Title: A reason for representation theorists to play billiards

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Abstract: <p>Representations of the fundamental groups of surfaces appear so often in geometry that it's tempting to see them primarily as geometric structures. In recent years, however, researchers have uncovered beautiful new features of these representations by thinking of them instead as dynamical systems. As an invitation to the dynamical point of view, I'll describe how geometric tools from the study of billiards can be used to build invariants of surface group representations.</p>

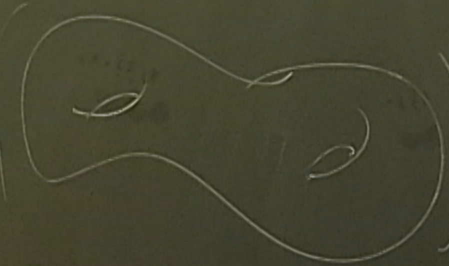
(I) Intro to billiards

(II) Billiards give a rep invariant

(III) We can pump it up!

(IV) Relation to Fock-Goncharov coordinates

A reason for representation theorists to play billiards

π_1 () $\longrightarrow$ Group, like $SL_2\mathbb{R}$

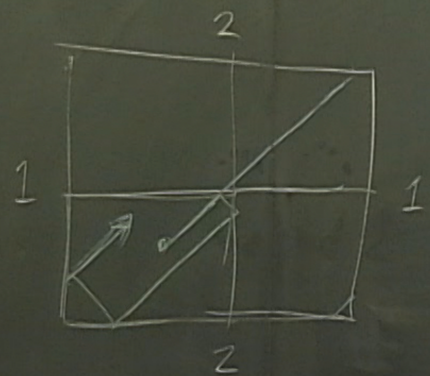
SURFACE GROUP — REPS

Geometry — Dynamics

er
rfaces
ts)

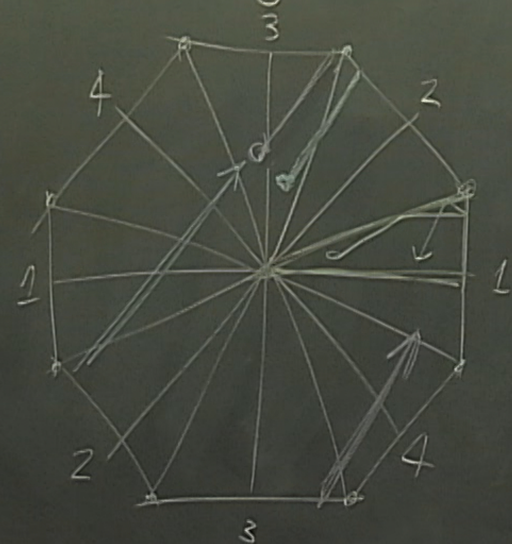
I Intro to billiards

1 Rectangular



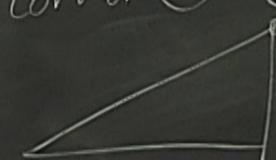
"unfolded table"
ex. of translation surface

2 Triangular



Convention: corners of octagon are "pockets" that absorb ball

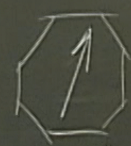
New feature: Nearby shots can diverge if they pass near this corner



II Billiards give a rep. invariant

- Pick a rep $\pi_1(\ ) \to SL_2\mathbb{R}$

$$n \longmapsto A_n$$

- Fix a shooting direction 

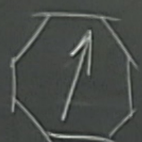
Technicality: direction must be "generic"

e.g. no shot connects two pockets

ep. invariant

$\bigcirc) \longrightarrow SL_2\mathbb{R}$

n $\longmapsto A_n$

ction 

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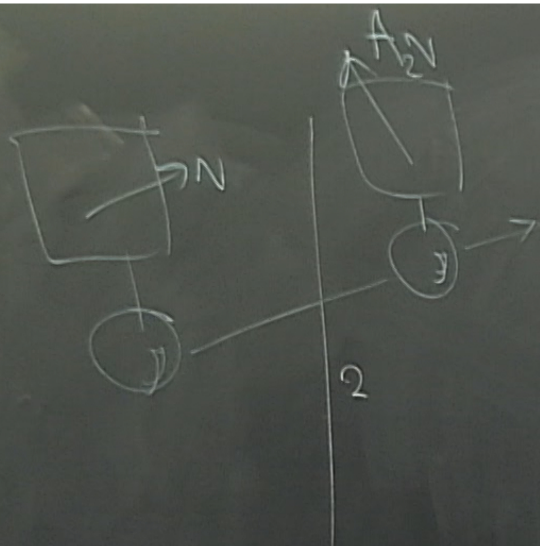
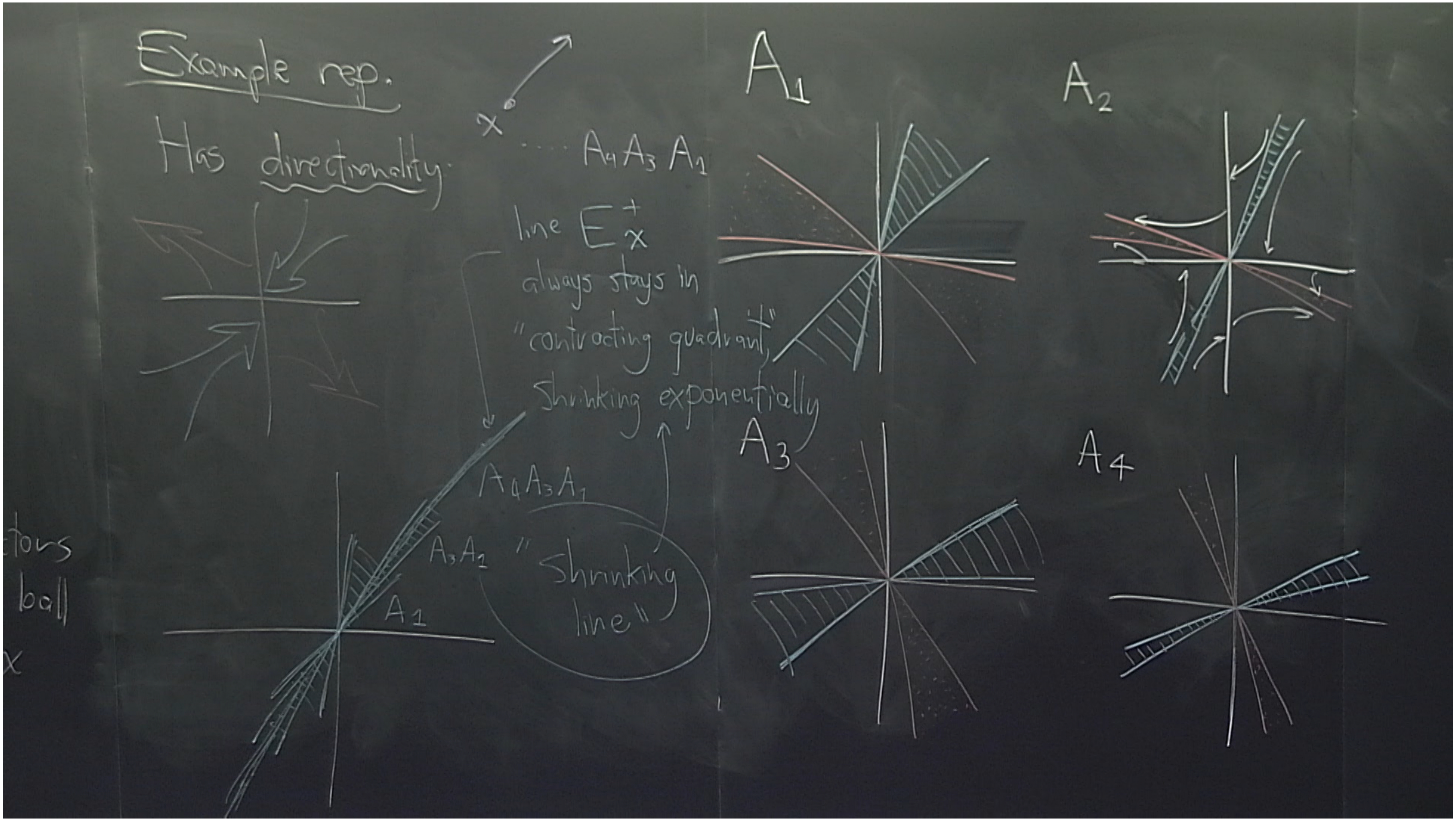


$$\Psi : \bigcirc \times \mathbb{R} \longrightarrow SL_2\mathbb{R}$$

$\Psi_x(t)$ = motion of vectors carried by a ball starting from x


Example rep.
Has directionality.
x
... A4 A3 A1
line E+x
always stays in
"contracting quadrant,"
shrinking exponentially
A4 A3 A1
A3 A1
A1
"Shrinking
line"
A1
A2
A3
A4
tors
ball
x

Example rep has a cool property:

For each $x \in$ ⬡, have line $E_x^{\pm}$ s.t.

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{\|\Psi_x(\pm t)\, v\|}{\|v\|} = -\Lambda \quad \text{for } v \in E_x^{\pm}$$

Under good dynamical conditions, equiv to a well-studied dynam. property, "uniform hyperbolicity" depend on start point x!

<u>Idea:</u> Keep track of $E_x^{\pm}$.

$\boxed{2}$

$v \in E_x^{\pm}$

a

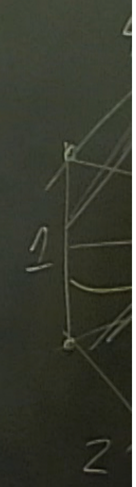
<u>Problem</u>: $E_x^{\pm}$ depend — badly — on x

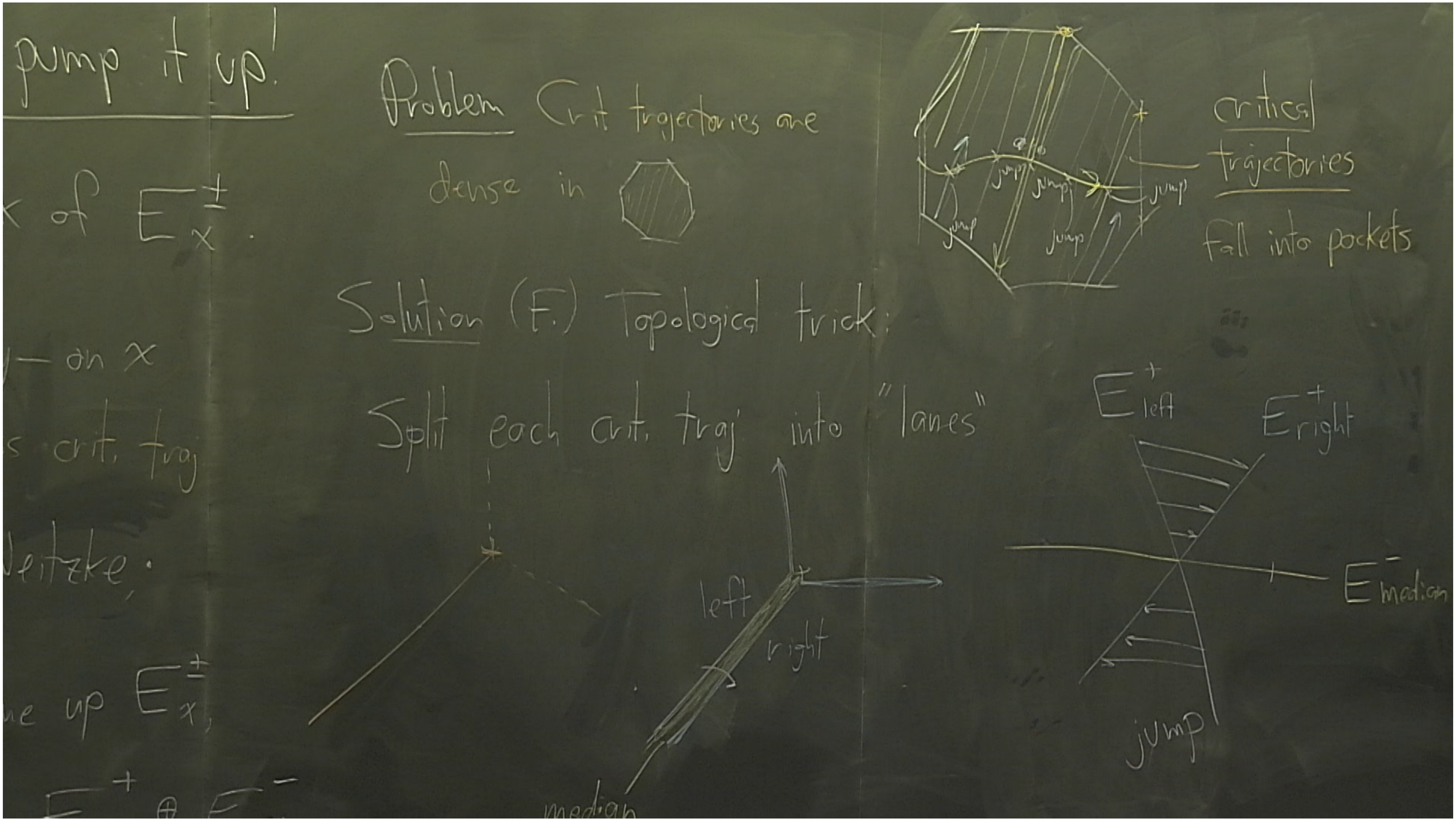
Change sharply when you cross crit. traj

<u>Solution</u> (Gaiotto, Moore, Neitzke, Bonahon, Dreyer)

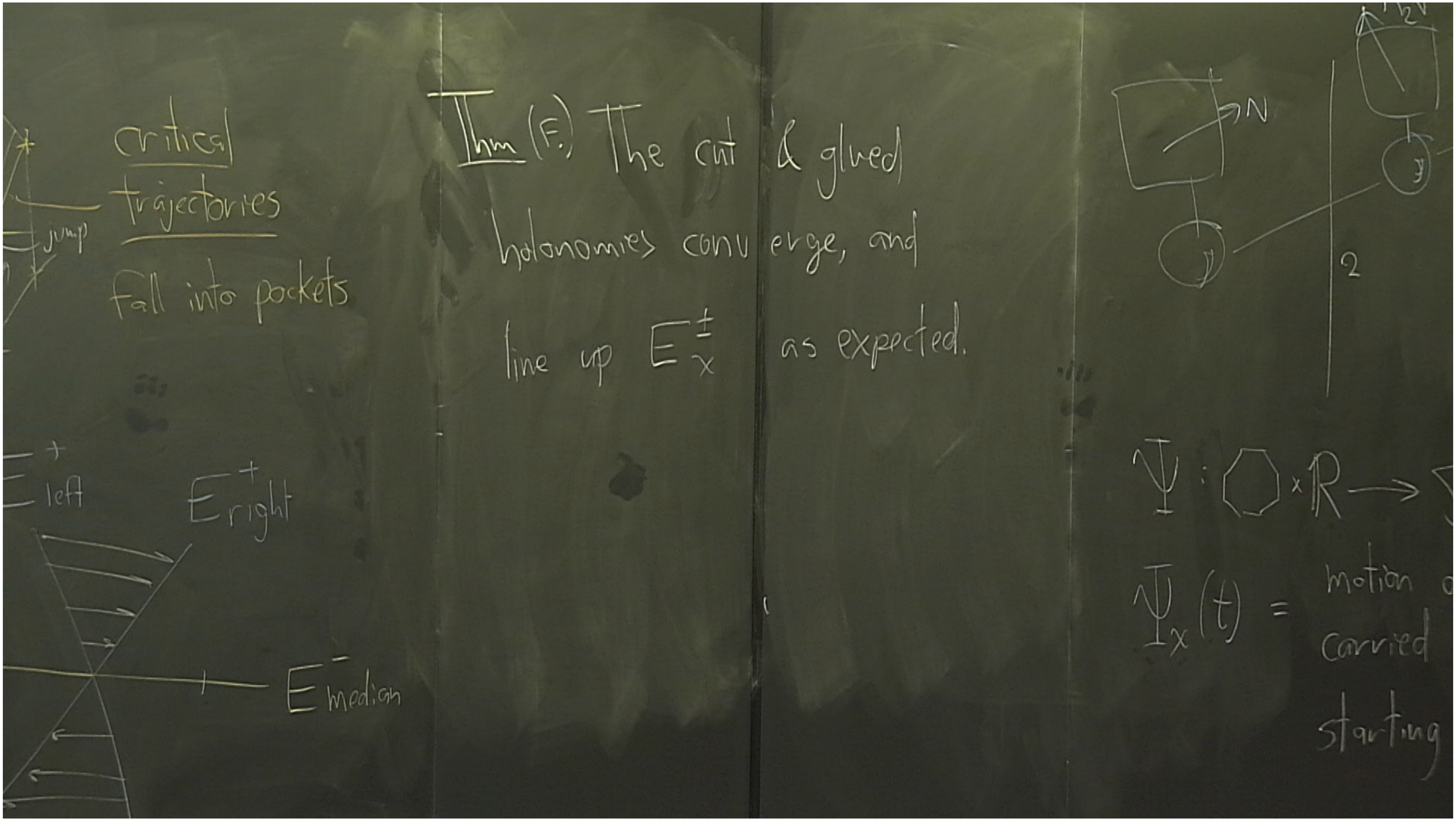
Cut & glue rep. to line up $E_x^{\pm}$,

splitting $SL_2\mathbb{R}$ rep into $\underset{\underset{GL_1}{\circlearrowright}}{E_x^{+}} \oplus \underset{\underset{GL_1}{\circlearrowright}}{E_x^{-}}$





pump it up!
x of $E_x^{\pm}$.
— on x
s crit. traj
Neitzke.
ne up $E_x^{\pm}$,
Problem Crit trajectories are
dense in
Solution (F.) Topological trick:
Split each crit. traj into "lanes"
left
right
median
jump
critical trajectories
fall into pockets
E^+_{left}
E^+_{right}
E^-_{median}
jump


critical
trajectories
fall into pockets
jump
Thm (F.) The cut & glued
holonomies converge, and
line up $E^{\pm}_{x}$ as expected.
E^{+}_{left}
E^{+}_{right}
E^{-}_{median}
N
2
$\Psi : \bigcirc \times \mathbb{R} \to$
$\Psi_x(t) =$ motion
carried
starting

$\in E_x^\pm$

Fact: Λ doesn't depend on start point x!

we can pump it up.

Idea: Keep track of $E_x^\pm$.

Problem: $E_x^\pm$ depend — badly — on x

Change sharply when you cross crit. traj

Solution (Gaiotto, Moore, Neitzke, Bonahon, Dreyer)

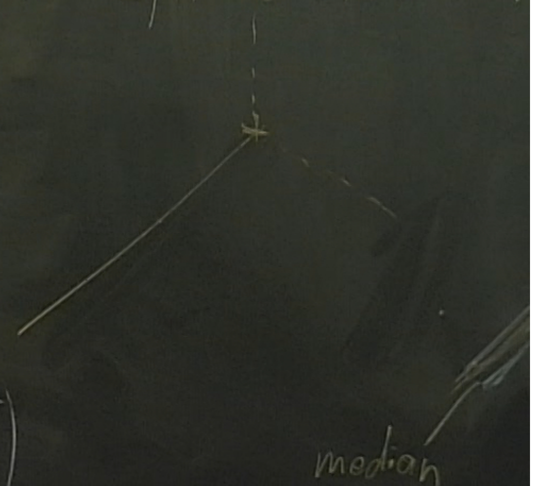
Cut & glue rep. to line up $E_x^\pm$, splitting $SL_2\mathbb{R}$ rep into

$$\boxed{\begin{matrix} E_x^+ & \oplus & E_x^- \\ \circlearrowleft & & \circlearrowleft \\ GL_1 & & GL_1 \end{matrix}}$$

Problem: Crit traj dense in

Solution (F.) Topo

Split each crit. t

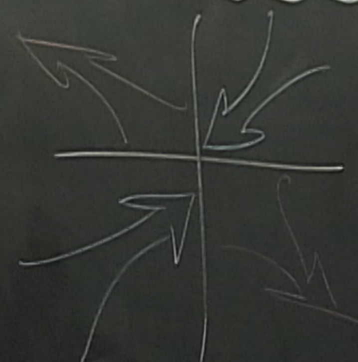


Expected: (Proven in many cases): abelianized rep. is a complete invariant if you fix the holonomy around each pocket at -1

Can easily extract Lyap. exp. by averaging holonomies of $(E^{\pm})^{ab}$ along trajectory

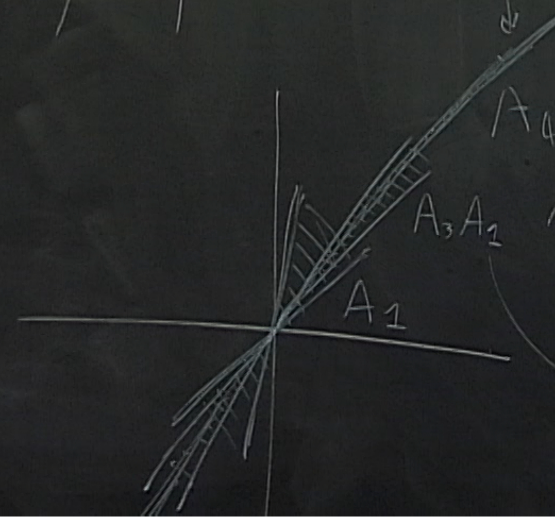
Example rep.

Has directionality:

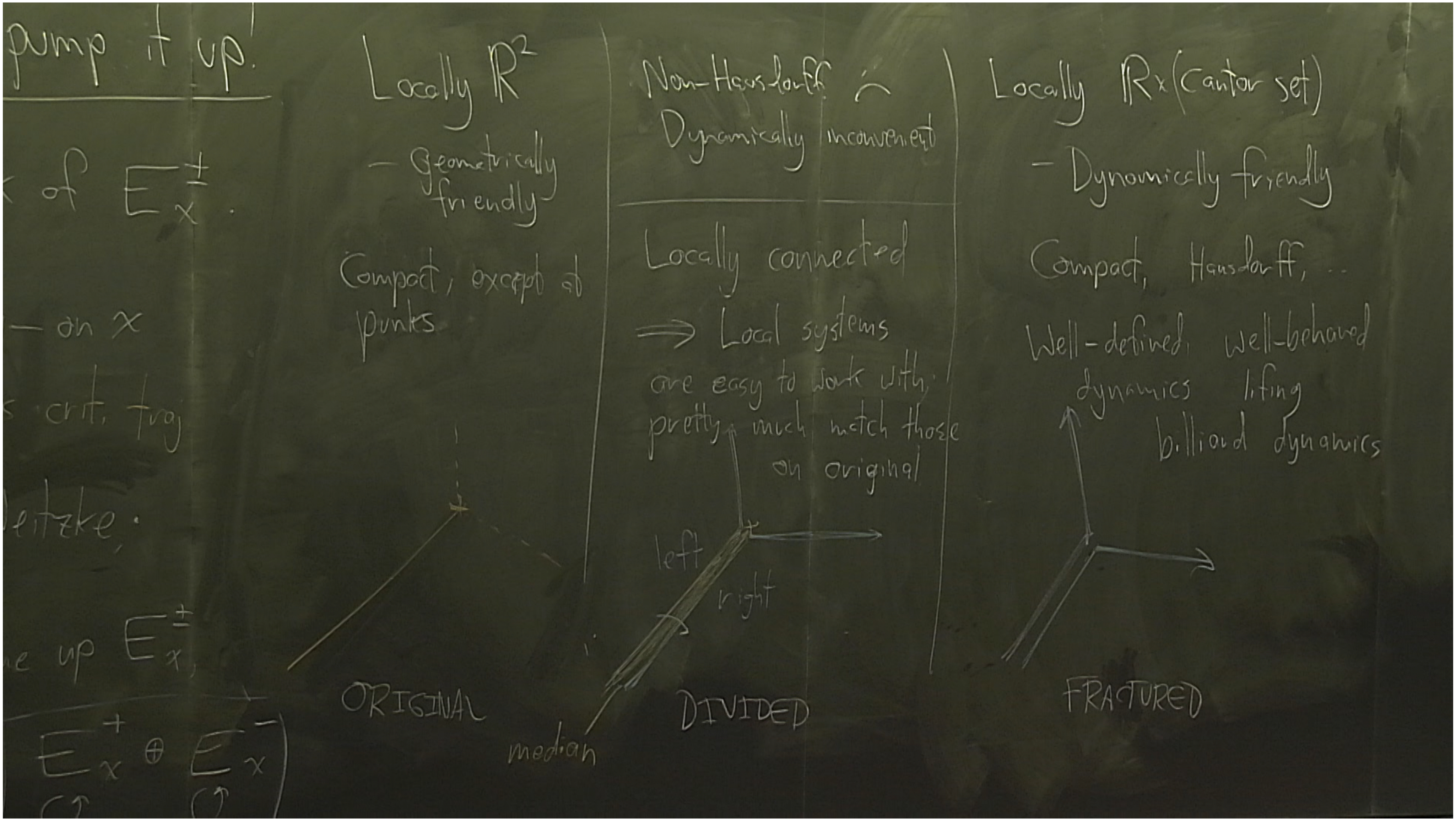


x .... $A_4 A_3 A_1$

line E^+_x always stays in "contracting quadrant", "Shrinking exponentially"



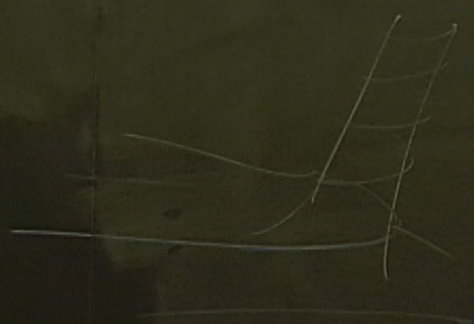
"Shrinking line"


pump it up!
of $E_x^{\pm}$.
— on x
crit. traj
eitzke.
e up $E_x^{\pm}$,
$E_x^{+} \oplus E_x^{-}$
Locally $\mathbb{R}^2$
— Geometrically friendly
Compact, except at punks
ORIGINAL
median
Non-Hausdorff :(
Dynamically inconvenient
Locally connected
⟹ Local systems are easy to work with; pretty much match those on original
left
right
DIVIDED
Locally $\mathbb{R}\times$(Cantor set)
— Dynamically friendly
Compact, Hausdorff, ..
Well-defined, well-behaved dynamics lifting billiard dynamics
FRACTURED

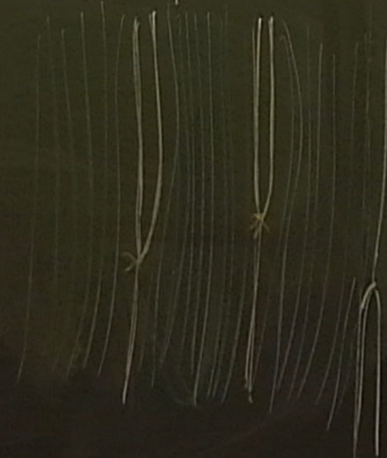
IV Relation to Fock-Goncharov coordinates

(Gaiotto, Moore, Neitzke)

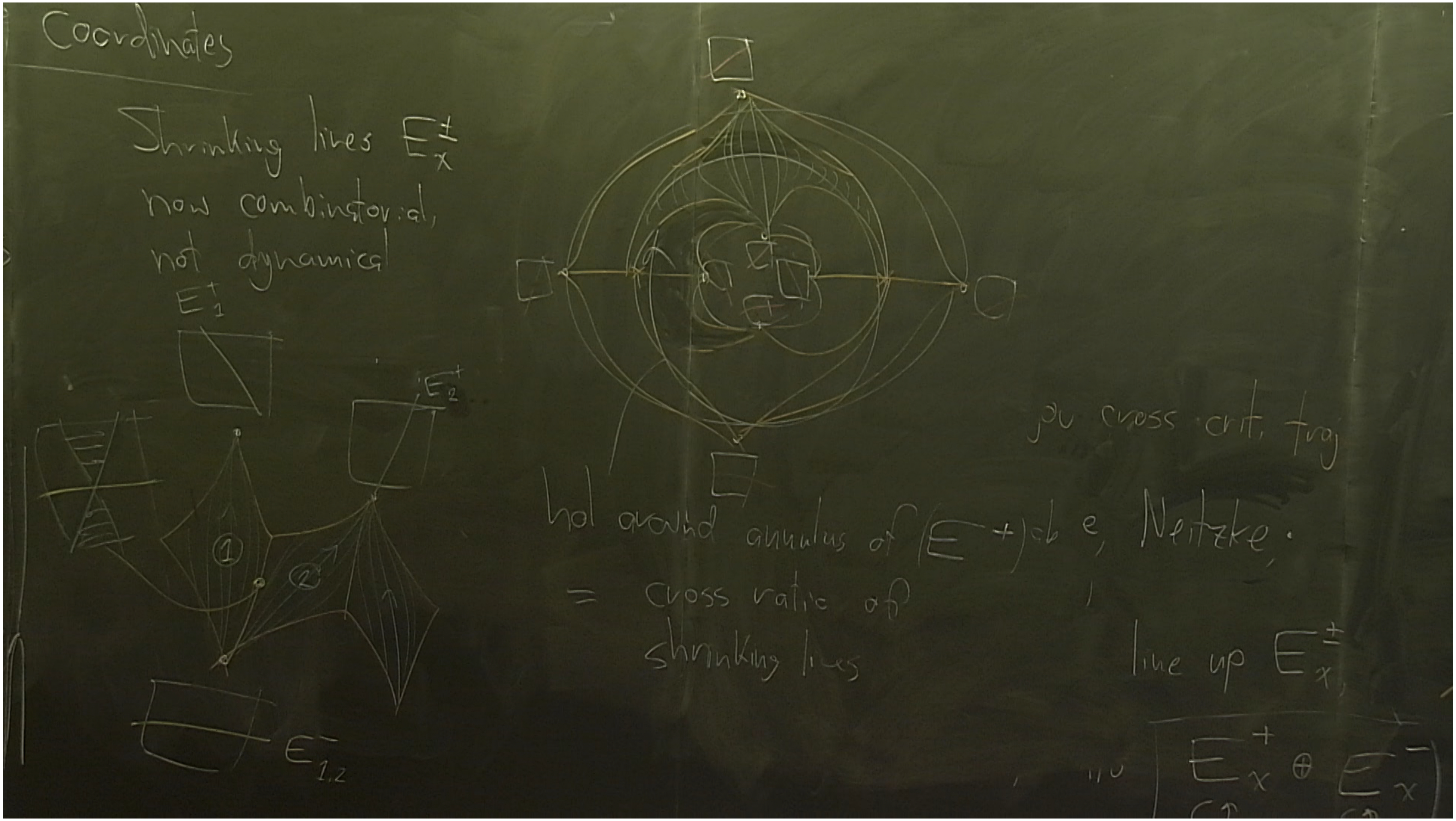
Working on translation surfaces with "cylindrical ends"



Flat picture:



Fact


Coordinates
Shrinking lines $E_x^\pm$
now combinatorial,
not dynamical
E_1^+
E_2^+
$E_{1,2}^-$
you cross crit. traj
hol around annulus of (E^+)
e, Neitzke.
= cross ratio of
shrinking lines
line up $E_x^\pm$
$E_x^+ \oplus E_x^-$