Title: PSI 2016/2017 Condensed Matter (Review) - Lecture 3

Date: Feb 02, 2017 09:00 AM

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Abstract:

methods: A) full diagonalization of $H$ $2^N \times 2^N$ matrix time $2^{3N}$

B) sparse diagonalization $|\psi\rangle \rightarrow |\psi'\rangle = H|\psi\rangle$

power method

Lanczos method

B1) building $H$ $2^N \times 2^N$ matrix time $2^{2N}$

B2) sparse vector update time $N2^N$

$2^N \times 2^N$ matrix time $2^{3N} \longrightarrow N \sim 10$

$|\psi\rangle \rightarrow |\psi'\rangle = H|\psi\rangle$

power method

Lanczos method

H $2^N \times 2^N$ matrix time $2^{2N}$ $N \sim 15$

vector update time $N2^N$ $N \sim 30 \longrightarrow 40$

Lecture 6 (th)

6.0 Previous lectures

methods

local Hamiltonian $H = \sum_l h_{l,l+1} = -\sum_l \sigma_l^x \sigma_{l+1}^x - h \sum_l \sigma_l^z$

ground state $|\psi_0\rangle$

Today: - entanglement - correlations

## 6.1 Reduced density matrix

A) n-level quantum system

$|\psi\rangle \in \mathbb{C}^n$ orthonormal basis $\{|1\rangle, |2\rangle \dots |n\rangle\}$ $\langle\alpha|\alpha$

$$|\psi\rangle = \sum_{\alpha=1}^{n} A_\alpha |\alpha\rangle$$

B) bipartite system

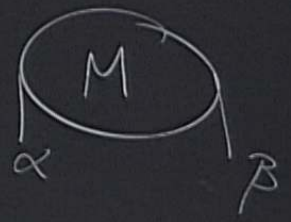
$|\psi^{AB}\rangle \in \mathbb{V}^A \otimes \mathbb{V}^B$

$\mathbb{C}^m \otimes \mathbb{C}^n$

$\{|1\rangle, |2\rangle, \dots |m\rangle\} \times \{|1\rangle, |2\rangle$

$|\alpha\beta\rangle \equiv |\alpha\rangle_A \otimes |\beta\rangle_B$

$$|\psi^{AB}\rangle = \sum_{\alpha\beta} M_{\alpha\beta} |\alpha\beta\rangle$$



$\{|1\rangle, |2\rangle \ldots |n\rangle\}$ $\quad \langle\alpha|\alpha'\rangle = \delta_{\alpha\alpha'}$

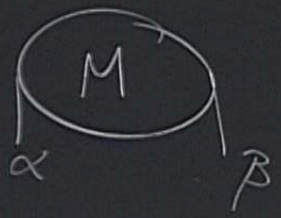
$\{|1\rangle, |2\rangle, \ldots |m\rangle\} \times \{|1\rangle, |2\rangle \ldots |n\rangle\}$

$|\alpha\beta\rangle \equiv |\alpha\rangle_A \otimes |\beta\rangle_B$

$|11\rangle, |12\rangle, \ldots |1n\rangle$
$|21\rangle, |22\rangle, \ldots |2n\rangle$
$\vdots$
$|m1\rangle, |m2\rangle \ldots |mn\rangle$

$m \cdot n$ basis vectors

$|\alpha\beta\rangle$



c) reduced density matrix

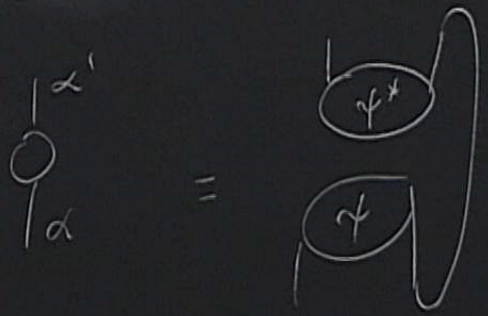
$$\rho^A \equiv tr_B \, |\psi^{AB}\rangle\langle\psi^{AB}| = \sum_{\beta=1}^{n} \langle\beta|_B \, |\psi\rangle\langle\psi|_{AB} \, |\beta\rangle_B = \sum_{\alpha\alpha'} \left( \sum_{\beta} \right.$$

$$\langle\beta'|_B \, |\alpha\beta\rangle_{AB} \equiv$$

$$\sum_{\alpha\alpha'} \left( \sum_{\beta} M_{\alpha\beta} M^{*}_{\alpha'\beta} \right) |\alpha\rangle\langle\alpha'| = \sum_{\alpha\alpha'} \rho^{A}_{\alpha\alpha'} |\alpha\rangle\langle\alpha'|$$

$$|\alpha_{\substack{B \\ A\,\bar{D}}}\rangle \equiv |\alpha\rangle \underbrace{\langle\bar{\beta}'|\beta\rangle}_{\delta_{\beta\beta'}}$$

$$\rho^{A}_{\alpha\alpha'} = \left(M \cdot M^{\dagger}\right)_{\alpha\alpha'}$$



62 Entanglement in bipartite systems

$V^{AB} \ni |\psi\rangle = \sum M_{\alpha\beta} |\alpha\beta\rangle$

- product state: $|\psi^{AB}\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$
- entangled state: $|\psi^{AB}\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$

examples

if $|\psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$ then $\rho^A = |\phi^A\rangle\langle\phi^A| \operatorname{tr}(|\varphi^B\rangle\langle\varphi^B|) = |\phi^A\rangle\langle\phi^A|$

$\Rightarrow$

$\Leftarrow$ ?

Examples

$|1\rangle \otimes |1\rangle = |11\rangle$

$|11\rangle + |22\rangle$

$|11\rangle + |12\rangle = |1\rangle \otimes (|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle + |22\rangle = (|1\rangle + |2\rangle) \otimes (|1\rangle + |2\rangle)$

$|11\rangle + |12\rangle + |21\rangle - |22\rangle$

$\neq X \otimes A$

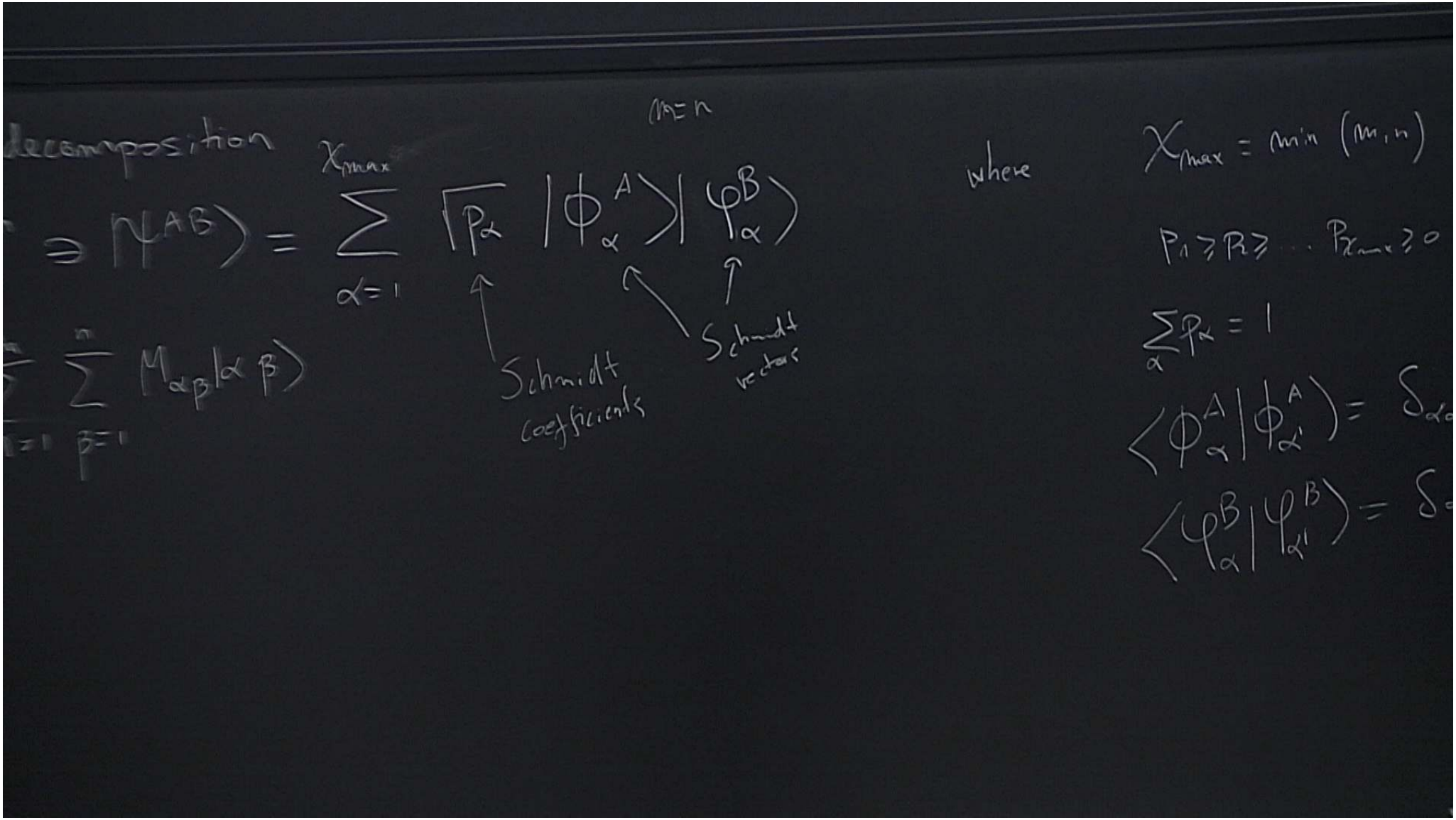
Schmidt decomposition

$$\mathbb{C}^M \otimes \mathbb{C}^n \ni |\psi^{AB}\rangle = \sum_{\alpha=1}^{\chi_{max}} \sqrt{p_\alpha}\, |\phi_\alpha^A\rangle |\varphi_\alpha^B\rangle \qquad M=n$$

where $\chi_{max} = \min(M, n)$

$p_1 \geq p_2 \geq \dots p_{\chi_{max}} > 0$

$$|\psi^{AB}\rangle = \sum_{\alpha=1}^{M} \sum_{\beta=1}^{n} M_{\alpha\beta} |\alpha\, \beta\rangle$$


decomposition
$\chi_{max}$
$n_A = n$
$\Rightarrow |\psi^{AB}\rangle = \sum_{\alpha=1}^{\chi_{max}} \sqrt{p_\alpha}\, |\phi_\alpha^A\rangle |\varphi_\alpha^B\rangle$
$\sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} M_{\alpha\beta} |\alpha\, \beta\rangle$
Schmidt coefficients
Schmidt vectors
where
$\chi_{max} = \min(m, n)$
$p_1 \geq p_2 \geq \ldots p_{\chi_{max}} \geq 0$
$\sum_\alpha p_\alpha = 1$
$\langle \phi_\alpha^A | \phi_{\alpha'}^A \rangle = \delta_{\alpha\alpha'}$
$\langle \varphi_\alpha^B | \varphi_{\alpha'}^B \rangle = \delta_{\alpha\alpha'}$

$\chi_{max} = \min(m, n)$

$p_1 \geq p_2 \geq \cdots p_{\chi_{max}} > 0$

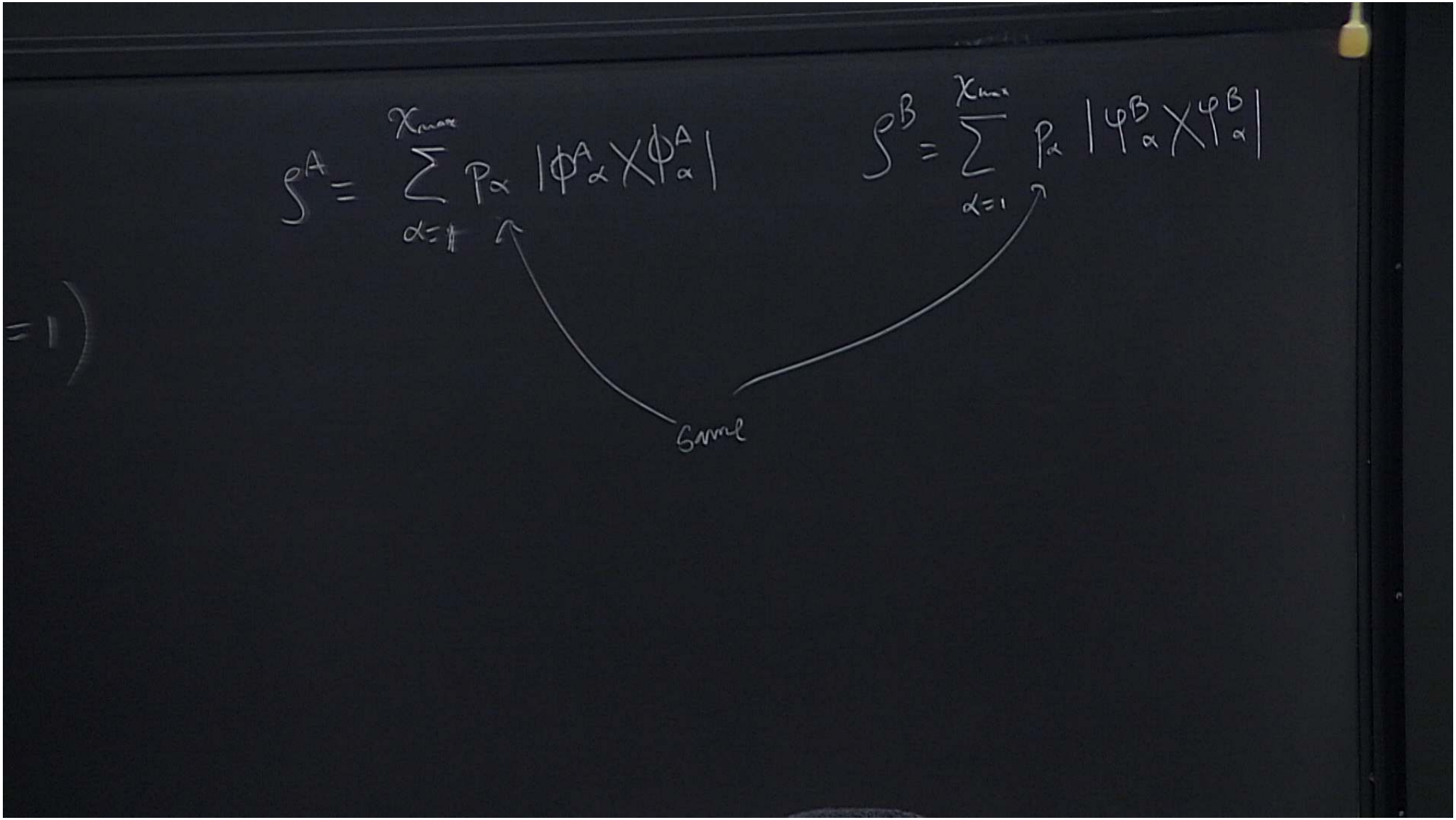
$\left( \langle \psi^{AB} | \psi^{AB} \rangle = 1 \right)$

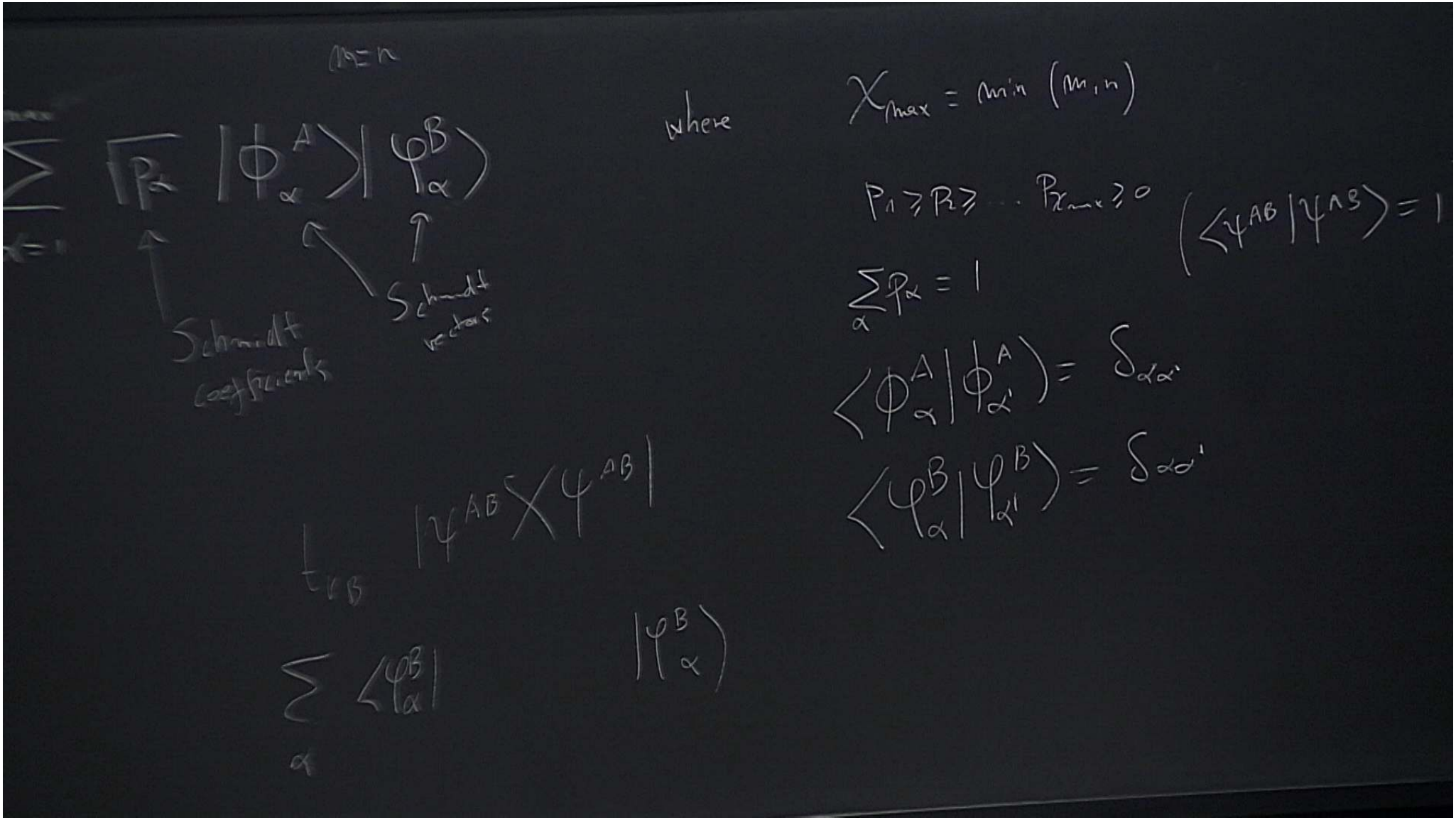
$\sum_\alpha p_\alpha = 1$

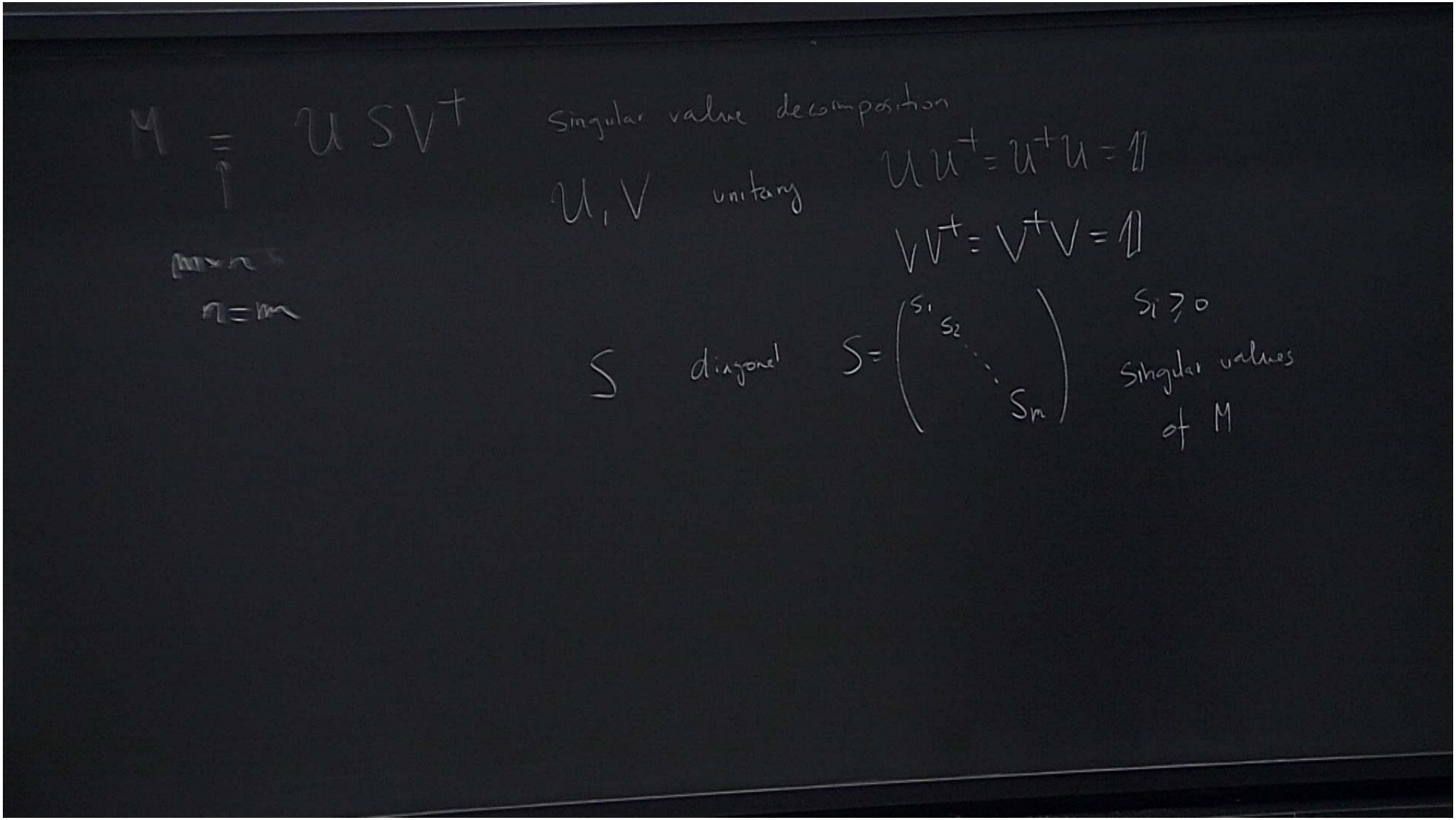
$\langle \phi^A_\alpha | \phi^A_{\alpha'} \rangle = \delta_{\alpha\alpha'}$

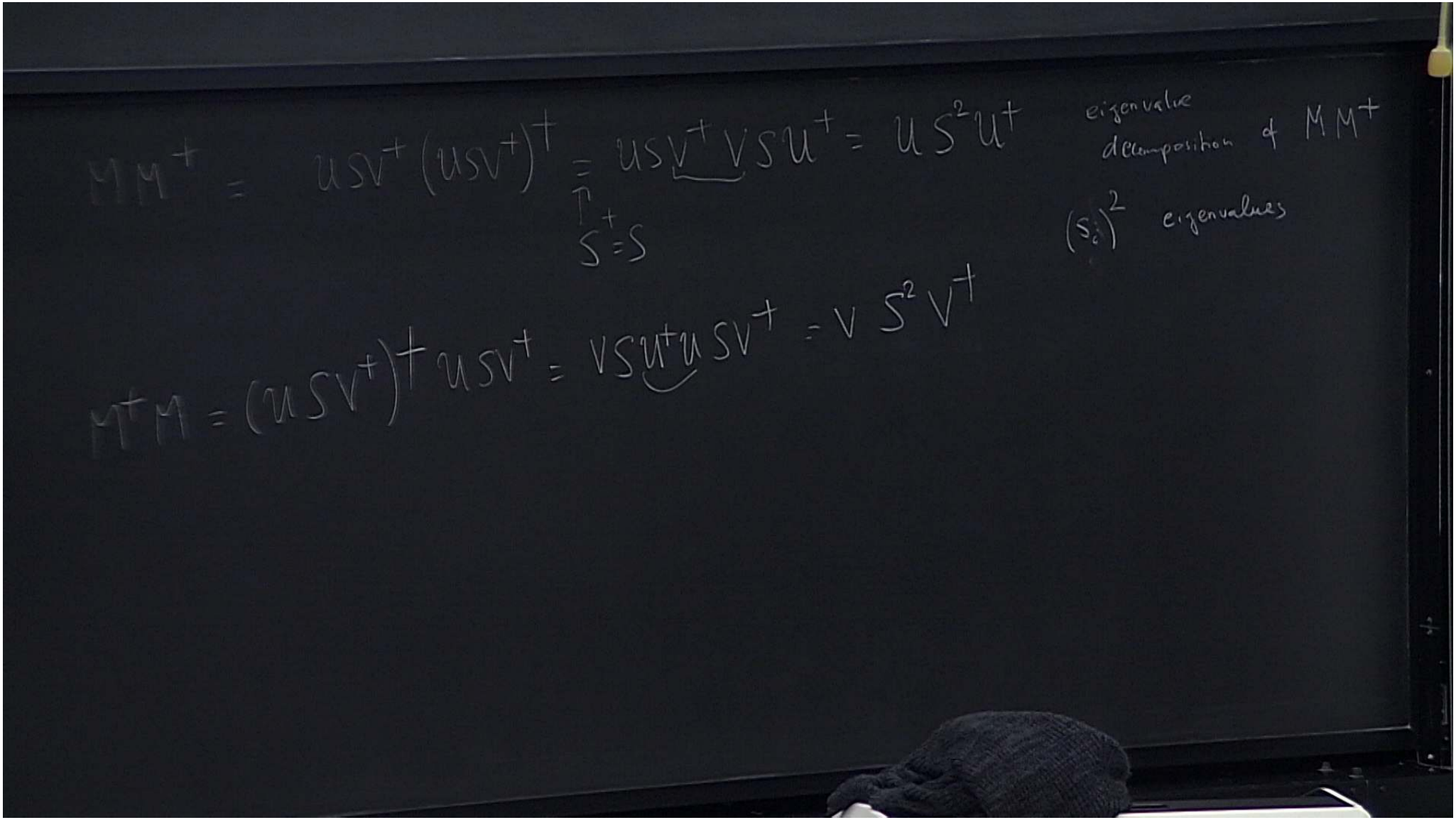
$\langle \phi^B_\alpha | \phi^B_{\alpha'} \rangle = \delta_{\alpha\alpha'}$

$\rho^A = \sum_{\alpha=1}^{\chi_{max}} p_\alpha |\phi^A_\alpha\rangle\langle\phi^A_\alpha|$


$\rho^A = \sum_{\alpha=1}^{\chi_{max}} p_\alpha |\phi^A_\alpha\rangle\langle\phi^A_\alpha|$
$\rho^B = \sum_{\alpha=1}^{\chi_{max}} p_\alpha |\varphi^B_\alpha\rangle\langle\varphi^B_\alpha|$
Same


m=n
where
$\chi_{max} = \min(m,n)$
$\sum_{\alpha=1}^{\chi} \sqrt{p_\alpha}\, |\phi_\alpha^A\rangle |\varphi_\alpha^B\rangle$
$p_1 \geq p_2 \geq \dots p_{\chi_{max}} > 0$
$\sum_\alpha p_\alpha = 1$
$\langle \phi_\alpha^A | \phi_{\alpha'}^A \rangle = \delta_{\alpha\alpha'}$
$\langle \varphi_\alpha^B | \varphi_{\alpha'}^B \rangle = \delta_{\alpha\alpha'}$
$(\langle \psi^{AB} | \psi^{AB} \rangle = 1$
Schmidt coefficients
Schmidt vectors
$t_{rB}\, |\psi^{AB}\rangle\langle\psi^{AB}|$
$\sum_\alpha \langle \varphi_\alpha^B |$
$|\varphi_\alpha^B\rangle$


M = U S V†
Singular value decomposition
U, V unitary
U U† = U† U = 𝟙
V V† = V† V = 𝟙
m × n
n = m
S diagonal
S = (s1, s2, ..., sm)
si ≥ 0
Singular values of M


$MM^\dagger = usv^\dagger(usv^\dagger)^\dagger = usv^\dagger vsu^\dagger = us^2u^\dagger$
eigenvalue decomposition of $MM^\dagger$
$I$
$S^\dagger = S$
$(s_i)^2$ eigenvalues
$M^\dagger M = (usv^\dagger)^\dagger usv^\dagger = vsu^\dagger usv^\dagger = vs^2v^\dagger$

$M = U S V^\dagger$ Singular value decomposition

$U, V$ unitary $\quad U U^\dagger = U^\dagger U = \mathbb{1}$

$V V^\dagger = V^\dagger V = \mathbb{1}$

$M$: $m \times n$, $n = m$

$S$ diagonal $\quad S = \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \\ & & & s_m \end{pmatrix}$ $\quad s_i \geq 0$

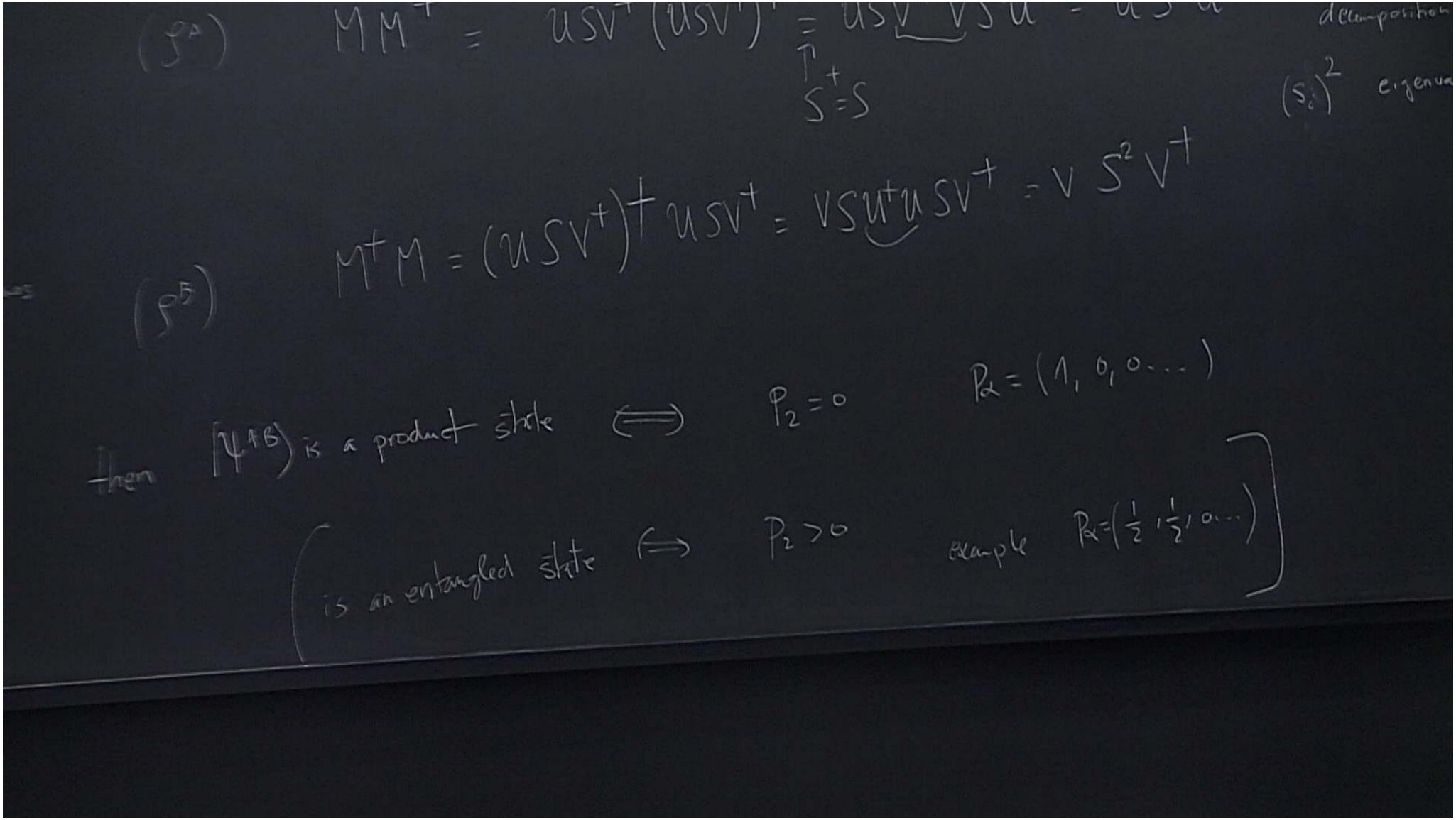
Singular values of $M$

**Theorem** let $|\psi^{AB}\rangle = \sum_{\alpha\beta} M_{\alpha\beta} |\alpha\beta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$

and let $P_\alpha = (P_1, P_2, \ldots)$ $\quad P_1 \geq P_2 \geq \ldots \geq 0$ $\quad \sum P_\alpha = 1$

be the eigenvalues of $MM^\dagger$

then


$(\rho^A)$ $MM^\dagger = usv^\dagger(usv^\dagger)^\dagger = usv^\dagger vsu = u s^2 u$
decomposition
$S^\dagger = S$
$(s_i)^2$ eigenva
$(\rho^B)$ $M^\dagger M = (usv^\dagger)^\dagger usv^\dagger = vsu^\dagger usv^\dagger = v s^2 v^\dagger$
then $|\psi_{AB}\rangle$ is a product state $\iff$ $P_2 = 0$ $R_2 = (1, 0, 0 \ldots)$
is an entangled state $\iff$ $P_2 > 0$ example $R_2 = (\frac{1}{2}, \frac{1}{2}, 0 \ldots)$

## 6.3 Characterization of entanglement

$P_\alpha = (p_1, p_2, \ldots, p_{k_{max}})$ $\quad e_\alpha \equiv \log_2 p_\alpha$ entanglement spectrum (E.S.)

$P_\alpha$ is invariant under local unitary transformations

$|\psi^{AB}\rangle \longrightarrow |\psi^{AB\prime}\rangle = U^A \otimes V^B |\psi^{AB}\rangle$

$\{P_\alpha\}$ $\quad \{P_\alpha\}$

entanglement spectrum (E.S.)

tions

$U^A \otimes V^B |\psi^{AB}\rangle$

example $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow p_\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$

$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

Entanglement entropy EE $\rho = U \begin{pmatrix} p_1 & & \\ & p_2 & \\ & & \ddots \\ & & & p_n \end{pmatrix} U^\dagger$

von Neumann entropy $S(\rho) \equiv -\operatorname{tr} \rho \log_2 \rho = -\sum_\alpha p_\alpha \log_2 p_\alpha$

$= S(p_\alpha)$

Shannon entropy

$p_\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$ E.E of $|\psi^{AB}\rangle$

$S(\rho^A) = S(\rho^B)$

Examples: Product state $S(p_\alpha) = -1 \log_2 1 - 0 \log_2 0 - \ldots = 0$

$p_\alpha = (1, 0, 0 \ldots)$

maximally entangled state of 2 qubits

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $p_\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$ $S(p_\alpha) = -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2}$

$= -\log_2 \frac{1}{2} = \log_2 2 = \underline{\underline{1 \text{ ebit}}}$

## 6.4 (bipartite) entanglement in many-body states

spin chain N spins

$$|\psi\rangle \in \underbrace{\mathbb{C}_1^2 \otimes \mathbb{C}_1^2 \otimes \cdots \otimes \mathbb{C}_1^2}_{N}$$

$$|\psi\rangle = \sum_{\alpha_1 \alpha_2 \cdots \alpha_N} \psi_{\alpha_1 \alpha_2 \cdots \alpha_N} |\alpha_1 \alpha_2 \cdots \alpha_N\rangle$$

$N=5$

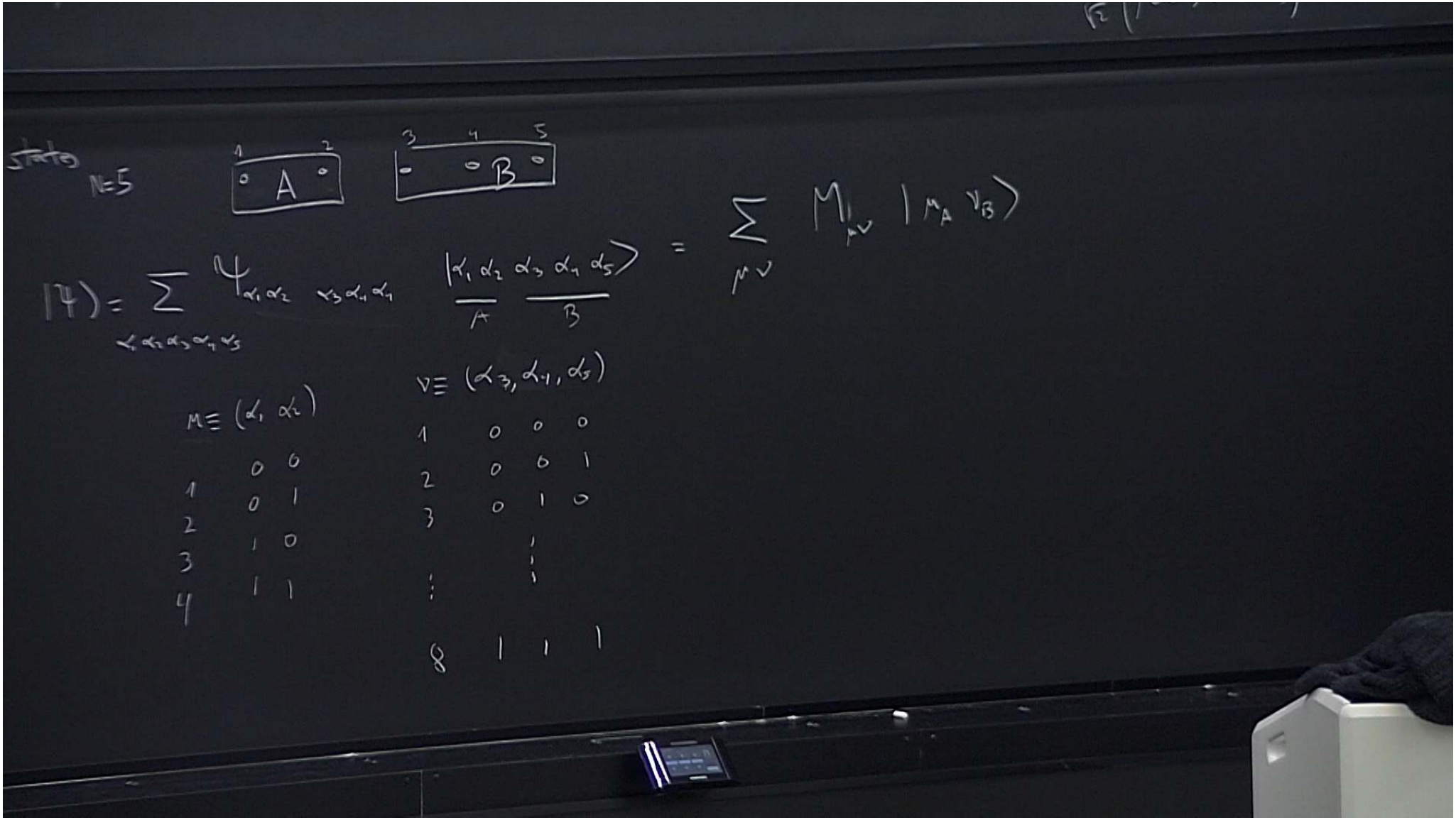
$$|\psi\rangle = \sum_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} \psi_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} |\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5\rangle$$

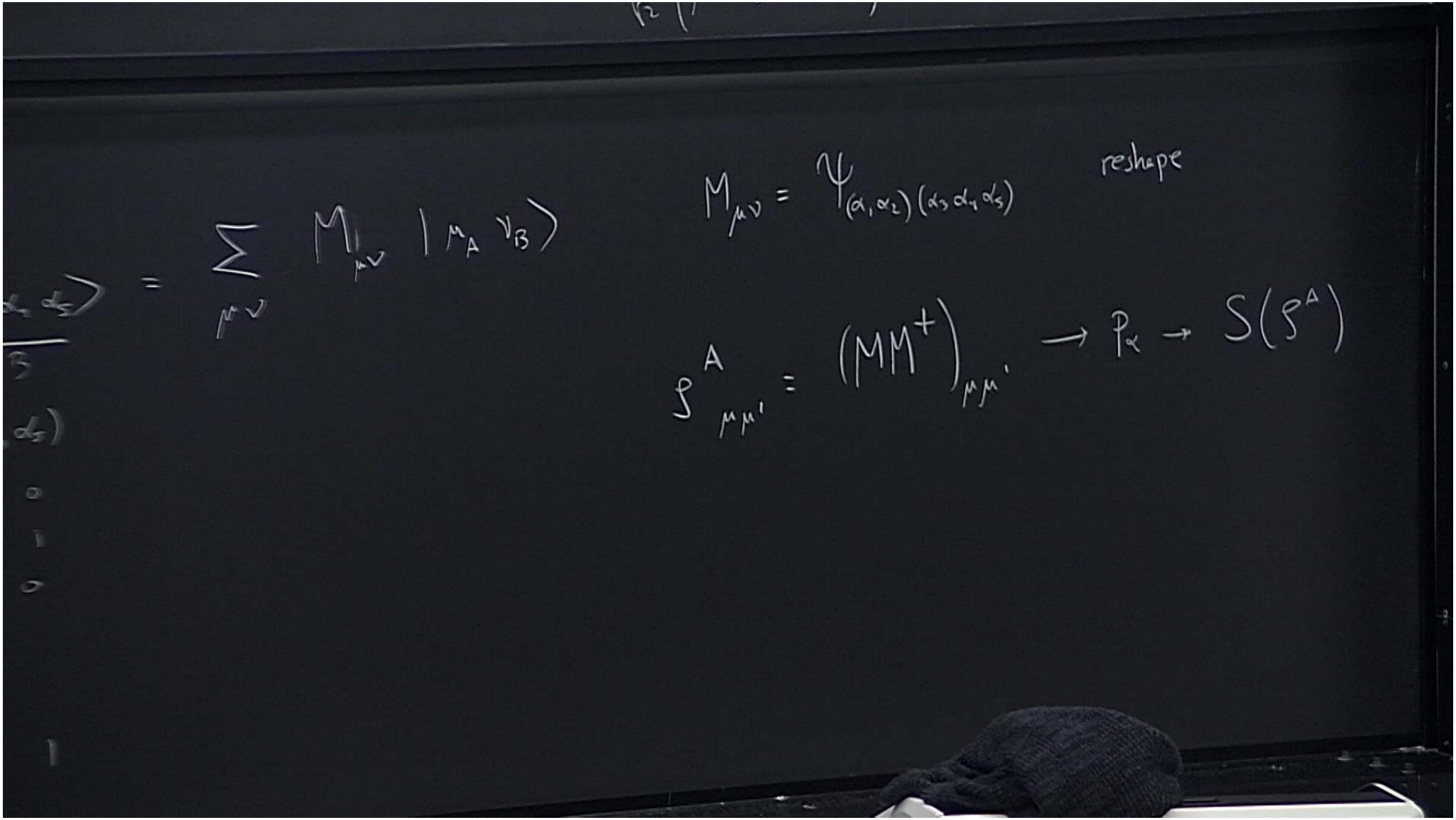
$M \equiv (\alpha_1, \alpha_2)$

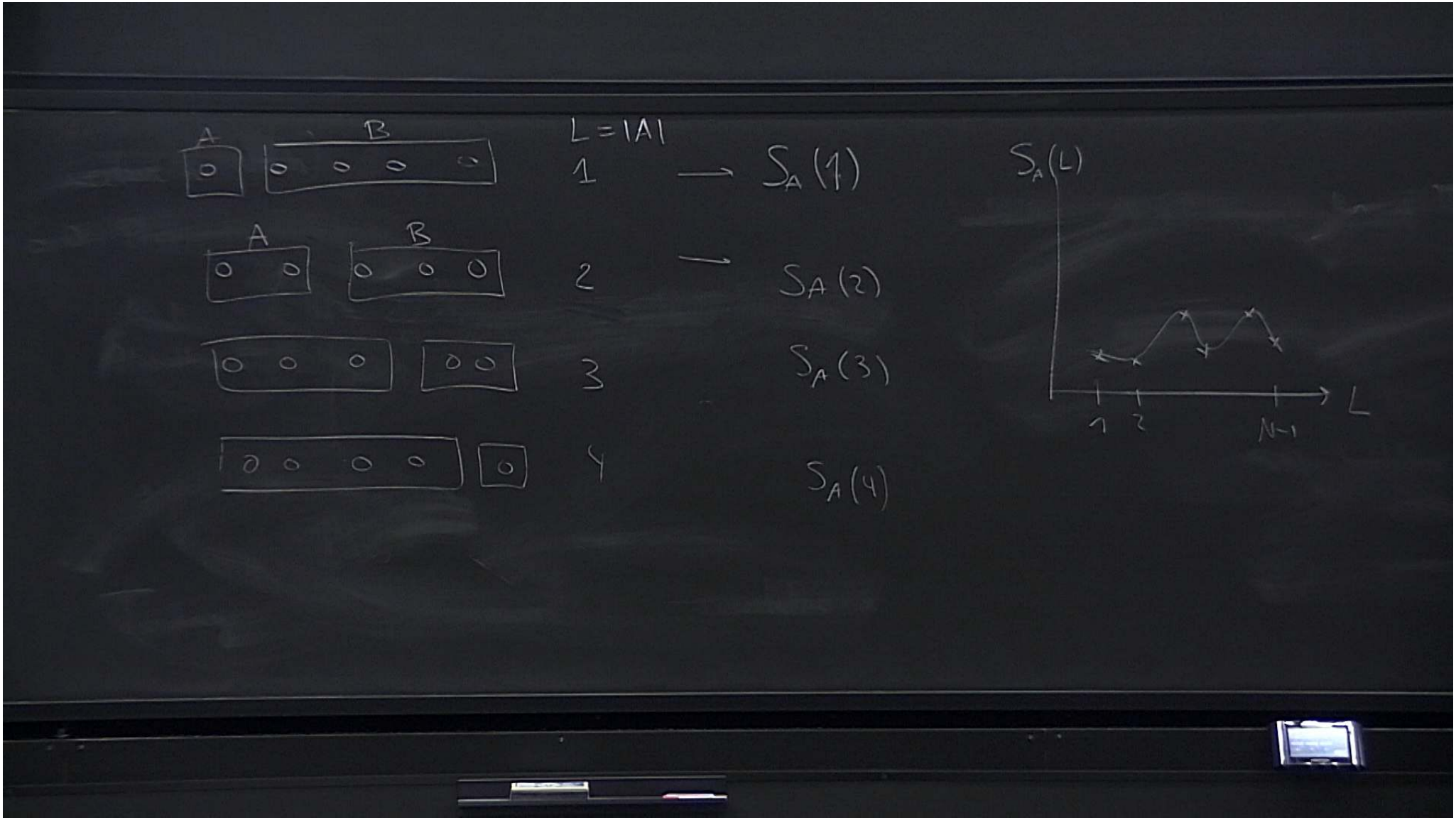
| | | |
|---|---|---|
| | 0 | 0 |
| 1 | 0 | 1 |
| 2 | 1 | 0 |
| 3 | 1 | 1 |
| 4 | | |

$\nu \equiv (\alpha_3, \alpha_4, \alpha_5)$

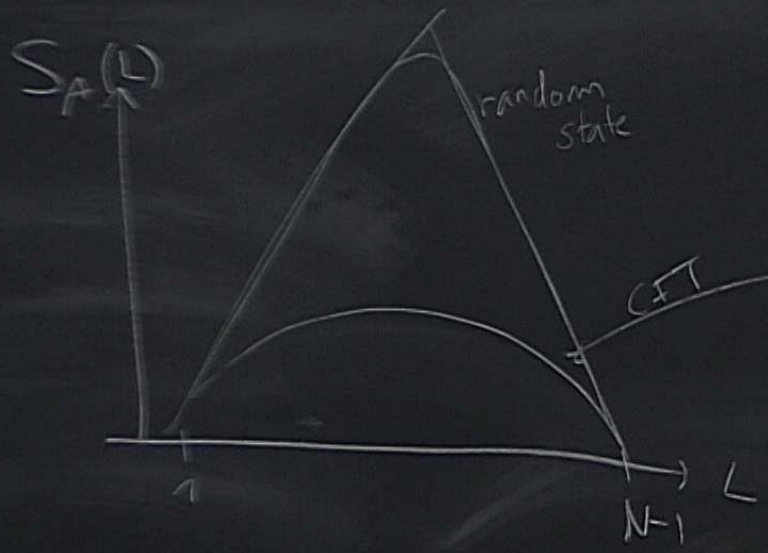
| | | | |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 |
| 3 | 0 | 1 | 0 |
| ⋮ | | ⋮ | |
| 8 | 1 | 1 | 1 |


stato
N=5
1
2
A
3
4
5
B
$|\psi\rangle = \sum_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} \psi_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} |\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5\rangle = \sum_{\mu\nu} M_{\mu\nu} |\mu_A \nu_B\rangle$
A
B
$\mu \equiv (\alpha_1, \alpha_2)$
$\nu \equiv (\alpha_3, \alpha_4, \alpha_5)$
1 0 0
2 0 1
3 1 0
4 1 1
1 0 0 0
2 0 0 1
3 0 1 0
8 1 1 1


$= \sum_{\mu\nu} M_{\mu\nu} |\mu_A \nu_B\rangle$
$M_{\mu\nu} = \psi_{(\alpha_1,\alpha_2)(\alpha_3,\alpha_4,\alpha_5)}$
reshape
$\rho^A_{\mu\mu'} = (MM^+)_{\mu\mu'} \rightarrow P_\alpha \rightarrow S(\rho^A)$


A
B
L = |A|
1
$S_A(1)$
A
B
2
$S_A(2)$
3
$S_A(3)$
4
$S_A(4)$
$S_A(L)$
L
1
2
N-1

CFT prediction (Calabrese & Cardy 2004)




$$S(L) = \frac{c}{3}\log_2\left[\frac{N}{\pi}\sin\frac{L\pi}{N}\right] + c_1$$

Numerical

Universal

central charge