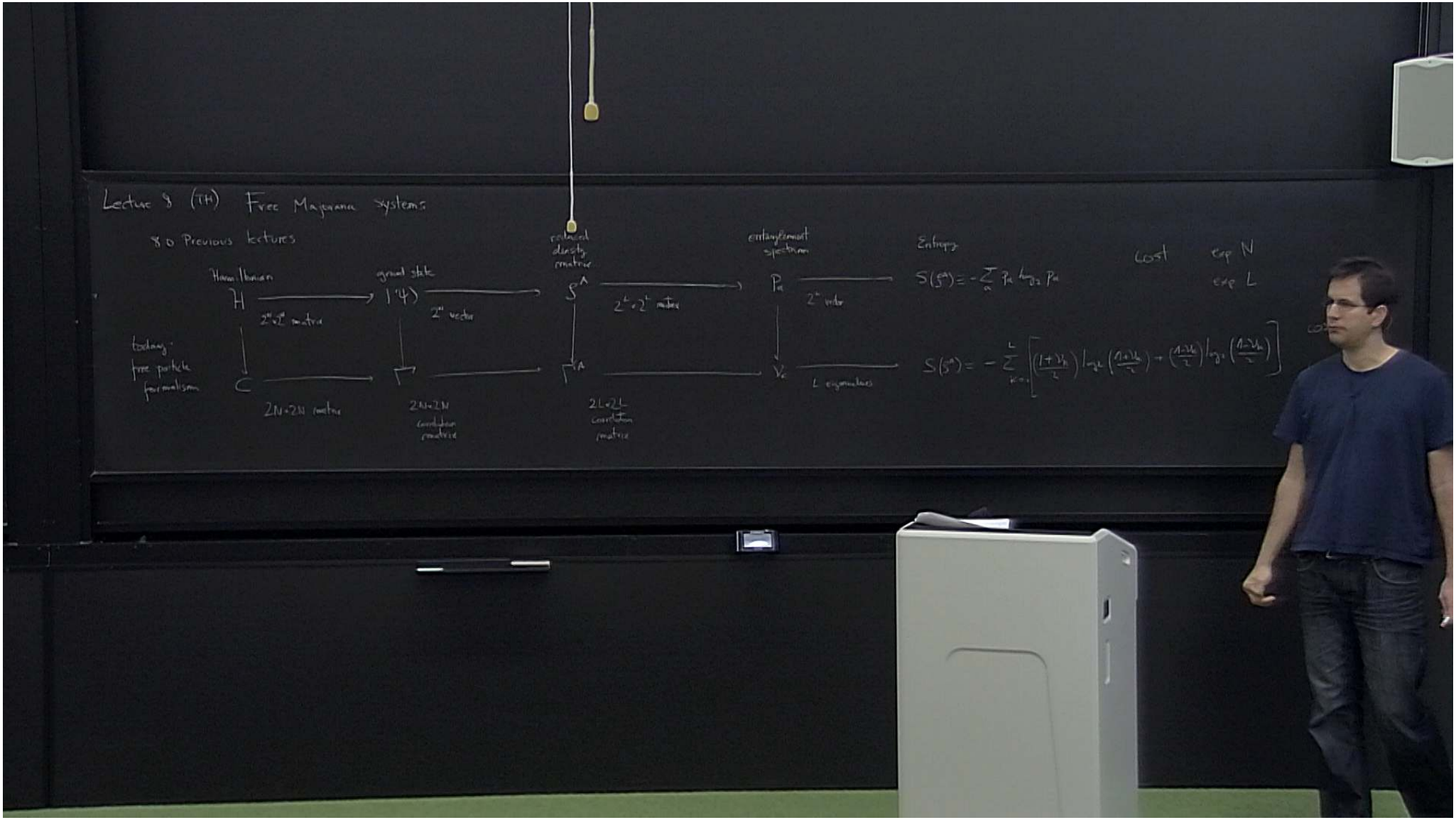
Title: PSI 2016/2017 Condensed Matter (Review) - Lecture 4

Date: Feb 03, 2017 09:00 AM

URL: http://pirsa.org/17020040

Abstract:


Lecture 8 (TH) Free Majorana systems
8.0 Previous lectures
Hamiltonian
ground state
reduced density matrix
entanglement spectrum
Entropy
cost
exp N
exp L
H
$2^N \times 2^N$ matrix
$|\psi\rangle$
$2^N$ vector
$\hat{\rho}$
$2^L \times 2^L$ matrix
$P_\alpha$
$2^L$ vector
$S(\hat{\rho}) = -\sum_\alpha P_\alpha \log_2 P_\alpha$
today: free particle formalism
C
$2N \times 2N$ matrix
$2N \times 2N$ correlation matrix
$\Gamma_A$
$2L \times 2L$ correlation matrix
$V_\alpha$
L eigenvalues
$S(\hat{\rho}) = -\sum_{k=1}^{L} \left[ \left(\frac{1+\nu_k}{2}\right) \log_2 \left(\frac{1+\nu_k}{2}\right) + \left(\frac{1-\nu_k}{2}\right) \log_2 \left(\frac{1-\nu_k}{2}\right) \right]$

ground state $|0\rangle$ $-\varepsilon/2$

excited state $|1\rangle$ $\varepsilon/2$

| HAMILTONIAN | DENSITY MATRIX | EIGENVALUES | ENTROPY |
|---|---|---|---|
| $H = -\frac{\varepsilon}{2}\sigma^z = \begin{pmatrix} -\frac{\varepsilon}{2} & 0 \\ 0 & \frac{\varepsilon}{2} \end{pmatrix}$ | $S = p\|0\rangle\langle 0\| + (1-p)\|1\rangle\langle 1\|$ <br> $\langle\sigma^x\rangle = \langle\sigma^y\rangle = 0$ <br> $\langle\sigma^z\rangle = 2p-1$ | $p, 1-p$ <br> $p = \frac{1+\langle\sigma^z\rangle}{2}$ | $S(\rho) = -p\log_2 p - (1-p)\log_2(1-p)$ |

$H = \varepsilon\left(a^\dagger a - \frac{1}{2}\right)$

$\langle a\rangle = \langle a^\dagger\rangle = 0$

$\langle a^\dagger a\rangle = 1-p$

$\langle a a^\dagger\rangle = p$

8.1 three operator algebras for one 2-level system $N=1$

ground state $|0\rangle$ $-\varepsilon/2$

exc. level state $|1\rangle$ $\varepsilon/2$

| ALGEBRA | HAMILTONIAN | DENSITY MATRIX | EIGENVALUES | ENTROPY |
|---|---|---|---|---|
| Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ $[\sigma^\alpha, \sigma^\beta] = 2i\varepsilon^{\alpha\beta\gamma}\sigma^\gamma$ | $H = -\frac{\varepsilon}{2}\sigma^z = \begin{pmatrix} -\frac{\varepsilon}{2} & 0 \\ 0 & \frac{\varepsilon}{2} \end{pmatrix}$ | $S = p\,\|0\rangle\langle 0\| + (1-p)\,\|1\rangle\langle 1\|$ | $p, 1-p$ | $S(S) = -p\log_2 p - (1-p)\log_2(1-p)$ |
| matrix representation $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | | $\langle\sigma^x\rangle = \langle\sigma^y\rangle = 0$ $\langle\sigma^z\rangle = 2p-1$ | $p = \frac{1+\langle\sigma^z\rangle}{2}$ | |
| Fermionic annihilation/creation operators $a, a^\dagger$ $\{a, a^\dagger\} = 1$ $\{a, a\} = \{a^\dagger, a^\dagger\} = 0$ $(aa^\dagger + a^\dagger a = 1,\ a^2 = a^{\dagger 2} = 0)$ | $H = \varepsilon\left(a^\dagger a - \frac{1}{2}\right)$ | $\langle a\rangle = \langle a^\dagger\rangle = 0$ $\langle a^\dagger a\rangle = 1-p$ $\langle a a^\dagger\rangle = p$ | | |
| matrix representation $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $a^\dagger a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $aa^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | | | | |

Pair of Majorana operators

$\psi_1, \psi_2$ $\psi_i^\dagger = \psi_i$ $\{\psi_i, \psi_j\} = \delta_{ij}$ $H = \frac{i}{}$

matrix representation

$\psi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\psi_1\psi_2 - \psi_2\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

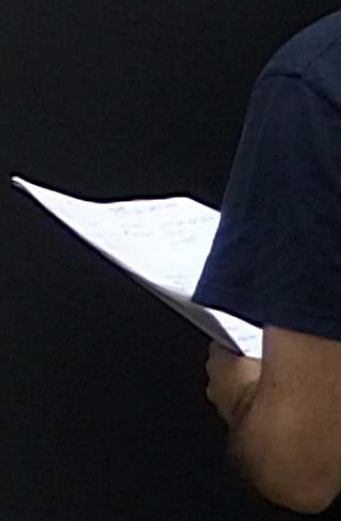
$$H = \frac{i}{2}\varepsilon\left(\psi_1\psi_2 - \psi_2\psi_1\right)$$

$$\langle\psi_1\rangle = 0$$
$$\langle\psi_2\rangle = 0$$

$$-i\,\langle[\psi_1, \psi_2]\rangle = 2p - 1 \equiv \nu$$

$$\Gamma_{mn} \equiv -i\,\langle[\psi_m, \psi_n]\rangle$$

$$\Gamma = \begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}$$



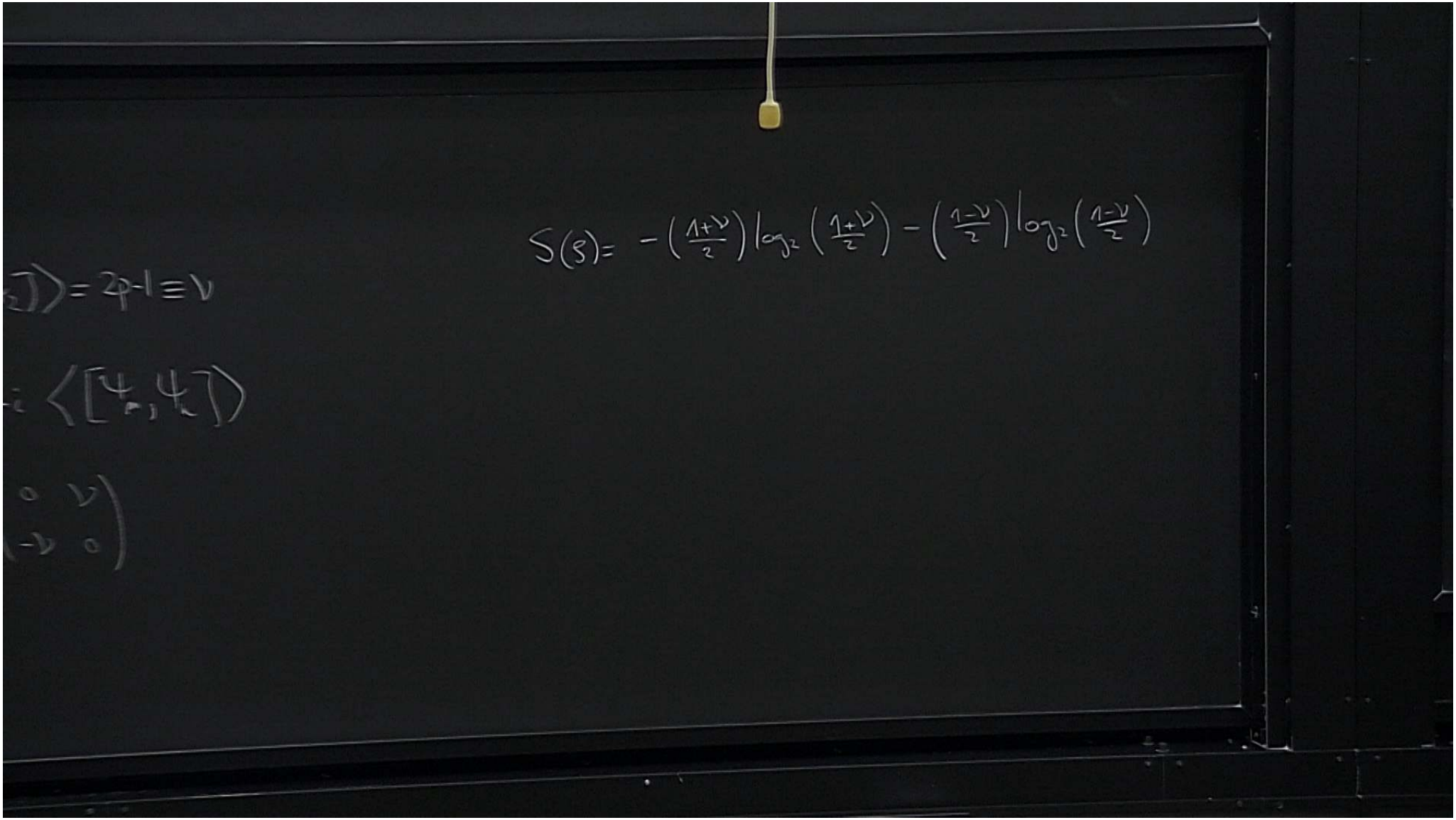
Pair of Majorana operators

$\psi_1, \psi_2 \qquad \psi_i^\dagger = \psi_i \qquad \{\psi_i, \psi_j\} = \delta_{ij}$

matrix representation

$$\psi_1 = \frac{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}{\sqrt{2}} \qquad \psi_2 = \frac{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{\sqrt{2}}$$

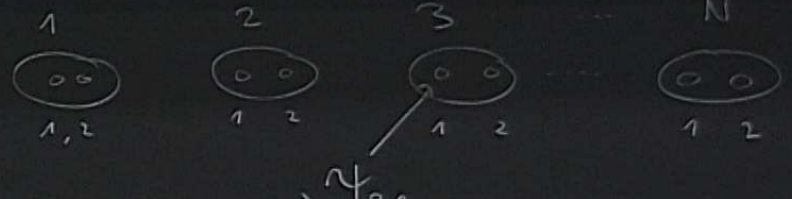
$$-i(\psi_1\psi_2 - \psi_2\psi_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$


$S(\beta) = -\left(\frac{1+\nu}{2}\right)\log_2\left(\frac{1+\nu}{2}\right) - \left(\frac{1-\nu}{2}\right)\log_2\left(\frac{1-\nu}{2}\right)$
$= 2\nu - 1 \equiv \nu$
$\langle [\psi_r, \psi_c] \rangle$

$\psi_1 = \frac{\sigma^x}{\sqrt{2}}$ $\psi_2 = \frac{\sigma^y}{\sqrt{2}}$ $\psi_2 = \frac{a - a^\dagger}{i\sqrt{2}}$ imaginary part

## 82 2N Majorana modes

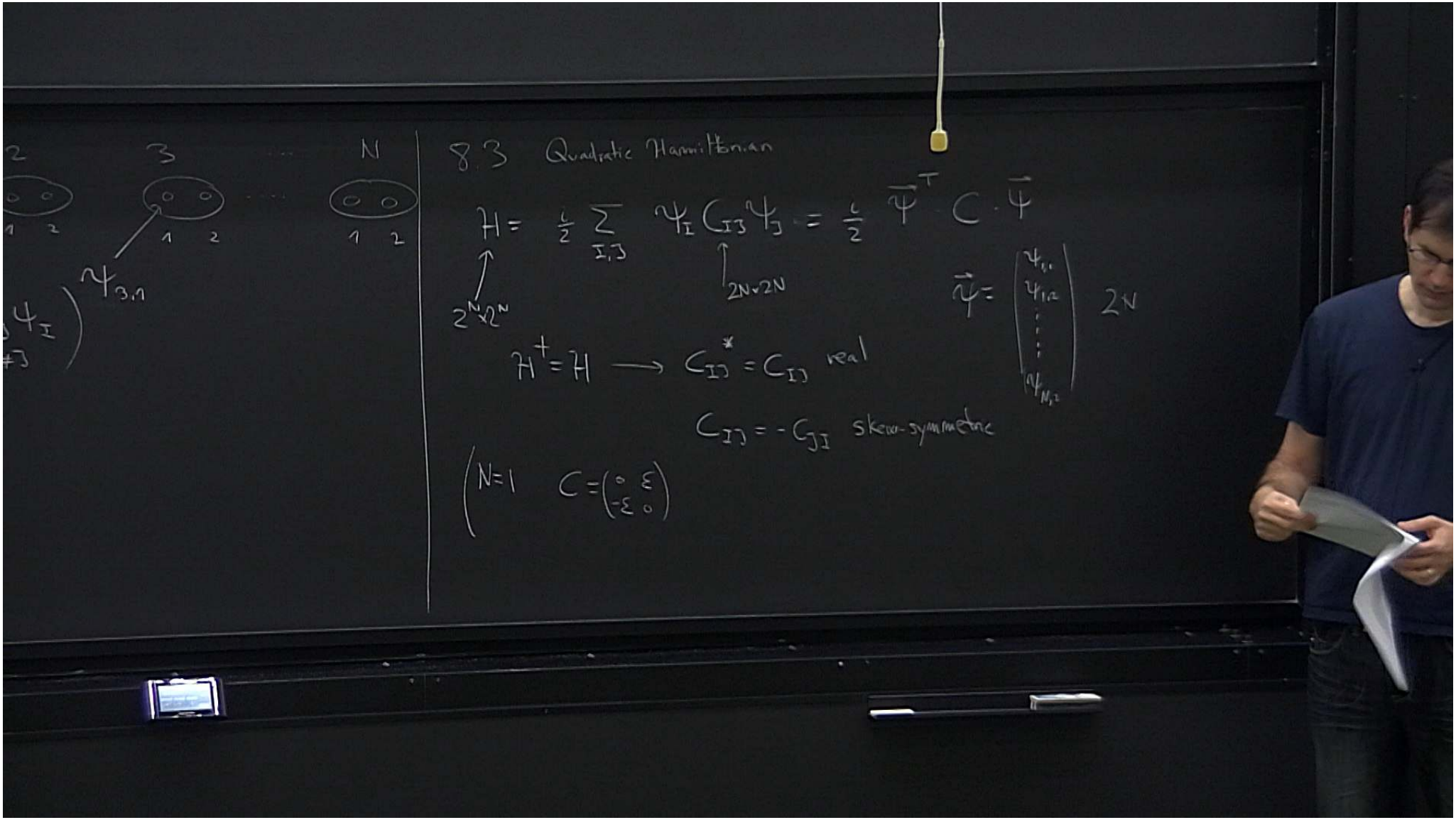
algebra $\psi_I$ $I \equiv (n, \alpha)$ $n = 1, 2, \ldots, N$ $\alpha = 1, 2$

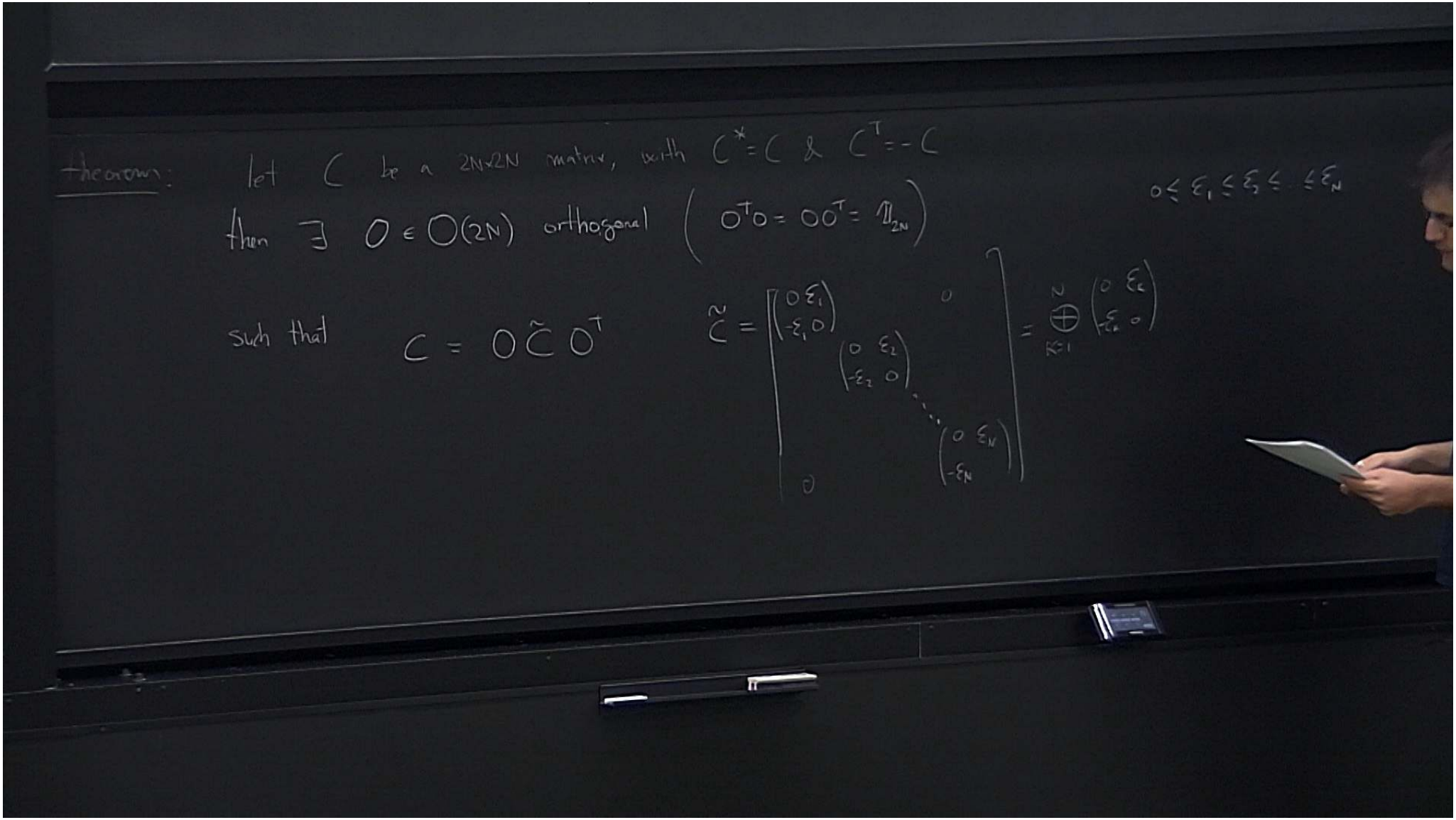


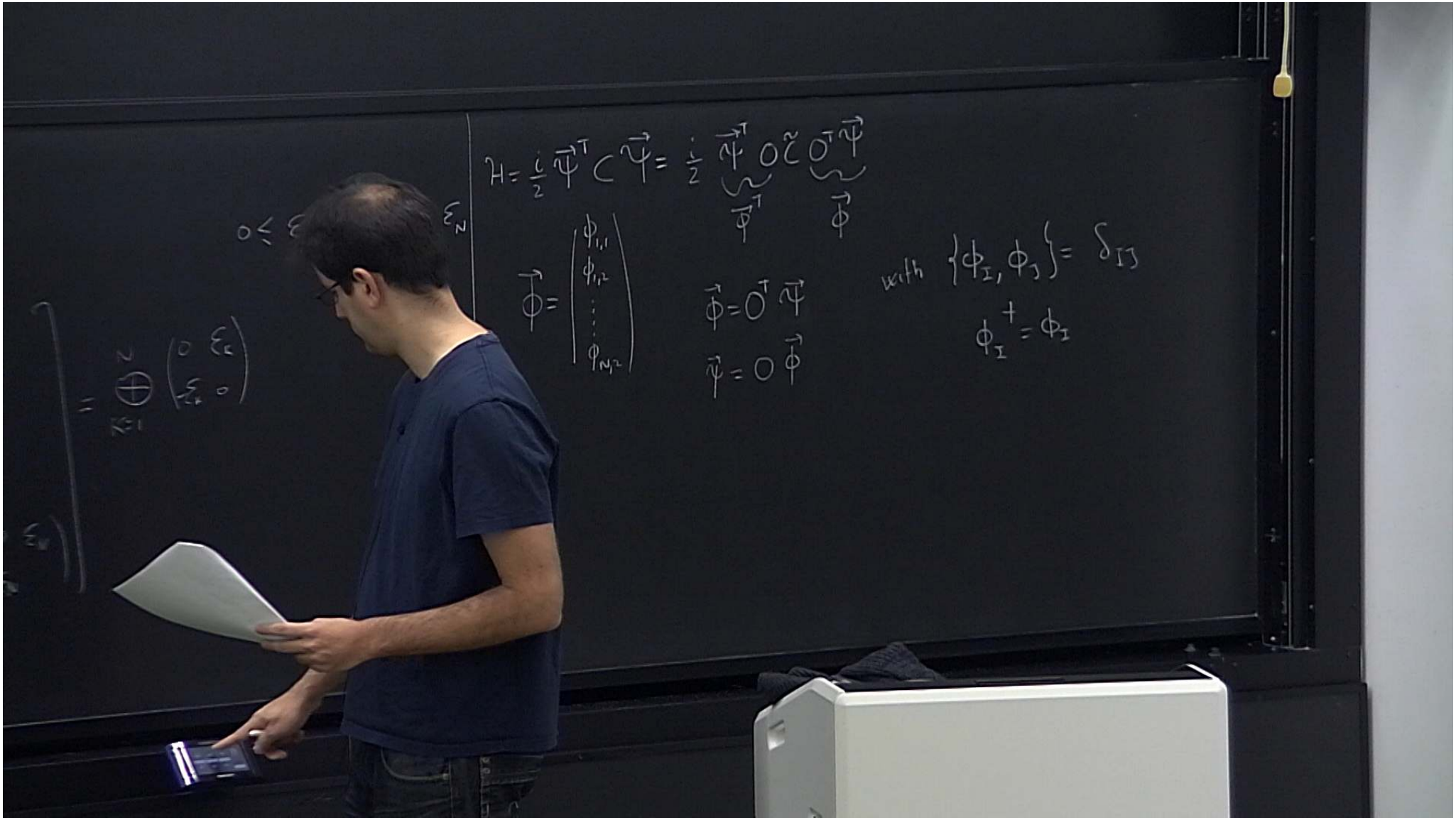

$\psi_I^\dagger = \psi_I$ $\{\psi_I, \psi_J\} = \delta_{IJ}$ $\left( \psi_I^2 = \frac{\mathbb{1}}{2}, \; \psi_I \psi_J = -\psi_J \psi_I \; (I \neq J) \right)$

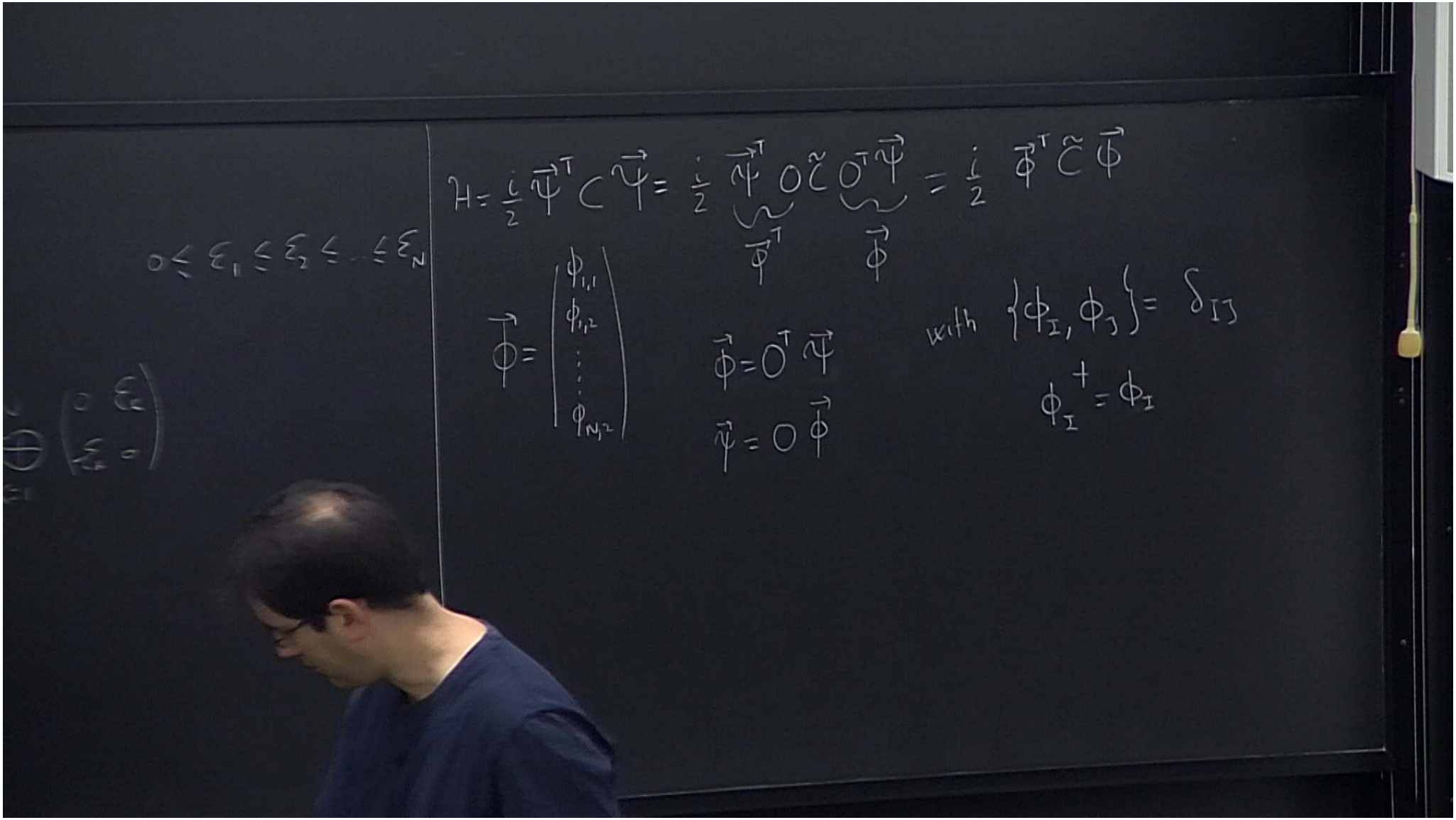
matrix representation $\rightarrow 2^N \times 2^N$ matrices

$N=1$ $\psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$


2
3
N
1 2
1 2
1 2
$\psi_{3,1}$
8.3 Quadratic Hamiltonian
$H = \frac{i}{2} \sum_{I,J} \psi_I C_{IJ} \psi_J = \frac{i}{2} \vec{\psi}^T C \vec{\psi}$
$2^N \times 2^N$
$2N \times 2N$
$\vec{\psi} = \begin{pmatrix} \psi_{1,1} \\ \psi_{1,2} \\ \vdots \\ \psi_{N,2} \end{pmatrix}$ $2N$
$H^\dagger = H \longrightarrow C_{IJ}^* = C_{IJ}$ real
$C_{IJ} = -C_{JI}$ skew-symmetric
$N=1 \quad C = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}$


theorem: let $C$ be a $2N \times 2N$ matrix, with $C^* = C$ & $C^T = -C$
then $\exists\ O \in O(2N)$ orthogonal $\left( O^T O = O O^T = \mathbb{1}_{2N} \right)$
$0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N$
such that $C = O \tilde{C} O^T$
$\tilde{C} = \begin{pmatrix} \begin{pmatrix} 0 & \varepsilon_1 \\ -\varepsilon_1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & \varepsilon_2 \\ -\varepsilon_2 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & \varepsilon_N \\ -\varepsilon_N & 0 \end{pmatrix} \end{pmatrix} = \bigoplus_{k=1}^{N} \begin{pmatrix} 0 & \varepsilon_k \\ -\varepsilon_k & 0 \end{pmatrix}$


$H = \frac{i}{2}\vec{\Psi}^T C \vec{\Psi} = \frac{i}{2}\vec{\Psi}^T O^T O \vec{\Psi}$
$\vec{\Phi} = O\vec{\Psi}$
$\vec{\Psi} = O^T\vec{\Phi}$
with $\{\Phi_I, \Phi_J\} = \delta_{IJ}$
$\Phi_I^\dagger = \Phi_I$


$H = \frac{i}{2}\vec{\Psi}^T C \vec{\Psi} = \frac{i}{2}\vec{\Psi}^T O \tilde{C} O^T \vec{\Psi} = \frac{i}{2}\vec{\Phi}^T \tilde{C} \vec{\Phi}$
$\vec{\Phi}^T$
$\vec{\Phi}$
$\vec{\Phi} = \begin{pmatrix} \phi_{1,1} \\ \phi_{1,2} \\ \vdots \\ \phi_{N,2} \end{pmatrix}$
$\vec{\Phi} = O^T \vec{\Psi}$
$\vec{\Psi} = O \vec{\Phi}$
with $\{\phi_I, \phi_J\} = \delta_{IJ}$
$\phi_I^\dagger = \phi_I$
$0 \leq \xi_1 \leq \xi_2 \leq \dots \leq \xi_N$

$$H = \frac{i}{2}\sum_{k=1}^{N} \varepsilon_k \left(\phi_{k,1}\phi_{k,2} - \phi_{k,2}\phi_{k,1}\right) = \sum_{k=1}^{N} \varepsilon_k \left(a_k^\dagger a_k - \frac{1}{2}\right) = -\sum_{k=1}^{N} \frac{\varepsilon_k}{2}\sigma_k^z$$

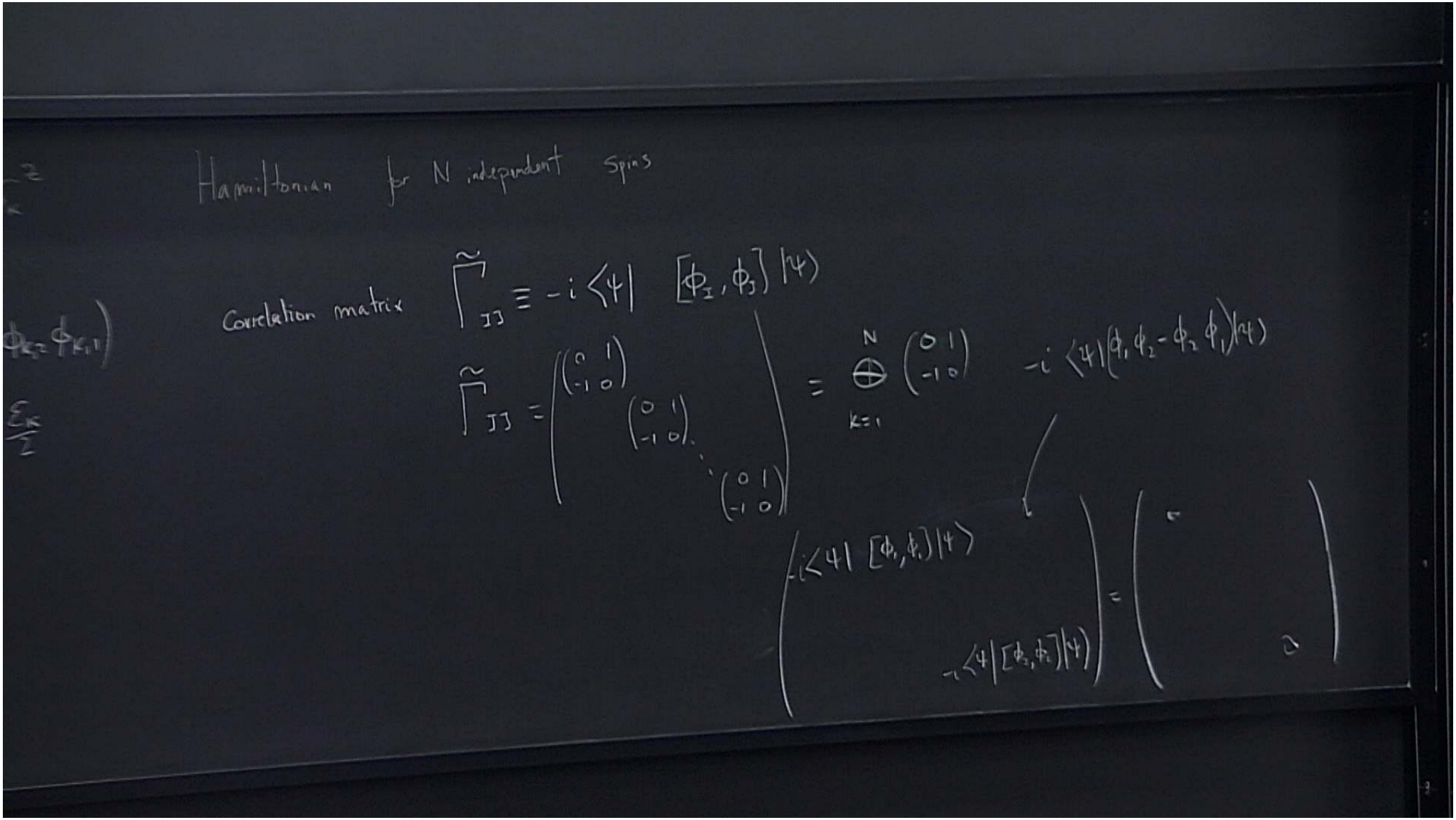
$$a_k \equiv \frac{\phi_{k,1} + i\phi_{k,2}}{\sqrt{2}}$$

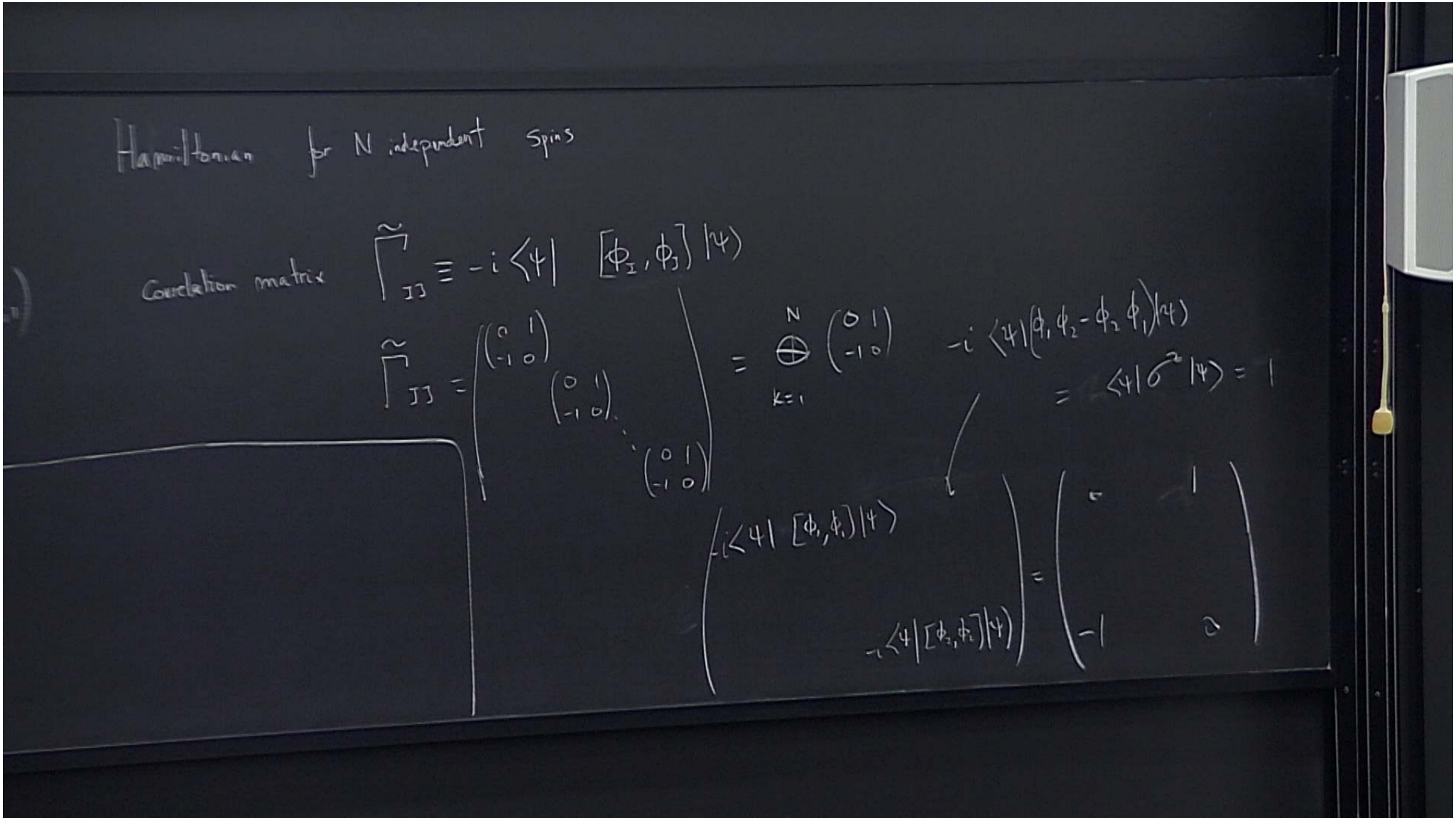
$$\sigma_k^z \equiv -i\left(\phi_{k,1}\phi_{k,2} - \phi_{k,2}\phi_{k,1}\right)$$

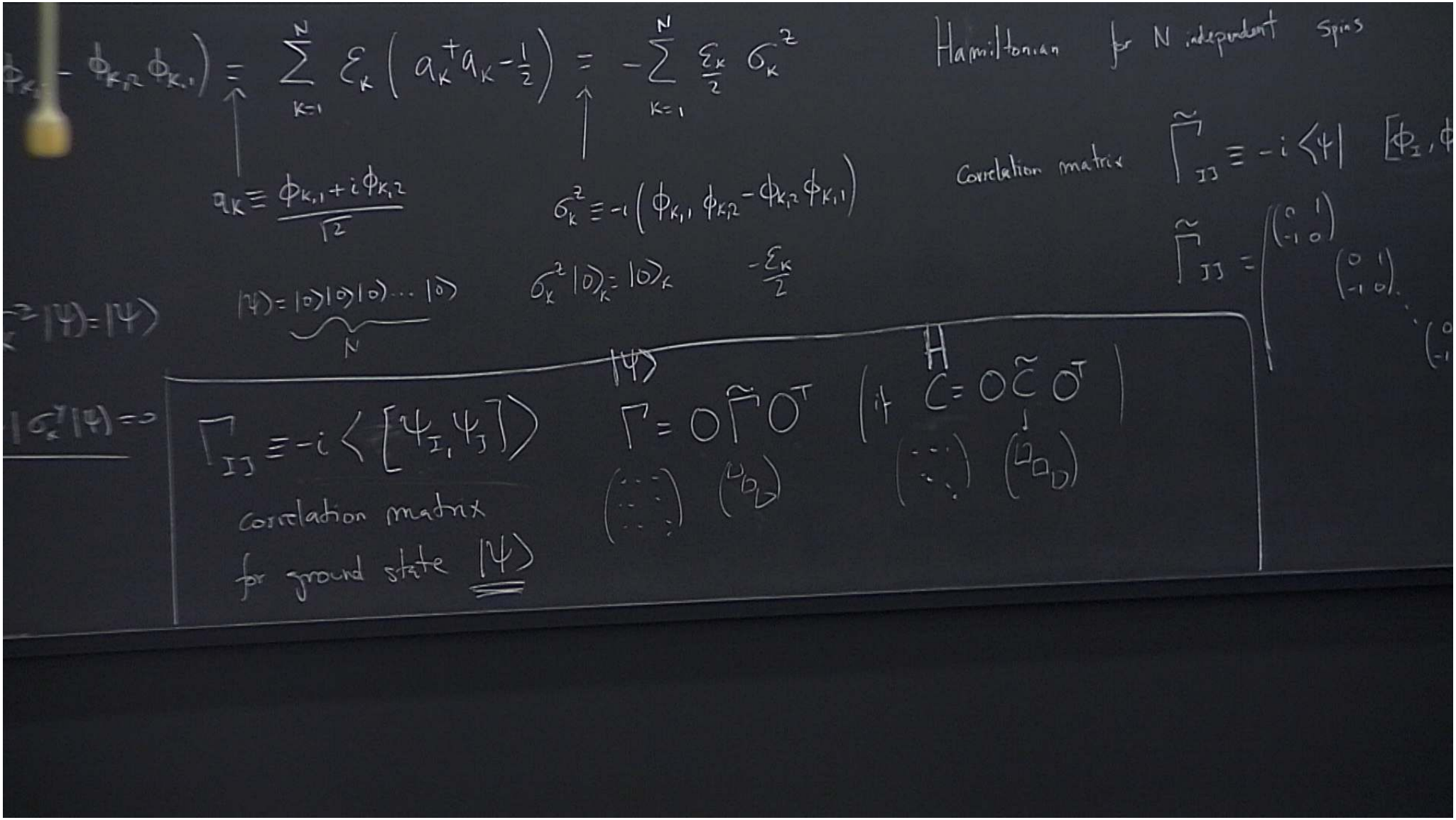
$$\sigma_k^z |0\rangle_k = |0\rangle_k \qquad -\frac{\varepsilon_k}{2}$$

8.4 Ground state $\quad \sigma_k^z |\psi\rangle = |\psi\rangle \qquad |\psi\rangle = \underbrace{|0\rangle|0\rangle|0\rangle\cdots|0\rangle}_{N}$

$$\langle\psi|\sigma_k^x|\psi\rangle = \langle\psi|\sigma_k^y|\psi\rangle = 0$$


Hamiltonian for N independent spins
Correlation matrix


Hamiltonian for N independent spins
Correlation matrix
$\tilde{\Gamma}_{IJ} \equiv -i\langle\psi|\,[\phi_I,\phi_J]\,|\psi\rangle$
$\tilde{\Gamma}_{IJ} = \begin{pmatrix} \begin{pmatrix}0&1\\-1&0\end{pmatrix} & & \\ & \begin{pmatrix}0&1\\-1&0\end{pmatrix} & \\ & & \ddots \end{pmatrix} = \bigoplus_{k=1}^{N} \begin{pmatrix}0&1\\-1&0\end{pmatrix}$
$-i\langle\psi|(\phi_1\phi_2-\phi_2\phi_1)|\psi\rangle$
$= \langle\psi|\sigma^z|\psi\rangle = 1$
$\begin{pmatrix} -i\langle\psi|[\phi_1,\phi_1]|\psi\rangle & \\ & -i\langle\psi|[\phi_2,\phi_2]|\psi\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$


$\phi_{K,2}\phi_{K,1}) = \sum_{K=1}^{N} \varepsilon_K \left(a_K^\dagger a_K - \frac{1}{2}\right) = -\sum_{K=1}^{N} \frac{\varepsilon_K}{2} \sigma_K^z$
Hamiltonian for N independent spins
$a_K \equiv \frac{\phi_{K,1} + i\phi_{K,2}}{\sqrt{2}}$
$\sigma_K^z \equiv -i\left(\phi_{K,1}\phi_{K,2} - \phi_{K,2}\phi_{K,1}\right)$
Correlation matrix
$|\psi\rangle = |0\rangle|0\rangle|0\rangle \cdots |0\rangle$
$N$
$\sigma_K^z |0\rangle_K = |0\rangle_K$
$-\frac{\varepsilon_K}{2}$
$\sigma_K^z|\psi\rangle = |\psi\rangle$
$\sigma_K^z|\psi\rangle \Rightarrow$
$\Gamma_{IJ} \equiv -i\langle[\psi_I, \psi_J]\rangle$
Correlation matrix
for ground state $|\psi\rangle$
$|\psi\rangle$
$\Gamma = O\tilde{\Gamma}O^T$
$H$
$C = O\tilde{C}O^T$

$H = \frac{1}{2} \sum_{k=1}$ $c_k$ ( $\psi_{k,1}$ $\psi_{k,2}$ ... )

$k=1$ $\qquad$ $k=1$ $\qquad$ $k=1$

$q_k \equiv \frac{\phi_{k,1} + i\phi_{k,2}}{\sqrt{2}}$ $\qquad$ $\sigma_k^z \equiv -i\left(\phi_{k,1}\phi_{k,2} - \phi_{k,2}\phi_{k,1}\right)$

## 8.4 Ground state

$\sigma_k^z |\psi\rangle = |\psi\rangle$ $\qquad$ $|\psi\rangle = \underbrace{|0\rangle|0\rangle|0\rangle \cdots |0\rangle}_{N}$ $\qquad$ $\sigma_k^z |0\rangle_k = |0\rangle_k$ $\qquad$ $-\frac{\varepsilon_k}{2}$

$\langle\psi|\sigma_k^x|\psi\rangle = \langle\psi|\sigma_k^y|\psi\rangle = 0$

Wick's theorem

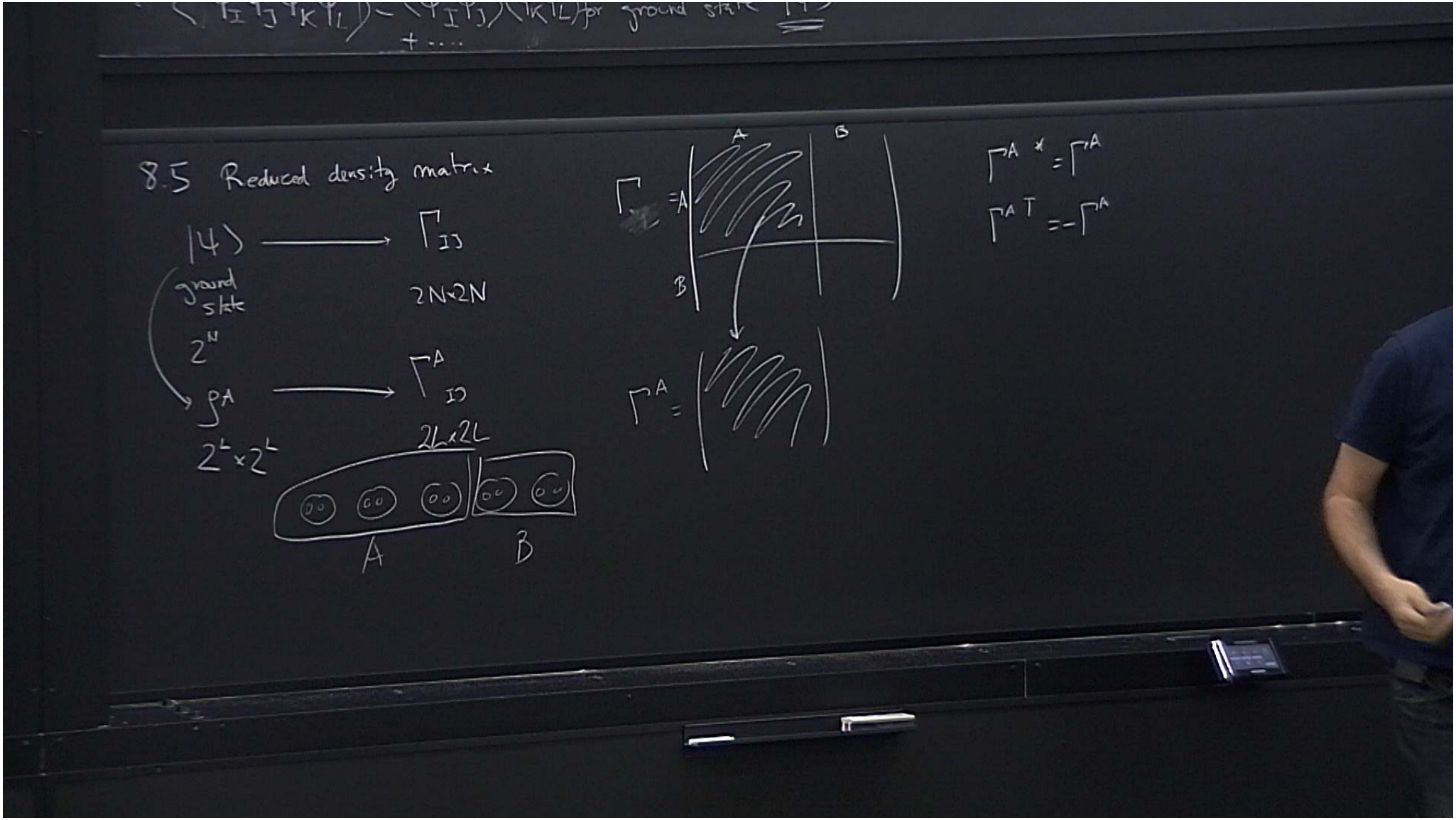
$\langle \psi_I \psi_J \psi_K \psi_L \rangle = \langle \psi_I \psi_J \rangle \langle \psi_K \psi_L \rangle$ + ... for ground state $|\psi\rangle$

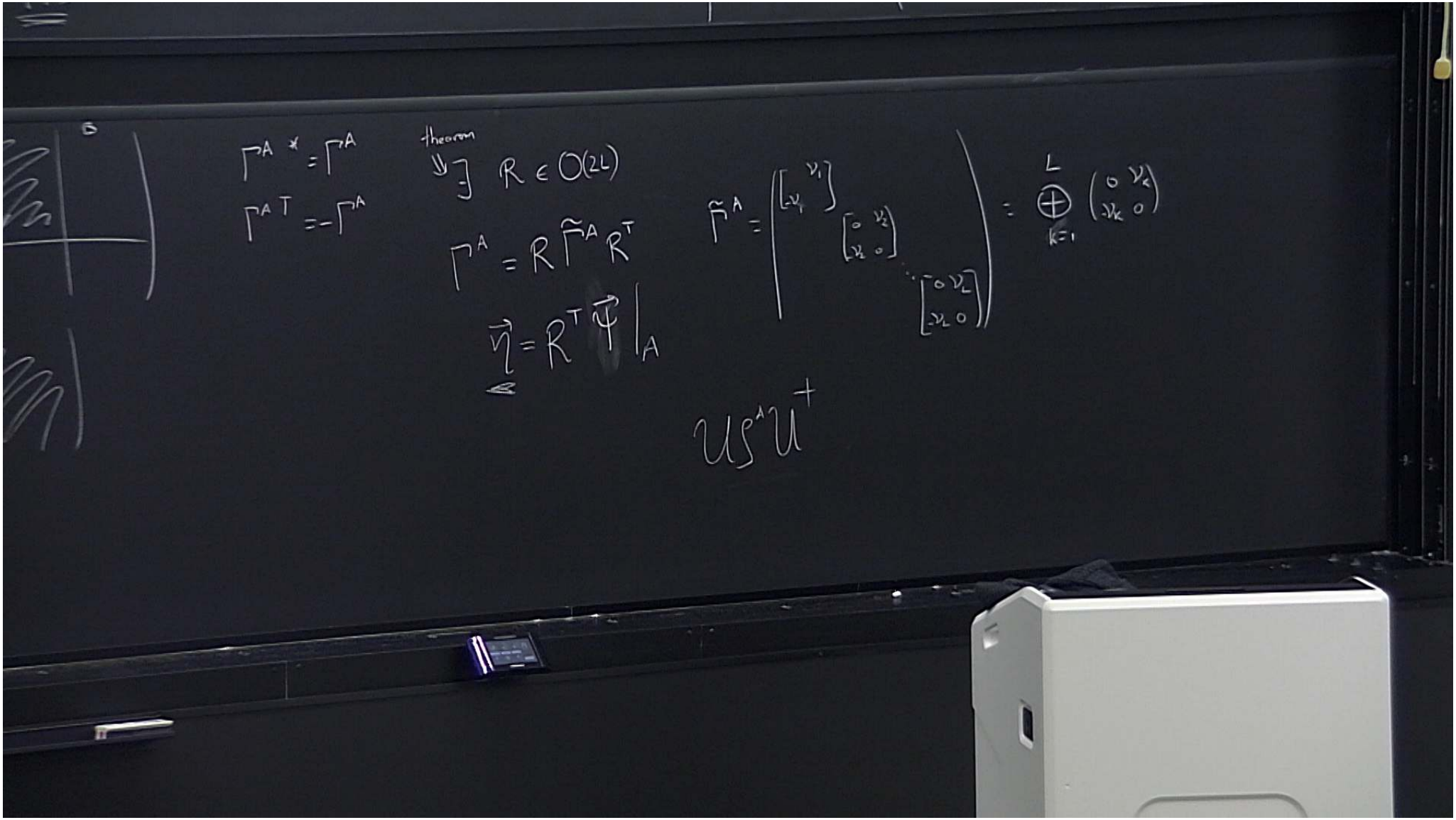
$\Gamma_{IJ} \equiv -i \langle [\psi_I, \psi_J] \rangle$

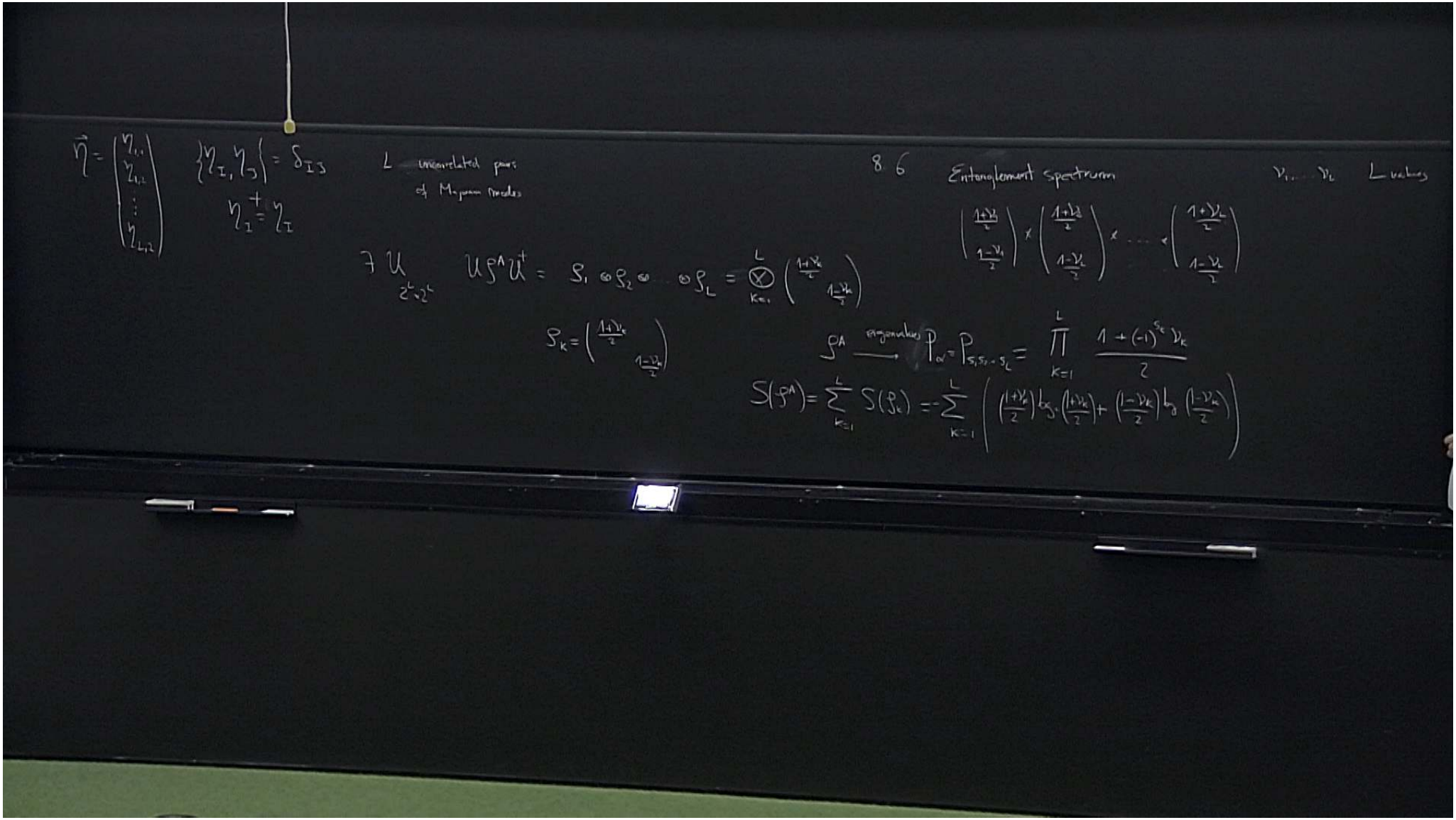
correlation matrix

$|\psi\rangle$ $\qquad$ $\Gamma = O \tilde{\Gamma} O^T$

$\begin{pmatrix} \cdots \\ \vdots \end{pmatrix}$ $\qquad$ $\begin{pmatrix} \\ \end{pmatrix}$


8.5 Reduced density matrix
|4⟩
ground state
$2^N$
$\rho_A$
$2^L \times 2^L$
$\Gamma_{IJ}$
2N×2N
$\Gamma^A_{IJ}$
2L×2L
A
B
$\Gamma^{A\,*} = \Gamma^A$
$\Gamma^{A\,T} = -\Gamma^A$


$\Gamma^{A\,*} = \Gamma^A$
$\Gamma^{A\,T} = -\Gamma^A$
theorem
$\exists\, R \in O(2L)$
$\Gamma^A = R\,\tilde{\Gamma}^A\,R^T$
$\vec{\eta} = R^T \vec{\Psi}\big|_A$
$\tilde{\Gamma}^A = \begin{pmatrix} \begin{bmatrix} & \nu_1 \\ -\nu_1 & \end{bmatrix} & & \\ & \begin{bmatrix} & \nu_2 \\ -\nu_2 & \end{bmatrix} & \\ & & \ddots \begin{bmatrix} & \nu_L \\ -\nu_L & \end{bmatrix} \end{pmatrix} = \bigoplus_{k=1}^{L} \begin{pmatrix} 0 & \nu_k \\ -\nu_k & 0 \end{pmatrix}$
$U S^A U^+$


$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{L+2} \end{pmatrix}$
$\{\eta_I, \eta_J\} = \delta_{IJ}$
$\eta_I^+ = \eta_I$
$L$ uncorrelated pairs of Majorana modes
8.6 Entanglement spectrum
$\nu_1 \dots \nu_L$ $L$ values
$\left(\begin{smallmatrix} \frac{1+\nu_1}{2} \\ \frac{1-\nu_1}{2} \end{smallmatrix}\right) \times \left(\begin{smallmatrix} \frac{1+\nu_2}{2} \\ \frac{1-\nu_2}{2} \end{smallmatrix}\right) \times \dots \times \left(\begin{smallmatrix} \frac{1+\nu_L}{2} \\ \frac{1-\nu_L}{2} \end{smallmatrix}\right)$
$2^L \times 2^L$
$U \rho^A U^+ = \rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_L = \bigotimes_{k=1}^{L} \left(\begin{smallmatrix} \frac{1+\nu_k}{2} \\ \frac{1-\nu_k}{2} \end{smallmatrix}\right)$
$\rho_k = \left(\begin{smallmatrix} \frac{1+\nu_k}{2} \\ \frac{1-\nu_k}{2} \end{smallmatrix}\right)$
$\rho^A \xrightarrow{\text{eigenvalues}} p_\alpha = p_{s_1, s_2, \dots s_L} = \prod_{k=1}^{L} \frac{1 + (-1)^{s_k} \nu_k}{2}$
$S(\rho^A) = \sum_{k=1}^{L} S(\rho_k) = -\sum_{k=1}^{L} \left( \left(\frac{1+\nu_k}{2}\right) \log_2 \left(\frac{1+\nu_k}{2}\right) + \left(\frac{1-\nu_k}{2}\right) \log_2 \left(\frac{1-\nu_k}{2}\right) \right)$