Title: FQHE and Hitchin Systems on Modular Curves

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Abstract:

FQHE and Hitchin systems on modular curves

S.C. [work in progress]

Hitchin Systems in Mathematics and Physics
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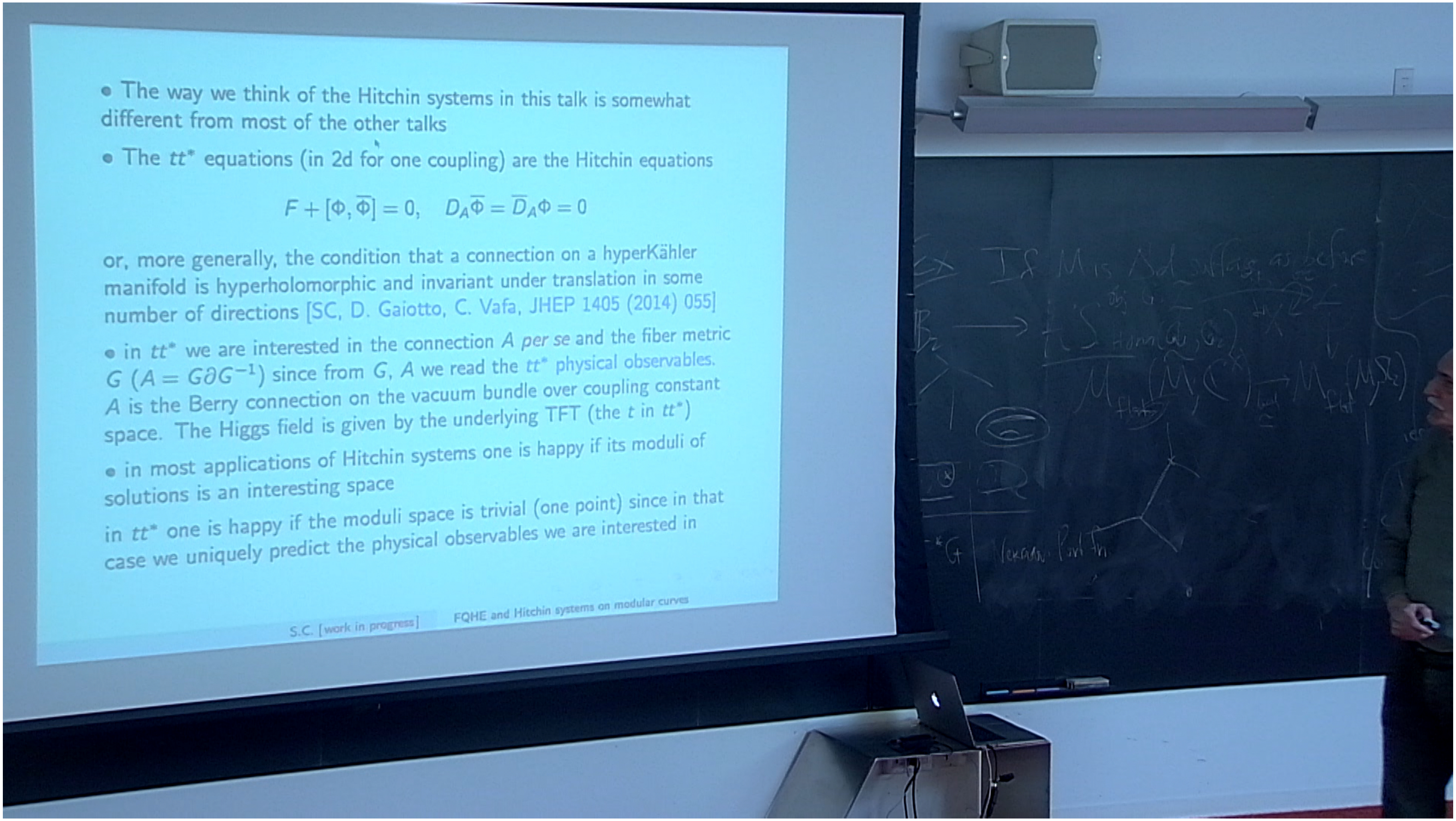
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• The tt^* equations (in 2d for one coupling) are the Hitchin equations
$F + [\Phi, \overline{\Phi}] = 0, \quad D_A \overline{\Phi} = \overline{D}_A \Phi = 0$
or, more generally, the condition that a connection on a hyperKähler manifold is hyperholomorphic and invariant under translation in some number of directions [SC, D. Gaiotto, C. Vafa, JHEP 1405 (2014) 055]
• in tt^* we are interested in the connection A *per se* and the fiber metric G ($A = G\partial G^{-1}$) since from G, A we read the tt^* physical observables. A is the Berry connection on the vacuum bundle over coupling constant space. The Higgs field is given by the underlying TFT (the t in tt^*)
• in most applications of Hitchin systems one is happy if its moduli of solutions is an interesting space
in tt^* one is happy if the moduli space is trivial (one point) since in that case we uniquely predict the physical observables we are interested in
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Cumrun Vafa [arXiv:1511.03372] has suggested that the

Fractional Quantum Hall Effect (FQHE),

as actually observed in the laboratory, may be modeled by the tt^* geometry of some complicated $\mathcal{N} = 4$ SQM systems

Object of main interest: the **Berry holonomy** of the vacuum bundle

$$\mathscr{V} \to \mathcal{U} \equiv \text{(space of universal parameters)}$$

- $\mathcal{U}$ a complex manifold
- $\mathscr{V}$ a holomorphic Hermitian bundle whose fiber $\mathscr{F}$ is the vector space of vacua (zero energy states) of the SQM model specified by parameters
- the associated TFT defines a holomorphic $\Phi \in \Omega^1(\mathrm{End}\,\mathscr{V})$ given by the action of the chiral fields on the vacua: $\mathscr{R} \subset \mathrm{End}\,\mathscr{V}$ and $\Phi \in \Omega^1(\mathscr{R})$
- the Berry connection A satisfies

$$[D_A, \overline{D}_A] + [\Phi, \overline{\Phi}] = D_A\overline{\Phi} = \overline{D}_A\Phi = D_A\Phi = \overline{D}_A\overline{\Phi} = 0$$

- These are Hitchin systems with actual technological implications

One of his motivations:

in FQHE phenomenology central role amplitudes of the form

$$\int_\gamma e^{-\sum_{i=1}^N V(x_i)} \prod_{1\le i<j\le N} (x_i - x_j)^{1/\nu}\, dx_1 \cdots dx_N$$

$V(z)$ one-particle potential

$$V(z) = \text{``}\sum_{y\in\Lambda} \log(z-y) + \sum_{s\in S} e(s)\log(z-s)\text{''}$$

$\Lambda \subset \mathbb{C}$ a lattice, $S \subset \mathbb{C}$ a discrete set where defects (quasi-holes) of various charges $e(s) \in \mathbb{Z}$ are placed. $0 < \nu \le 1$ is the *filling fraction*.

Proper definition: finite volume and then thermodynamical limit

Amplitudes of the form

$$\int_\gamma e^{-\sum_{i=1}^N V(x_i)} \prod_{1\le i<j\le N} (x_i - x_j)^{1/\nu}\, dx_1 \cdots dx_n \qquad (*)$$

arise in (2,2) systems as BPS brane amplitudes in a double scaling limit $\Lambda \to 0$, $\zeta \to 0$ with $\Lambda/\zeta =$ fixed (Λ mass scale, $\zeta \in \mathbb{P}^1$ twistor parameter)

Idea: take seriously the SQM and its tt^* which give $(*)$ in the limit

Many possibilities for the $\mathcal{N} = 4$ SQM which yields $(*)$: Basically, possible $\mathcal{N} = 4$ models classified by their Witten index as function of N [$w(1) = \#$(one-particle low-lying states)]

$$w_B(N) = \binom{N + w(1) - 1}{N} \qquad \text{“Bose statistics”}$$

$$w_F(N) = \binom{w(1)}{N} \qquad \text{“Fermi statistics”}$$

We focus on the “fermionic” version: much simpler! (but still quite hard)

Defined if $\nu > 0$, natural when $0 < \nu \le 1$ (physical range)

In the "fermionic" model we are effectively reduced to study the one-particle tt^* geometry, i.e. the LG model with superpotential

$$V(z) = \text{``}\sum_{y\in\Lambda} \log(z-y) + \sum_{s\in S} e(s)\log(z-s)\text{''}$$

Solving tt^* ($\equiv$ **Hitchin eqns.**) is simpler when the SQM model has:
Abelian symmetry group $\mathcal{A}$ acting freely and transitively on the vacua
Fiber $\mathscr{F}$ of vacuum bundle $\mathscr{V}$ regular representation of $\mathcal{A}$

$$\mathscr{F} = L^2(\mathcal{A}) \simeq L^2(\mathrm{Hom}(\mathcal{A}, U(1))$$

- $\mathscr{A}$ centralizes tt^* metric and Berry holonomy

$\Rightarrow$ both are diagonal in the character basis

- infinite number of vacua (in the thermodynamic limit)

$\Rightarrow$ $\mathcal{A}$ should also get infinite: a group of translations:

$\Lambda \cup S = L \subset \mathbb{C}$ a lattice, $\quad e\colon L/\Lambda \to U(1)$ an additive character

$$e(s+s') = e(s)\,e(s'), \qquad e(s+\Lambda) = e(s)$$

- position of quasi-holes well-defined on the elliptic curve $E(\Lambda) \equiv \mathbb{C}/\Lambda$

$$V(z,\tau) = \sum_{y \in L/\Lambda} e(y) \, \log \theta_1\big((z-y)/2, \tau\big), \qquad \Lambda = 2\pi\mathbb{Z} \oplus 2\pi\tau\mathbb{Z}, \;\; \tau \in \mathbb{H}$$

$$V'(z,\tau) = \sum_{y \in L/\Lambda} e(y) \left(\zeta(z-y,\tau) - \frac{\eta_1}{\pi}(z-y) \right),$$

$\zeta(z,\tau)$ Weierstrass ζ-function, $\zeta(z+2\pi,\tau) = \zeta(z) + 2\eta_1$.

To preserve invariance under translation by the lattice Λ, $V'(z,\tau)$ should be an elliptic function $\Longleftrightarrow$ $e(s)$ not the trivial character

Unfortunately $e(s)$ **trivial** is the most interesting case for FQHE
$e(s)$ roots of 1: OK for SQM. **Quadratic characters**: real charges

Superpotential $V(z,\tau)$ still multi-valued on $E(\Lambda)$ for 2 reasons:
i) θ_1 just quasi-periodic for Λ,
ii) branches of log

$\Rightarrow$ SQM model has the symmetry L but in a very subtle way
the group L/Λ acts as discrete R-symmetry

Classification: We may assume e to be faithful (otherwise $\Lambda \to \ker e$).

$$L/\Lambda \simeq \mathbb{Z}/M_1\mathbb{Z} \oplus \mathbb{Z}/M_2\mathbb{Z} \quad \text{with} \quad M_1 \mid M_2.$$

L/Λ has a faithful character iff $\gcd(M_1, M_2) = 1$ so

$$L/\Lambda \simeq \mathbb{Z}/M\mathbb{Z}, \qquad M \geq 2$$

Models with the required symmetry are parametrized by

1) *an elliptic curve* $E \equiv E(\Lambda)$
2) *a torsion subgroup* $T \equiv L/\Lambda \subset E, \quad T \simeq \mathbb{Z}/M\mathbb{Z}, \quad (M \geq 2)$
3) *a faithful character* $e \colon T \to U(1)$ *up to equivalence* $e \sim e^{-1}$

Equivalently by the pairs (E, p) *with* $p \in E$ *the unique point of order strictly* M *such that* $e(p) = e^{2\pi i/M}$ *(a fixed primitive* M*-root)*

Pairs (E, p): elliptic curve with a level M structure of type $\Gamma_1(M)$

Space of models of given level $M \equiv$ **moduli space of elliptic curves with structure $\Gamma_1(M)$ (mod isomorphism)**

$$\Gamma_1(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod M \right\} \subset SL(2,\mathbb{Z})$$

moduli space of elliptic curve with $\Gamma_1(M)$ *structure* $= Y_1(M) \equiv \mathbb{H}/\Gamma_1(M)$

better compactify the space: $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

$$\text{compactified moduli space } X_1(M) \equiv \overline{\mathbb{H}}/\Gamma_1(M)$$

added points: **cusps** $\mathbb{P}^1(\mathbb{Q})/\Gamma_1(M)$

Modular curve $X_1(M)$ is the space of models (coupling constant space)

$X_1(M)$ a Riemann surface of genus

$$g(X_1(M)) = 1 + \frac{M^2}{24} \prod_{p|M} \left(1 - p^{-2}\right) - \frac{1}{4} \sum_{d|M} \phi(d)\,\phi(M/d),$$

(ϕ Euler totient function $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$)

To compactify the coupling constant space $Y_1(M)$ we added the **cusps**

$$\#\text{cusps}(X_1(M)) = \frac{1}{2}\sum_{d|M} \phi(d)\,\phi(M/d).$$

Cusps: points at infinite distance in the natural hyperbolic metric from any regular theory. They are singular limits. Various kinds:

U type cusps: a BPS particle gets zero mass closing the mass-gap

I type cusps: a BPS particle gets infinite mass and decouples

I/U type cusps: both mechanisms. **Not possible for M prime**

Other "bad" points in $Y_1(M) = X_1(M) \setminus \{\text{cusps}\}$ where the mass gap closes or states decouple?

- Not expected since they are at finite distance from regular models
- for $M = 2$ one checks that all non-cusp points are regular
- likely to be true for general M

Additional structures

The modular curve $X_1(M)$ has an important group of automorphisms

$$(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\} \equiv \mathrm{Gal}(\mathbb{Q}[\cos(2\pi/M)]/\mathbb{Q})$$

given by the *diamond automorphisms*

$$\langle m\rangle : X_1(M) \to X_1(M), \quad m \in (\mathbb{Z}/M\mathbb{Z})^\times, \quad \langle m\rangle(E,p) = (E, mp)$$

Since the curve $X_1(M)$ is the space of theories, $\langle m\rangle$ send one theory to another: *it is a duality.* Its effect is to change the character (charge assignment of 'quasi-holes') $e \mapsto e^m$. All models with a given E obtained from any one by acting with $\mathrm{Gal}(\mathbb{Q}[\cos(2\pi/M)]/\mathbb{Q})$:
weakly-coupled model with charges e is a strongly-coupled limit of the model with charges e^m for all choices of $m \neq 1$

Solutions to tt should be* $\mathrm{Gal}(\mathbb{Q}[\cos(2\pi/M)]/\mathbb{Q})$*-covariant*

Similar story with *Hecke correspondences* T_m $(m \in \mathbb{N})$ (subtler dualities)

$$X_1(M) \longleftarrow X_1^1(M,m) \longrightarrow X_1(M)$$

tt^* equations same as Hitchin equations in coupling space

The tt^* equations for the level M models: *a family of Hitchin systems* parametrized by the characters of Λ,

$$\mathrm{Hom}(\Lambda, U(1)) \simeq S^1 \times S^1,$$

over the modular curve $X_1(M)$ with prescribed singularities at the cusps and covariant under the diamond automorphisms $\langle m \rangle$ (Hecke ?)

- $\mathrm{Hom}(\Lambda, U(1)) \simeq S^1 \times S^1$ depends on a choice of generators for Λ (or L)

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi \\ \theta \end{pmatrix} \quad \Rightarrow$$

action of $\Gamma_1(M)$
on the family of Hitchin systems:
an invariance

Using the symmetry L,

$$\mathrm{End}(\mathscr{F}) \simeq \mathrm{End}(L^2(S^1 \times S^1)) \otimes \mathrm{End}(L^2(\mathrm{Hom}(L/\Lambda, U(1))$$

In the TFT trivialization, both A and Φ act as multiplicative operators on $\mathrm{End}(L^2(S^1 \times S^1))$ and Φ is proportional to $\mathrm{Id} \in \mathrm{End}(L^2(S^1 \times S^1))$

A is a $S^1 \times S^1$ family of connections in *the Cartan of* $\mathfrak{sl}(M)$

$$A(\phi, \theta) = \mathrm{diag}(A(\phi,\theta)_1, A(\phi,\theta)_2, \cdots, A(\phi,\theta)_M),$$
$$\Phi \in \Omega^1(\mathfrak{sl}(M))$$

The spectral curve in $K_{X_1(M)}$ has the form

$$\det[\lambda - \Phi] = \lambda^M - \rho$$

for a meromorphic M-differential $\rho \in \Gamma\left(X_1(M), K^M_{X_1(M)}\right)$ which has an arithmetic construction:

Topological side is "arithmetic"

Arithmetic construction of spectral cover

Notation: $e(k) = e^{2\pi i \ell k/M}$ with $\ell \in (\mathbb{Z}/M\mathbb{Z})^{\times}$, $\ell \sim (M-\ell)$

By a *spectral cover* $\widetilde{X} \xrightarrow{\pi} X_1(M)$ I mean a cover over which the eigenvalue of Φ_ℓ is a globally defined holomorphic one-form μ_ℓ

$$\pi^* \det[\lambda - \Phi_\ell] = \prod_{k=0}^{M-1} \left(\pi^*\lambda - e^{2\pi i k/M}\mu_\ell\right)$$

In physical terms the spectral cover is a curve parametrizing pairs

(SQM model, vacuum mod Λ) $\equiv$ (point in $X_1(M)$, eigenvalue of Φ)

$V'(z,\tau)$ is an elliptic function with simple poles at $z = 2\pi k/M$, $k \in \mathbb{Z}/M\mathbb{Z}$, such that $V'(z + 2\pi/M, \tau) = e(1)\, V'(z,\tau)$. Thus vacua $z_0 + 2\pi k/M$, $k \in \mathbb{Z}/M\mathbb{Z}$, and $Mz_0 = 0$ (Abel's thm)

classical vacua have a simple characterization in terms of the Weil pairing

Arithmetic construction of spectral cover

$E[M] \subset E$ the group of M-torsion points,

$$\langle -, - \rangle_{\mathrm{Weil}} : E[M] \times E[M] \to \mathbb{Z}/M\mathbb{Z} \text{ is the Weil pairing}$$

SQM model parametrized by (E, p) $p \in T \subset E[M]$, $e(p) = e^{2\pi i/M}$,

$$\boxed{q \in E \text{ is a classical vacuum} \;\Leftrightarrow\; q \in E[M] \text{ and } \langle p, q \rangle_{\mathrm{Weil}} = 1}$$

$$\boxed{(model, vacuum) \equiv (E, p, q : p, q \in E[M],\ \langle p, q \rangle_{\mathrm{Weil}} = 1)}$$

Spectral cover: moduli space of such triples (E, p, q) which are called *Elliptic curves with a level M structure of type* $\Gamma(M)$

moduli space of such triples is yet another modular curve

Arithmetic construction of spectral cover

Principal congruence subgroup $\Gamma(M) \subset SL(2,\mathbb{Z})$

$$\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod M \right\} \subset SL(2,\mathbb{Z})$$

$$1 \to \Gamma(M) \to \Gamma_1(M) \xrightarrow{b} \mathbb{Z}/M\mathbb{Z} \to 0,$$
$$[\Gamma_1(M) : \Gamma(M)] = M,$$

$$\text{spectral cover} \equiv Y(M) \equiv \mathbb{H}/\Gamma(M) \xrightarrow{\text{comp.}} X(M) \equiv \overline{\mathbb{H}}/\Gamma(M)$$

the (compactified) spectral cover is the *principal modular curve of level* M, $X(M)$. The M-fold spectral cover

$$X(M) \xrightarrow{\pi} X_1(M) \quad \text{is the canonical projection}$$
$$\overline{\mathbb{H}}/\Gamma(M) \xrightarrow{\pi} \overline{\mathbb{H}}/\Gamma_1(M), \quad M\text{-fold cover}$$

On $X(M)$ the eigenvalue μ_ℓ of Φ_ℓ is well defined $\mu_\ell \in \Omega^1_{X(M)}(\log D_{\mathrm{I}})$,
$D_{\mathrm{I}} \equiv$ divisor of **type I cusps**

Explicit form of eigenvalue one-form μ_ℓ on $X(M) \equiv \overline{\mathbb{H}}/\Gamma(M)$

$$\mu_\ell = \sum_{k=0}^{M-1} e^{2\pi i \ell k/M} \frac{d\mathcal{Z}_{\ell,k}}{\mathcal{Z}_{\ell,k}}$$

where $\mathcal{Z}_{\ell,k}(\tau)$ is the partition function of a complex free chiral fermion on a torus of periods $(2\pi, 2\pi\tau)$ subjected to the boundary conditions

$$\psi(z+2\pi) = e^{2\pi i \ell/M}\psi(z), \qquad \psi(z+2\pi\tau) = -e^{2\pi i k/M}\psi(z),$$

$$\mathcal{Z}_{\ell,k} = q^{B_2(\ell/M)/2} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i k/M} q^{m-\ell/M}\right)\left(1 - e^{-2\pi i k/M} q^{m-(M-\ell)/M}\right)$$

$$\frac{d\mathcal{Z}_{\ell,k}}{\mathcal{Z}_{\ell,k}} \qquad \text{meromorphic one-form on } X(M)$$

Modular properties

$$\overline{\mathbb{H}} \xrightarrow{\varpi} X(M) \equiv \overline{\mathbb{H}}/\Gamma(M)$$

$$\varpi^* \mu_\ell = F_\ell(\tau)\, d\tau$$

$F_\ell(\tau)$ is a meromorphic (poles at cusps) modular function with character for the congruence subgroup $\Gamma_1(M)$ and good properties under $\Gamma_0(M)$

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod M \right\} \subset SL(2,\mathbb{Z})$$

$$1 \to \Gamma_1(M) \to \Gamma_0(M) \xrightarrow{a} (\mathbb{Z}/M\mathbb{Z})^\times \to 1$$

$$F_\ell\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2\, e^{2\pi i a b \ell (M-\ell)/M}\, F_{a\ell}(\tau),$$

Behavior at cusps

At cusps on $X(M)$ $(A(\phi,\theta),\Phi)$ have regular singularities

$$A(\phi,\theta) = \frac{1}{2}\boldsymbol{q}(\phi,\theta)\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) + \text{regular},$$

$$\Phi = C\,\frac{dz}{z} + \text{regular}$$

$$\text{U cusp} \begin{cases} \boldsymbol{q}(\phi,\theta) \text{ non-trivial} \\ C \text{ nilpotent} \end{cases}$$

$$\text{I cusp} \begin{cases} \boldsymbol{q}(\phi,\theta) = 0 \\ C \text{ semi-simple} \end{cases}$$

- Eigenvalues of $\boldsymbol{q}(\phi,\theta)$: the states which become massless at a U cusp are described by a SCFT, the $\boldsymbol{q}(\theta,\varphi)$ are the $U(1)_R$ charges of the susy vacua in this SCFT
- C: action on chiral ring $\mathscr{R}_{\text{SCFT}}$ of operator $\mathcal{O}$ perturbing away from cusp point

U vs. I cusps: Example

$X(5)$ has 12 cusps which map into **4** inequivalent cusps for the physical coupling curve $X_1(5)$. Which ones are **U** respectively **I** type?

Schur (1917) considered the infinite product:

$$K(q) = q^{-1/5}\prod_{n\geq 0}\frac{(1-q^{5n+1})(1-q^{5n+3})}{(1-q^{5n+2})(1-q^{5n+3})} = \frac{G(q)}{q^{1/5}\,H(q)} = \left(\frac{q^{1/5}}{1+\frac{q}{1+\frac{q^2}{1+\cdots}}}\right)^{-1}$$

$G(q)$, $H(q)$ Rogers-Ramanujan funct. $\equiv$ characters of (2,5) minimal CFT

and asked for which roots of unity it converges. Answer:

it converges at $q = e^{2\pi i a/b}$ ($\frac{a}{b}\in\mathbb{Q}$) $\Leftrightarrow$ $\frac{a}{b}$ a **I** cusp for the $M = 5$ model

$K(q)$ *Hauptmodul* of $X(5)$. Eigenvalue μ_ℓ rational differential in $K(q)$. Icosahedral group $SL(2,\mathbb{Z})/\Gamma(5)$ acts on $K(q)$ by Möbius maps. Its action determines μ_ℓ

$$\text{I cusps (width 5)} = \left\{0, \tfrac{1}{2}\right\}, \quad \text{U cusps (width 1)} = \left\{\tfrac{2}{5}, \infty\right\}$$

M odd prime: $\frac{1}{2}\phi(M)$ U cusps (width 1), $\frac{1}{2}\phi(M)$ I cusps (width M)

width of cusp related to emergent $U(1)_R$ symmetry in the limit theory

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Behavior at cusps

Residue of μ_ℓ at cusp $(a:c) \in \mathbb{P}^1(\mathbb{Q})$, $\gcd(a,c)=1$

$$\kappa_{M,\ell}(a:c) \stackrel{\text{def}}{=} \frac{1}{2} M \sum_{k=0}^{M-1} e^{2\pi i \ell k/M} \widetilde{B}_2\left(\frac{a\ell + ck}{M}\right) \in \frac{1}{2M}\mathbb{Z}[e^{2\pi i/M}],$$

$$\widetilde{B}_2(x) = \{x\}^2 - \{x\} + \tfrac{1}{6}, \qquad \text{with} \quad \{x\} \equiv x - [x].$$

The cusp $(a:c)$ is U type iff $\kappa_{M,\ell}(a:c) = 0$

$$\kappa_{M,\ell}(a+sM : c+tM) = \kappa_{M,\ell}(a:c) \qquad \forall\, s,t \in \mathbb{Z},$$
$$\kappa_{M,\ell}(a:c) = 0 \qquad \text{if and only if} \quad \gcd(M,c) > 1,$$
$$\gcd(M,c) = 1 \;\Rightarrow\; \kappa_{M,\ell}(a+a':c) = \varrho(a'\bar{c})\, \kappa_{M,\ell}(a:c),$$
$$\gcd(M,b) = 1 \;\Rightarrow\; \kappa_{M,\ell}(a:c) = \kappa_{M,b\ell}(a\bar{b} : bc)$$

($\bar{a} \equiv$ inverse in $\mathbb{Z}/M\mathbb{Z}$, $\bar{a}a = 1 \mod M$, $\varrho(k) = e^{2\pi i k \ell (M-\ell)/M}$)

$\tau = i\infty$ always a U cusp, $\tau = 0$ always a I cusp

μ_ℓ: **expressions are much simpler when** $M \leq 5$

genus covering curve $X(M)$

$$g(X(M)) = \begin{cases} 1 + \frac{M-6}{24} M^2 \prod_{p|M} \left(1 - p^{-2}\right) & M > 2 \\ 0 & M = 2, \end{cases}$$

For $M \leq 5$, $g = 0$ and $X(M) \simeq \mathbb{P}^1$, isomorphism given by **Hauptmodul** $z = z(\tau)$. There exists a Hauptmodul such that

$$z(\tau + 1) = e^{2\pi i/M} z(\tau)$$

μ_ℓ rational differential of the form

$$\mu_\ell = \sum_{\substack{(a:c) \text{ I type} \\ \text{cusp of } X_1(M)}} \kappa_\ell(a:c) \sum_{k=0}^{M-1} \frac{e^{2\pi i k \ell (M-\ell)} dz}{z - e^{2\pi i k/M} z(a:c)}$$

Example: $M = 5$, $z(\tau) = K(e^{2\pi i \tau})^{-1}$; $z(a:c) = \left(\frac{5}{c}\right)\left[e^{2\pi i ac/5}\, \frac{\sqrt{5}-1}{2}\right]^{\left(\frac{5}{c}\right)}$

Example $M = 2$: $z(\tau) = 1 - \lambda(\tau)$

where $\lambda(\tau)$ Legendre modular function

$$z(0) = 0, \qquad z(1) = \infty, \qquad z(i\infty) = 1.$$

$z(\tau)$ modular invariant for $\Gamma(2)$ and in fact a *Hauptmodul*

$$z\colon X(2) \xrightarrow{\sim} \mathbb{P}^1,$$

$$z(T\tau) \equiv z(\tau + 1) = \frac{1}{z(\tau)}.$$

In terms of the $\mathbb{P}^1$ coordinate $z = z(\tau)$, projection $X(2) \to X_1(2)$ is

$$z \sim z^{-1}.$$

- $z = 0$ and $z = \infty$ map to the unique I type cusp on $X_1(2) \equiv X_0(2)$
- $z = 1$ is a U type cusp

The eigenvalue 1-form μ on $\mathbb{P}^1 \simeq X(2)$ has simple poles at $z = 0$ and $z = \infty$, this fixed it up to overall coefficient

$$\mu = -\frac{1}{4}\frac{dz}{z}$$

tt^* equations/Hitchin equations for $M=2$

tt^* metric along the fiber
$$G(\phi,\theta,z)=\Big|\wp(\pi\tau)-\wp(\pi(\tau+1))\Big|\exp\Big(\sigma_3 L(\phi,\theta,z)\Big)$$

$$\partial_{\bar z}\partial_z L(\phi,\theta,z)=\frac{1}{16|z|^2}\sinh\Big(2L(\phi,\theta,z)\Big)$$
$$L(\phi,\theta,z)=-L(-\phi,-\theta,z)$$

Setting $x=-\frac{1}{4}\log z$ we reduce to the well known $\widehat{A}_1$ **Toda equation**

$$\partial_{\bar x}\partial_x L(\phi,\theta,x)=\sinh\Big(2L(\phi,\theta,x)\Big)$$

many special solutions are known (often reduces to Painlevé III)

x not univalued on $X(2)$, but univalued when pulled back to $\mathbb{H}$
We need an **automorphic family of solutions to $\widehat{A}_1$ Toda**

$$L\Big(\phi,\theta,x(\tau)\Big)=L\left(a\phi+b\theta,c\phi+d\theta,x\left(\frac{a\tau+b}{c\tau+d}\right)\right),\quad\forall\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1(2)$$

No solution to Toda (known to me) has the right automorphic properties

We can solve the equations near the cusps:

- $z = 1$ is a U type cusp. The model is asymptotic to the LG model

$$W(X) = -2q^{1/2}\left(e^{X} - e^{-X}\right), \quad q = e^{2\pi i \tau} \to 0, \quad \text{(related to the } \mathbb{P}^1 \ \sigma\text{-model)}$$

whose tt^* equations are also $\widehat{A}_1$ Toda

$$\partial_{\bar{x}} \partial_x L(\theta, x) = \sinh\big(2L(\theta, x)\big)$$

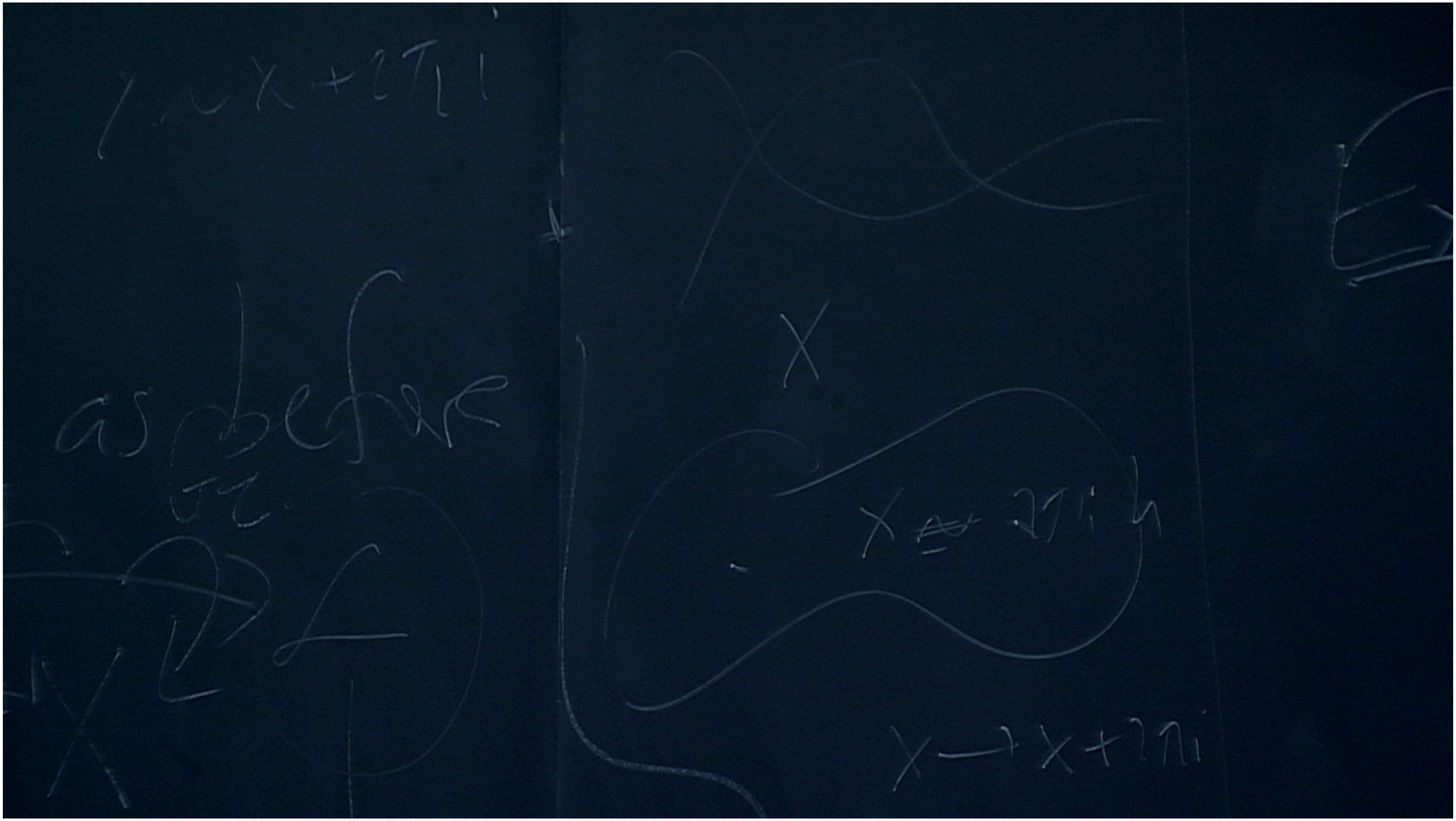
where $L(\theta, x)$, $0 \leq \theta < 2\pi$, is the family of **all** solutions which vanish at infinity and are regular for $x \neq 0$ (they are Painlevé transcendents)

setting $L(\phi, \theta, x) = L(\theta, x)$ solves the equations, reality constraints, regularity conditions, has the right asymptotics as $z \to 1$, and passes other consistency checks

yet it cannot be the correct solution since it is not automorphic

- $\Rightarrow$ the solution cannot become trivial at the I cusp $z = 0$

tt^* eqns. linearize. Their solutions very reminiscent of Maass form


as before
x → x+2πi

No solution to Toda (known to me) has the right automorphic properties

We can solve the equations near the cusps:

- $z = 1$ is a U type cusp. The model is asymptotic to the LG model

$$W(X) = -2q^{1/2}\left(e^{X} - e^{-X}\right), \quad q = e^{2\pi i\tau} \to 0, \quad \text{(related to the } \mathbb{P}^1 \text{ } \sigma\text{-model)}$$

whose tt^* equations are also $\widehat{A}_1$ Toda

$$\partial_{\bar{x}}\partial_x L(\theta, x) = \sinh\big(2L(\theta, x)\big)$$

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tt^* eqns. linearize. Their solutions very reminiscent of Maass form

The case of general M similar: tt^* eqns. are $\widehat{A}_{M-1}$ **Toda equations**

$$G(\phi,\theta) = \left|\sum_{k=0}^{M-1} e^{2\pi i\ell k/M}\,\wp\!\left(\frac{2\pi(\ell\tau-k)}{M}\right)\right| \exp\Big[\mathrm{diag}\big(L_k(\phi,\theta,x)\big)\Big]$$

$$\sum_{k=1}^{M} L_k(\phi,\theta,x) = 0, \qquad L_k(\phi,\theta;\tau) + L_{M-\ell-k}(-\phi,-\theta;\tau) = 0$$

Change of variable

$$\tau \longmapsto x(\tau) = \int_{i\infty}^{\tau} \mu_\ell,$$

$$\partial_{\bar{x}}\partial_x L_k(\phi,\theta,x) =$$
$$= \exp\Big[L_k(\phi,\theta,x) - L_{k+1}(\phi,\theta,x)\Big] - \exp\Big[L_{k-1}(\phi,\theta,x) - L_k(\phi,\theta,x)\Big]$$

The solutions which are everywhere regular and vanish at ∞ are more or less known (and fully determined as a by-product of the present analysis) **but they are not automorphic**

Asymptotics at U cusps more interesting:
non-trivial action of the diamond duality group

- it has important implications for general tt^* geometry as well as for the theory of regular solutions to Toda eqns. (cfr. A. Its and co-worker)
- For M an **odd prime** U cusps form a single orbit of $(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\}$

The $\frac{1}{2}\phi(M)$ $\Gamma_1(M)$-inequivalent U cusps are

$$\tau = i\infty, \text{ and } \tau = a/M \quad \text{with } 2 \leq a \leq (M-1)/2 \equiv \phi(M)/2$$

The asymptotic behavior of model with character $e(k) = e^{2\pi i \ell k/M}$

$$q_\infty \equiv e^{2\pi i\tau} \sim 0 \qquad W(X) \approx -M\, q_\infty^{\ell(M-\ell)/M} \left[\frac{e^{(M-\ell)X}}{M-\ell} + \frac{e^{-\ell X}}{\ell}\right].$$

$$q_{a/M} \equiv \exp\left[-2\pi i\,\frac{\bar{a}\tau - s}{M\tau - a}\right] \sim 0$$

$$W(X) \approx -M\, q_{a/M}^{\lfloor a\ell\rceil\left(M-\lfloor a\ell\rceil\right)/M} \left[\frac{e^{\left(M-\lfloor a\ell\rceil\right)X}}{M-\lfloor a\ell\rceil} + \frac{e^{-\lfloor a\ell\rceil X}}{\lfloor a\ell\rceil}\right],$$

$\lfloor n\rceil$: the integer n mod M such that $0 \leq \lfloor n\rceil \leq M-1$.

In other words: for M **odd prime** we get at the several U type cusps all affine $\widehat{A}(p,q)$ models with $p+q=M$ and $\gcd(p,q)=1$

These models form an orbit of the duality $(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\}$

They play a crucial role in classification of $N=2$ susy in 2d and 4d:

- if $X \sim X + 2\pi i$ they are mirror to the σ-model with target the weighted projective line $\mathbb{P}(p,q)$
- if $X \sim X + 2\pi i K$ (K an integer $K \to \infty$ in the thermodynamic limit) they are associated to the (mutation class of the) quiver obtained by orienting the affine Dynkin graph $\widehat{A}_{MK-1}$ with pK (qK) arrows in the positive (negative) direction
- the 2d BPS spectrum (in some chamber) is the Dynkin quiver
- the 2d quantum monodromy is minus the Coxeter of the affine quiver
- in 4d: $SU(2)$ SYM coupled to two Argyres-Douglas of types D_p and D_q

The tt^* equations of all these models are the same $\widehat{A}_{M-1}$ Toda equations

$$\partial_{\bar{x}}\partial_x L_k(\theta,x) = \exp\Big[L_k(\theta,x) - L_{k+1}(\theta,x)\Big] - \exp\Big[L_{k-1}(\theta,x) - L_k(\theta,x)\Big]$$

but with different reality constraints for different p

$$L_k(\theta,x) + L_{p-k}(-\theta,x) = 0$$

and different regularity conditions

It was a surprise that the regular solutions to these different PDE system are indeed related by the diamond duality $(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\}$

Regular solutions to the PDE recently described for $M = 5$ by A. Its *et al*

Unexpected action of $(\mathbb{Z}/M\mathbb{Z})^\times/\{\pm 1\}$ explains their results
and generalize them to arbitrary M

The **automorphic property** of solutions to tt^* for the modular $\mathcal{N} = 4$ SQM models is actually useful (for a totally different problem)