

Title: Abelianization in complex Chern-Simons theory and a hyperholomorphic line bundle

Date: Feb 17, 2017 11:00 AM

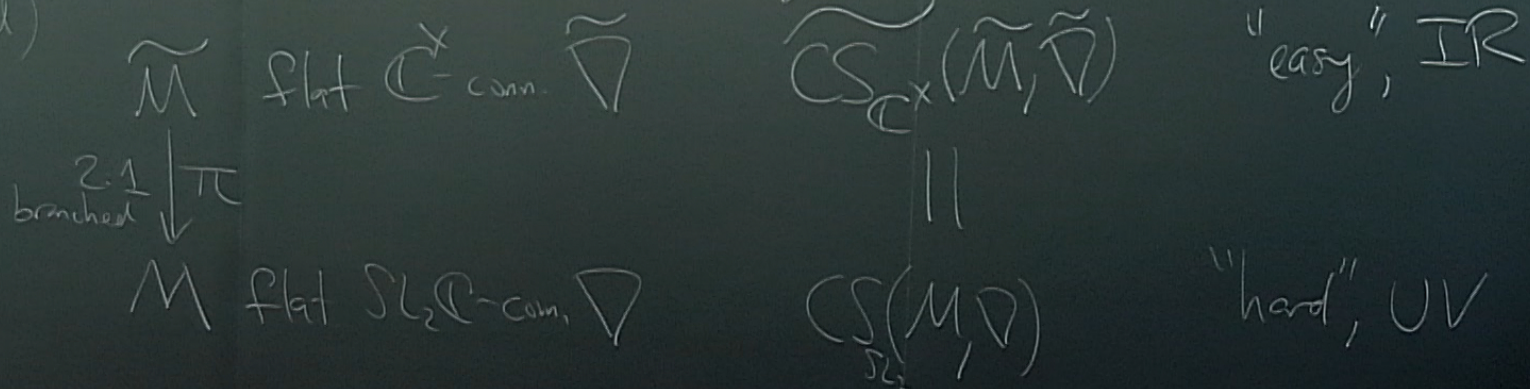
URL: <http://pirsa.org/17020034>

Abstract: I will describe an approach to classical complex Chern-Simons theory via "abelianization", relating flat  $SL(N)$ -connections over a manifold of dimension  $d \leq 3$  to flat  $GL(1)$ -connections over a branched  $N$ -fold cover. This is joint work with Dan Freed. When applied in dimension  $d=2$  this construction leads to an alternative description of a hyperholomorphic line bundle over the moduli space of Higgs bundles, studied e.g. by Haydys, Hitchin, Alexandrov-Persson-Pioline.

# Abelianization in complex classical Chern-Simons theory

(WIP w/ D. Freed)

Basic picture:





## Classical CS theory

$M$  cpt 3-mfd  $\nabla$  flat  $SL_2(\mathbb{C})$ -conn. over  $M$

Trivialize, write  $\nabla = d + A$   $A \in \Omega^1(M, sl_2(\mathbb{C}))$

$$S = \frac{1}{4\pi i} \int_M \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

$$CS_{SL_2}(M, \nabla) = \exp(S) \in \mathbb{C}^\times$$

This is



This is part of an invertible TFT for  $d$ -manifolds w/ flat  $SL_2(\mathbb{C})$ -conn:  
(dSS)

2  $\mathbb{C}$ )

$$\text{closed 3-manifold } M \quad (M, \nabla) \leadsto CS_{SL_2}(M, \nabla) \in \mathbb{C}^\times$$

$$\text{closed 2-manifold } M \quad (M, \nabla) \leadsto CS_{SL_2}(M, \nabla) \in \text{a 1-dim vector space}$$

Lines

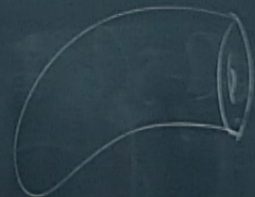


This is part of an invertible TFT for  $d$ -manifolds w/ flat  $SL_2(\mathbb{C})$ -conn:  
 $(d \leq 3)$

closed 3-mfd  $M$   $(M, \nabla) \leadsto CS_{SL_2}(M, \nabla) \in \mathbb{C}^\times$

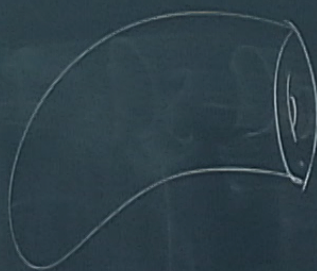
closed 2-mfd  $M$   $(M, \nabla) \leadsto CS_{SL_2}(M, \nabla) \in \underline{\text{Lines}}$  a 1-dim vector space

Package as:  $CS: \text{Bord}_{SL_2} \rightarrow \underline{\text{Lines}}$





m:



space

Rk

① If we let  $\nabla$  be non-flat  
then flatness of  $\nabla \leftrightarrow \nabla$  is crit. pt. of  $S$ .

② Motivation: the lines  $CS_{SL_2}(M, \nabla)$   
( $M$  fixed Riemann surface)

make a line bundle over  $M_{\text{flat}}(C, SL_2)$

in fact hyperbol (TSIS) on  $M_H(C, SL_2)$

[Haydys, A.P., Hitchin] [AN]



## Classical CS theory

Also have

$$CS_{\mathbb{C}^X}: \text{Bord}_{\mathbb{C}^X} \rightarrow \underline{\text{Lines}}$$

$\text{Bord}_{\mathbb{C}^X}$  = cat. of  $(3,2)$ -mfds with  
flat  $\mathbb{C}^X$ -conn. and spin structures

Much simpler than  $CS_{S^1}$ . Ex of  $M$  opt 3-mfd,  $\nabla$  top. trivial,  $CS_{\mathbb{C}^X}(M, \nabla) = 1$ .

This is part of an inv

closed 3-mfd

closed 2-mfd

Pack



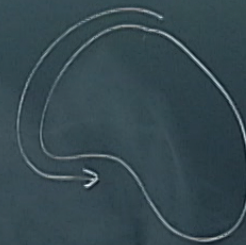
# Deformed $\mathbb{C}^*$ -CS theory

Manif 3-manifold

Fix circles  $L_1, \dots, L_n \subset M$   $\nabla = d + A$

$$\text{set } S = \frac{1}{4\pi i} \int A \wedge dA + \frac{1}{2\pi i} \sum_{i=1}^n L_{i2}(X_i)$$

$$X_i = \text{hol}_{\nabla}(L_i) \in \mathbb{C}^*$$



Rk

① If then

② Mot



# Deformed $\mathbb{C}^*$ -CS theory

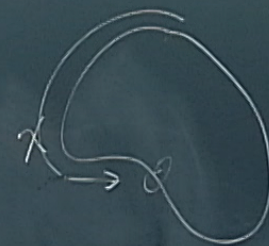
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crit pts. of  $S$  (formally)  $\boxed{F = - \sum_i \log(1 - X_i) \delta_{L_i}}$

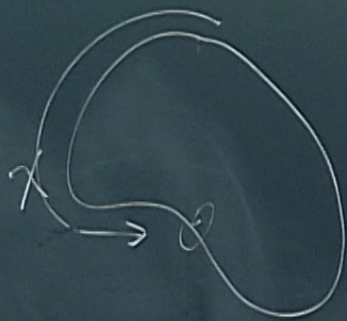


Rk

① If then

② Mot





Rk This kind of deformation of CS theory appears in A model top. string [Witten, Ooguri-Vafa]

$$M \subset N$$

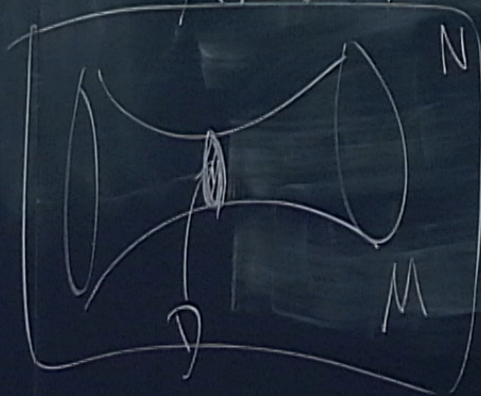
Lag 3-dim  
sympl. 6-dim

$X_i$

$$X_i = \text{hol}_\nabla(L_i) \in \mathbb{C}^\times$$

$\delta L_i$

A model on  $N$ :  $Z_{\text{open}} = \text{deformation of } Z_{\text{CS}} \text{ on } M$





theory

$$\widetilde{\text{CS}}_{\text{ex}}(\widetilde{M}, \widetilde{\nabla})$$

"easy", IR

||

$$\text{CS}_{\text{SUSY}}(M, \nabla)$$

"hard", UV

[cf. Cecotti-Cordova-Vafa]

Classical CS th

Also have

$$\text{CS}_{\text{ex}}: \mathbb{R}$$

$$\text{Bord}_{\text{ex}} = \mathbb{C}$$

flat  $\mathbb{D}$

Much simpler than



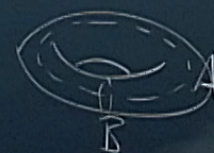
# Classical CS theory

So, we make a category  $\text{Bord}_{\mathbb{C}^X}$

"manifolds  $M$  w/ flat  $\mathbb{C}^X$ -conn  
w/ singularities"

i.e. allow defects in 3-mfds:

non-manifold points on  $\tilde{M}$ , whose link is  $T^2$



$$\text{hol}_A \nabla = X \in \mathbb{C}^X$$

$$\text{hol}_B \nabla = 1 - X \in \mathbb{C}^X$$



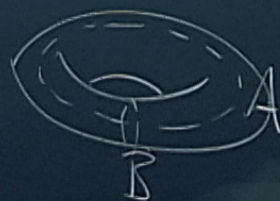
theory

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defects in 3-mfds:

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$$\underline{A, B} - A - B$$

$$\begin{aligned} \nabla &= X \in \mathbb{C}^X \\ \nabla_A &= 1 - X \in \mathbb{C}^X \end{aligned}$$



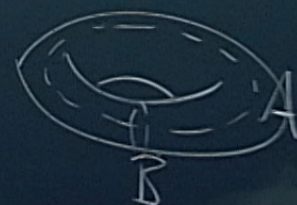
# Classical CS theory

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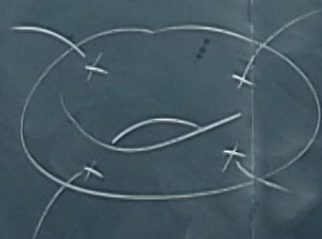


$$\text{hol}_A \nabla = X \in \mathbb{C}^X$$

$$\text{hol}_B \nabla = 1 - X \in \mathbb{C}^X$$



(Also need codim-1 defects  $\oint_A \text{hol}_A(\nabla) = -1$



Let's ignore this.)

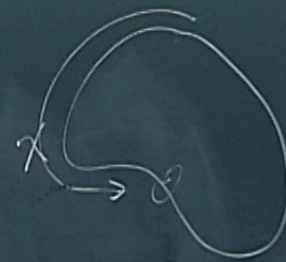
Then have  $\widetilde{CS}_{\mathbb{C}^X} : \widetilde{\text{Bord}}_{\mathbb{C}^X} \rightarrow \underline{\text{Lines}}$

$$h_{\nabla}(L_i) \in \mathbb{C}^X$$

Away from hypers,  $\widetilde{CS}_{\mathbb{C}^X} = CS_{\mathbb{C}^X}$

need an elt in  $Z_S(\bigotimes, \nabla)$

a canonical such elt exists:  $\exp\left(\frac{Li_2(\chi_1)}{2\pi i} + c\right)$





let's ignore this.)

$x: \text{Board}_x \rightarrow \text{Lines}$

pers,  $\text{CS}_x = \text{CS}_{\text{CS}_x}$

$Z_{\text{CS}}(\text{CS}_x, \nabla)$

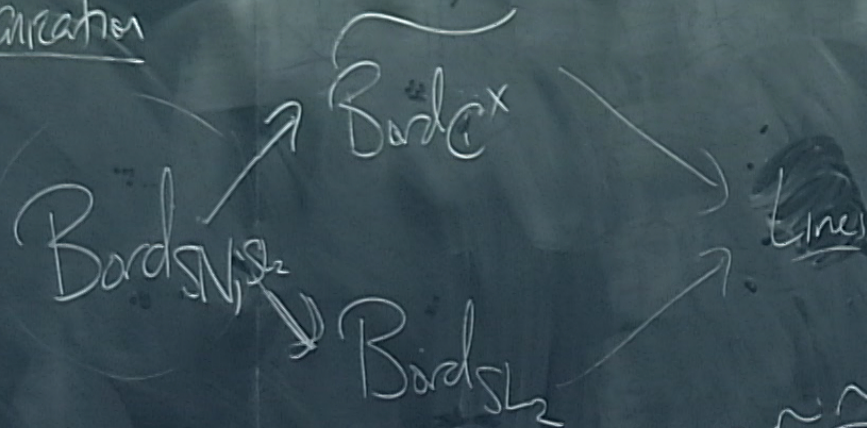
ch elt exists:  $\exp\left(\frac{\text{Li}_2(x_1)}{2\pi i} + c\right)$

A model on  $N$ :

The diagrams include a large irregular shape with points  $x_i$  inside, a small diagram of a point  $x$  with an arrow, and a diagram of a surface  $M$  with a boundary  $D$  and a point  $N$ .



# Abelianization



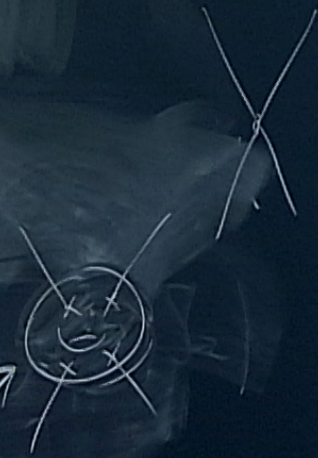
Objects/morphisms of  $\text{BordSL}_2$ :  $(M, \nabla, \tilde{M}, \tilde{\nabla}, \pi, L)$

- $(M, \nabla) \in \text{BordSL}_2$

- $(\tilde{M}, \tilde{\nabla}) \in \text{BordC}^x$

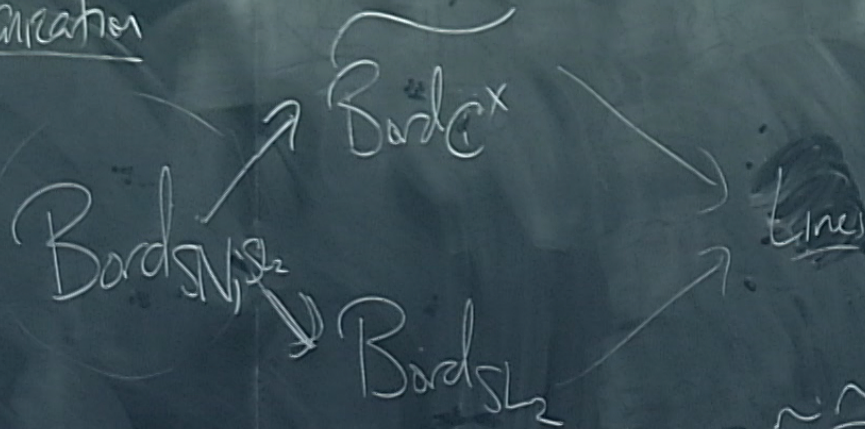
- $\pi: \tilde{M} \rightarrow M$

2:1 branched cover, <sup>subfld</sup>branch locus codim 2, meets hypers as





# Abelianization



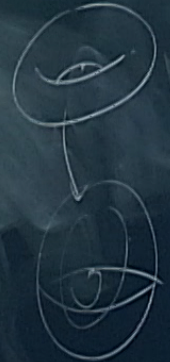
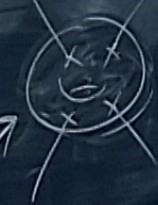
Objects/morphisms of  $\text{BordSL}_2$ :  $(M, \nabla, \tilde{M}, \tilde{\nabla}, \pi, [L])$

- $(M, \nabla) \in \text{BordSL}_2$

- $(\tilde{M}, \tilde{\nabla}) \in \text{BordSL}_2^x$

- $\pi: \tilde{M} \rightarrow M$

$\tilde{M} \rightarrow M$  2:1 branched cover, <sup>submfd</sup> branch locus codim 2, meets hypers as





$L: \pi_* \widetilde{\nabla} \xrightarrow{\sim} \nabla$  piecewise flat, jumps at  
 codim-1 walls  $W \subset M$ ,  
 jump unipotent  $S \in \text{End}(\pi_* \widetilde{\nabla})$

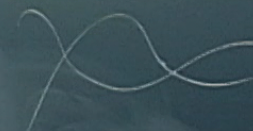
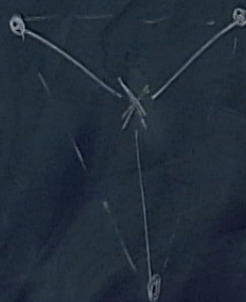
walls  $W_i$  have some standard shape on  $S^2$  around hyper  
 — triangulate  $S^2$  as tetrahedron  $M$



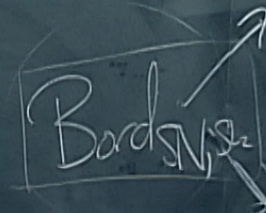


Ex If  $M$  is a punctured surface  
w/ ideal  $\Delta_N$ , then every generic  
 $\nabla$  w/ unip. monodromy & punctures lifts  
canonically to  $\text{Bord}_N$ .

Holonomies of  $\tilde{\nabla}$   
on



Abelianization

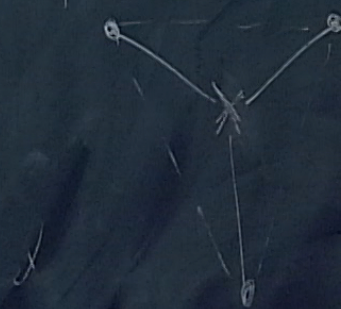


Objects/morphisms of  $\mathcal{T}$

- $(M, \nabla) \in \text{Bord}_N$
- $(\tilde{M}, \tilde{\nabla}) \in \text{Bord}_N$
- $\pi: \tilde{M} \rightarrow M$



Ex If  $M$  is a punctured surface  
w/ ideal  $\Delta_N$ , then every generic  
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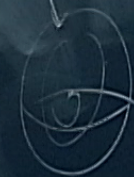
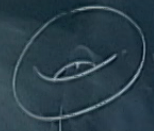
Holonomies of  $\tilde{\nabla}$   
on  $\tilde{M}$  are  
"cluster coords" of  $\nabla$   
[Fock-Goncharov, Penner, ...]



Ex If  $M$  is ideally  $\Delta^d$  3-manifold  
(e.g. figure-8 knot complement)

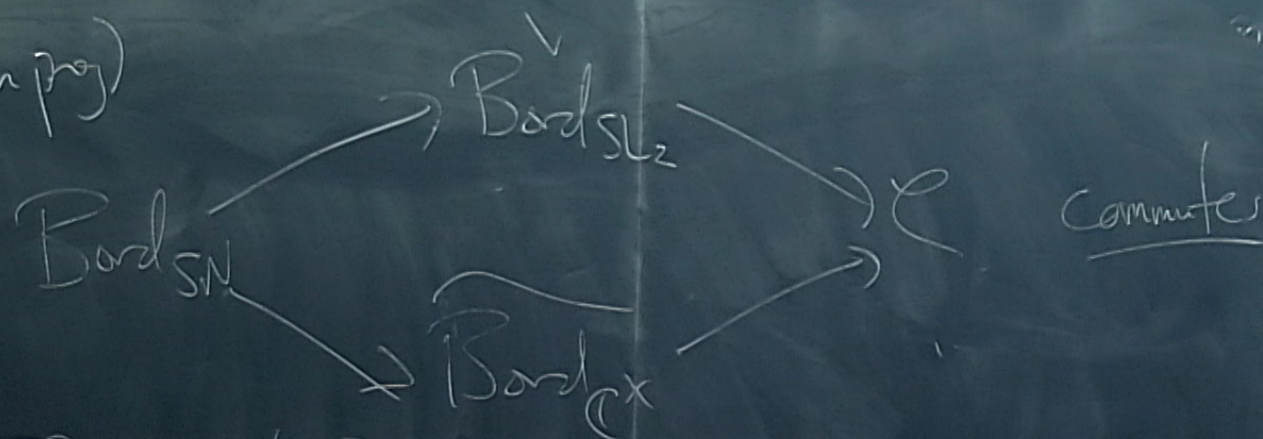
then 'many'  $SL_2(\mathbb{C})$ -con  $\nabla$  on  $M$   
lift to  $\tilde{Bord}_N$  (each  $\nabla$  has  
are hyper)

Hol. of  $\tilde{\nabla}$  on  $\tilde{M}$   
are "shape parameters" of  $\nabla$  [Thurston]





Thm (in prog)

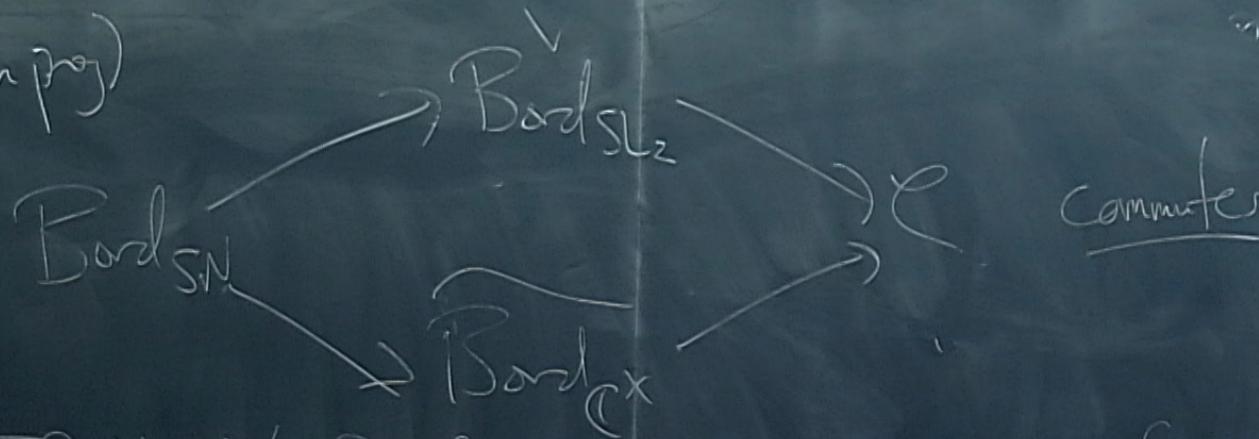


Ex If  $M \triangleleft d$  Intd  $\dagger$   
 $\nabla$  lifts to  $\text{Bord}_{\text{SN}}$

then  $\| \in S_{\text{SL}_2}(M, \nabla) = \exp\left(\frac{1}{2\pi i} \sum_i L_2(X_i)\right)$



Thm (in prog)



Ex If  $M$  Ad 3-mfd +  
 $\nabla$  lifts to  $\text{Bord}_{SU}$

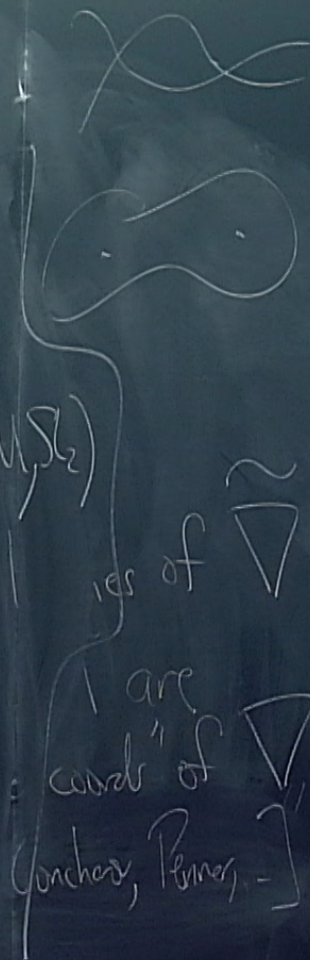
(Dupont, Guichard, Neumann, Zickert-)

then  $\tau_{SL_2}(M, \nabla) = \exp\left(\frac{1}{2\pi i} \sum_i L_2(X_i)\right)$



Ex If  $M$  is  $\Delta d$  surface as before

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{\quad} & \mathbb{L} \\ \downarrow & & \downarrow \\ M_{\text{flat}}(\widehat{M}, \mathbb{C}^{\times}) & \xrightarrow[\sim]{\text{local}} & M_{\text{flat}}(M, \text{SL}_2) \end{array}$$



Ex If  $M$  is  $i$   
(e.g. figure)

then 'man'

lift to  $\mathbb{R}$

Hol. of  $\nabla$  on  
are "shape"



$\downarrow$   
 $\rightarrow \text{Bord}_{\text{SL}_2} \rightarrow \mathcal{C}$   
 $\rightarrow \text{Bord}_{\mathbb{C}^\times}$   
 Id  $\rightarrow$   $\mathbb{C}^\times$   $\rightarrow$   $\mathcal{C}$   
 is to  $\text{Bord}_{\text{SW}}$   
 $\in S_{\text{SW}}(M, \nabla) = \exp\left(\frac{1}{2\pi i} \sum_i L_{1,2}(X_i)\right)''$

Commuter

(Dupont, Guichard, Neumann, Zickert...)

Ex