Title: TBA

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Abstract:

BPS states, torus links, and wild character varieties

- String theoretic approach to the cohomology of wild character varieties; with Ron Donagi and Tony Pantev.

- Generalizes previuous work with Wu-yen Chuang and Guang Pan, and Wu-yen Chuang, Ron Donagi and Tony Pantev.

- Physical derivation and generalization of conjectures of Hausel, Mereb, Wong and Shende, Treumann, Zaslow.

1. Wild Character varieties
[Witten, Boalch]

- (C, p) smooth projective curve with marked point
- Q irregular type at p with values in $\mathfrak{t} \subset \mathfrak{gl}(r, \mathbb{C})$

i.e. $\mathfrak{t}$-valued meromomorphic fct germ at p modulo holomorphic terms

$$Q = \sum_{k=1}^{n-1} \frac{A_k}{z^k}, \qquad A_k \in \mathfrak{t}$$

for some $n \in \mathbb{Z}$, $n \geq 2$.

- R $\in \mathfrak{t}$ diagonal matrix
- Flat $\mathfrak{gl}(r, \mathbb{C})$-connections on $C \setminus \{p\}$ locally gauge

equivalent to

$$dQ + \frac{\mathsf{R}}{z} + \text{holomorphic terms}$$

at p.

- $\mathcal{S}_{\mathsf{Q},\mathsf{R}}$ the variety of Stokes data [Witten, Boalch]. Rigorous construction [Boalch] by quasi-hamiltonian reduction.

- $\mathcal{S}_{\mathsf{Q},\mathsf{R}}$ smooth quasi-projective for generic R. Dimension

$$d_{\mu,n} = 2r^2(g-1) + n(r^2 - \sum_{i=1}^{\ell} m_i^2) + 2$$

where μ is the partition of r determined by the eigenvalues of R.

- $WP(\mathsf{Q},\mathsf{R};u,v)$ the mixed Poinvaré polynomial:

$$WP(\mathsf{Q},\mathsf{R};u,v)=\sum_{k,j}\dim Gr_k^W H^j(\mathcal{S}_{\mathsf{Q},\mathsf{R}})u^{k/2}v^j$$

where W_kH^j weight filtration.

- Wild non-abelian Hodge correspondence [Boalch,Biquard], [Mochizuki]

Irregular flat connections $\leftrightarrow$ Irregular Higgs bundles

Moduli spaces related by hyper-Kähler rotation.

- $P=W$ correspondence [de Cataldo, Hausel, Migliorini]

$W_{2k}H^j(\text{character variety})=P_kH^j(\text{Higgs bundle moduli space})$

where P_kH^j perverse Leray filtration from Hitchin map.

Home Page
Title Page
Page 4 of 24
Go Back
Full Screen
Close
Quit
4 of 24

2. The conjecture of Hausel, Mereb, and Wong

- R regular and sufficiently generic

$$Z_{HMW}(z,w) = \sum_{\lambda} \Omega_{\lambda}^{g,n}(z,w)\widetilde{H}_{\lambda}(\mathsf{x}; z^2, w^2)$$

where:

- the sum in the right hand side is over all Young diagrams λ,
- for each λ

$$\Omega_{\lambda}^{g,n} = \prod_{\square\in\lambda} \frac{(-z^{2a(\square)}w^{2l(\square)})^r (z^{2a(\square)+1} - w^{2l(\square)+1})^{2g}}{(z^{2a(\square)+2} - w^{2l(\square)})(z^{2a(\square)} - w^{2l(\square)+2})},$$

and

- $\mathsf{x} = (x_1, x_2, \ldots)$ is an infinite set of formal variables and $\widetilde{H}_\lambda(\mathsf{x}; z^2, w^2)$ are the modified Macdonald polynomials.

- Define $\mathbb{H}_{\mu,n}(z,w)$ by

$$Z_{HMW}(z,w) = \sum_{k \geq 1} \sum_{\mu} \frac{(-1)^{n|\mu|} w^{k d_{\mu,n}} \mathbb{H}_{\mu,n}(z^k, w^k)}{(1 - z^{2k})(w^{2k} - 1)} m_\mu(\mathsf{x}^k)$$

where $m_\mu(\mathsf{x})$ are the monomial symmetric functions and $\mathsf{x}^k = (x_1^k, x_2^k, \ldots)$. Then one has the following conjectural formula

$$WP(\mathsf{Q}, \mathsf{R}; u, v) = \mathbb{H}_{(1^r),n}(u^{1/2}, -u^{-1/2} v^{-1})$$

for any $r, n \geq 1$ and any Q and any generic regular R.

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2. The conjecture of Shende, Treumann and Zaslow

• $\Sigma \subset \mathbb{A}^2$ is a reduced rational plane curve with one singular point ν.

• $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ projection onto one of the coordinate axes.

• $\Sigma \simeq$ affine part of a spectral curve for a meromorphic Hitchin system on $\mathbb{P}^1$ with a pole at ∞.

• Wild non-abelian Hodge correspondence $\Rightarrow$ character variety $\mathcal{S}_\Sigma$.

• L link of singular point $\nu \in \Sigma$.

• $P_L^{(0)}(u)$ leading term in Homfly polynomial of L.

Home Page
Title Page
Page 7 of 24
Go Back
Full Screen
Close
Quit

Then

$$P_L^{(0)}(u) = WP(\mathcal{S}_\Sigma, u, -1)$$

for a specific normalization of $P_L^{(0)}(u)$

- Non-abelian Hodge 'mirror' of the conjecture of Oblomkov and Shende

$$P_L^{(0)}(u) = \sum_{n\in\mathbb{Z}} u^n \chi(Hilb_n(\Sigma, \nu)).$$

- Refined generalization [Oblomkov, Rassmusen,Shende]

- Refined colored generalization conjectured by [D,Hua,Soibelman] using the stable pair theory of the conifold. Will be used later in the talk.

- Unrefined colored generalization proven by [Maulik]

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2. Irregular parabolic Higgs bundles

- (C, p) curve with marked point, $D = np$, $n \geq 1$, $M = K_C(D)$.
- $\underline{\xi} = (\xi_1, \ldots, \xi_\ell)$ $\ell \geq 1$ sections of $K_C(D)|_D$.
- Irregular Higgs $\underline{\xi}$-parabolic Higgs bundle $(E, \Phi, E_D^\bullet, \underline{\alpha})$

E : vector bundle on C, $\quad E_D = E \otimes \mathcal{O}_D$

$\Phi : E \longrightarrow E \otimes M$, $\quad \Phi_D = \Phi|_D$

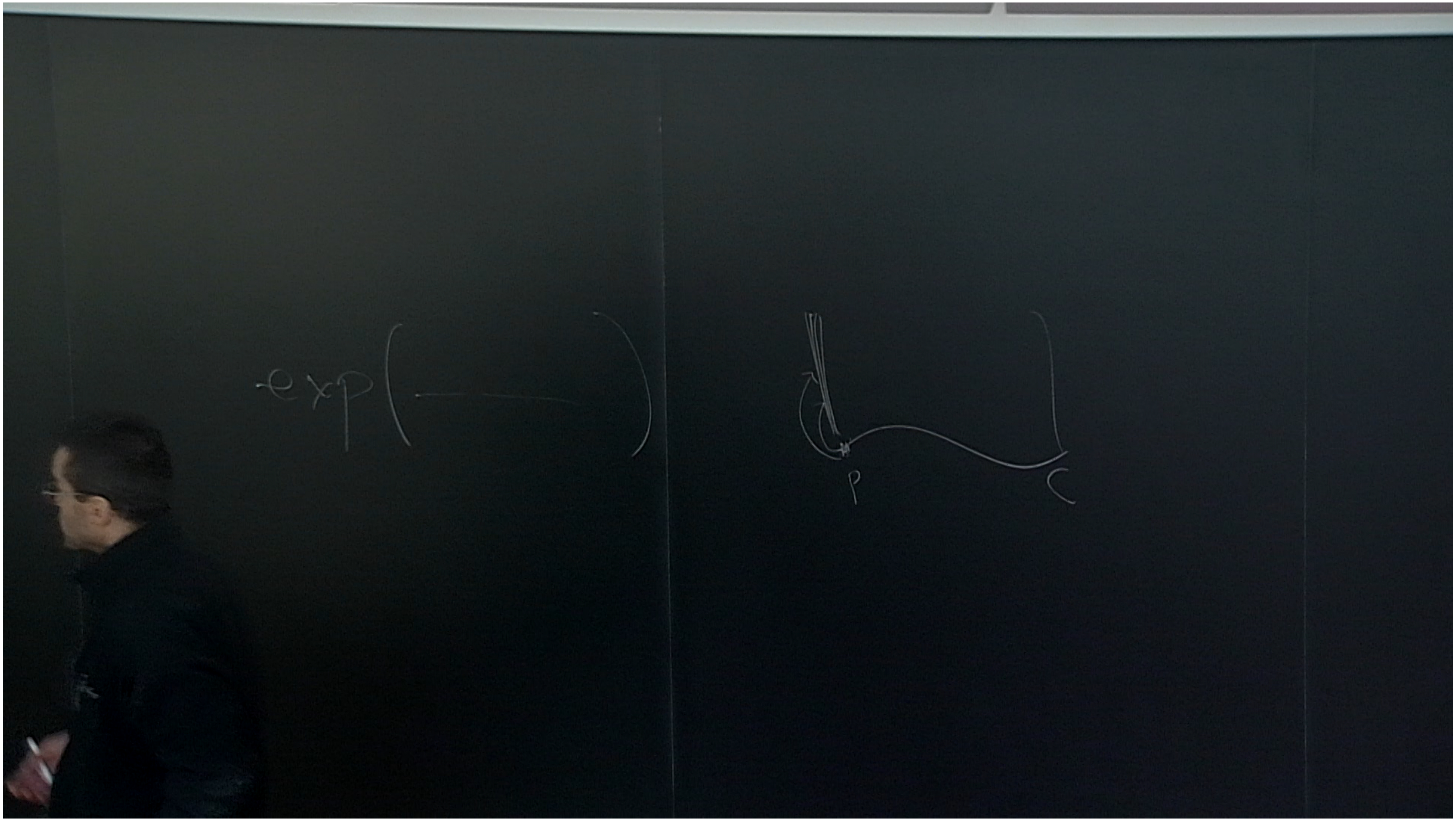
$0 \subset E_D^1 \subset \cdots \subset E_D^\ell = E_D \quad E_D^i / E_D^{i-1}$ loc. free $\mathcal{O}_D$–modules

$\Phi_D(E_D^i) \subseteq E_D^i$, $\quad \mathrm{gr}^i(\Phi_D) = \xi_i \otimes \mathbf{1}$

$\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ realparabolic weights

$0 < \alpha_\ell < \cdots < \alpha_1 < 1.$

Home Page
Title Page
Page 9 of 24
Go Back
Full Screen
Close
Quit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- Parabolic stability condition [Maruyama, Yokogawa] $\Rightarrow$ moduli stack $\mathcal{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ where

$$\underline{m} = (m_1, \ldots, m_\ell), \quad m_i = \mathrm{length}_{\mathcal{O}_D}(E^i_D / E^{i-1}_D)$$

$$d = \deg(E), \quad \sum_{i=1}^{\ell} m_i = \mathrm{rk}(E)$$

- for generic weights $\mathbb{C}^\times$-gerbe over smooth coarse moduli space
- similar objects introduced by [Inaba, Saito]

2. Irregular parabolic Higgs bundles

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$$E : \text{vector bundle on } C, \quad E_D = E \otimes \mathcal{O}_D$$
$$\Phi : E \longrightarrow E \otimes M, \quad \Phi_D = \Phi|_D$$
$$0 \subset E_D^1 \subset \cdots \subset E_D^\ell = E_D \quad E_D^i / E_D^{i-1} \text{ loc. free } \mathcal{O}_D\text{-modules}$$
$$\Phi_D(E_D^i) \subseteq E_D^i, \quad \mathrm{gr}^i(\Phi_D) = \xi_i \otimes \mathbf{1}$$
$$\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \quad \text{realparabolic weights}$$
$$0 < \alpha_\ell < \cdots < \alpha_1 < 1.$$

3. Spectral correspondence

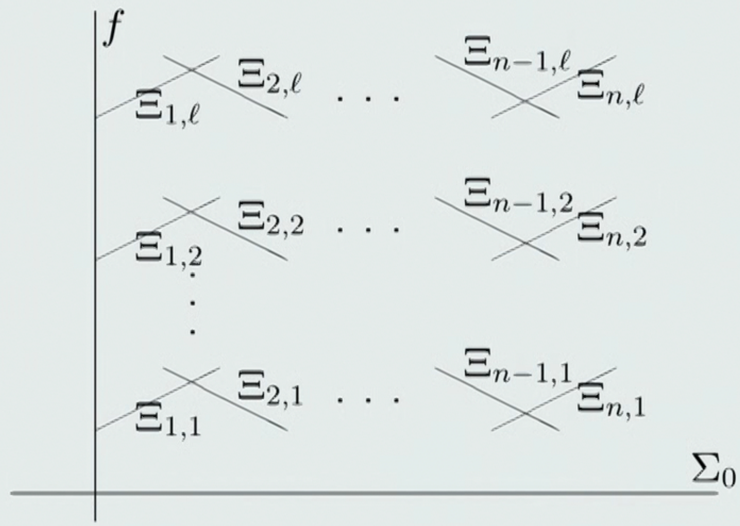
- Goal: construct holomorphic symplectic surface $S_{\underline{\xi}}$ such that

$$\begin{array}{c} \mathcal{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d) \\ \wr\! \mid \\ \text{Moduli stack Bridgeland stable dim 1 sheaves on } S_{\underline{\xi}} \end{array} \qquad (*)$$

- [Kontsevich, Soibelman] construct $S_{\underline{\xi}}$ such that

$$\begin{array}{c} \text{Hitchin base} \\ \wr\! \mid \\ \text{Linear system on } S_{\underline{\xi}} \end{array}$$

- [Szabo] proves $(*)$ for certain open subsets



- $T_{\underline{\xi}}$ = blow-up of the images of sections $\xi_1, \ldots, \xi_\ell : D \to M_D$ on M

(assumed pairwise distinct and nonzero)

- $S_{\underline{\xi}} = T_{\underline{\xi}} \setminus$anti-canonical divisor
- For any $\underline{m} = (m_1, \ldots, m_\ell)$ there is a linear system on $S_{\underline{\xi}}$ consisting of compact divisors which are finite covers of C.
- There is a compactly supported B-field $\beta \in H^2_c(S_{\underline{\xi}}, \mathbb{R})$

$$\beta(\Sigma_{\underline{m}}) = n \sum_{i=1}^{\ell} m_i \beta_i$$

for any divisor $\Sigma_{\underline{m}}$ in this linear system, where $\beta_i \in \mathbb{R}$, $1 \leq i \leq \ell$.

- β-stability for dimension one sheaves F on $S_{\underline{\xi}}$ with

$$\mathrm{ch}_1(F) = \Sigma_{\underline{m}}, \qquad \chi(F) = c.$$

- $\mathfrak{M}^{ss}_{\beta}(S_{\underline{\xi}}; \underline{m}, c)$ moduli stack of β-semistable sheaves

Spectral correspondence

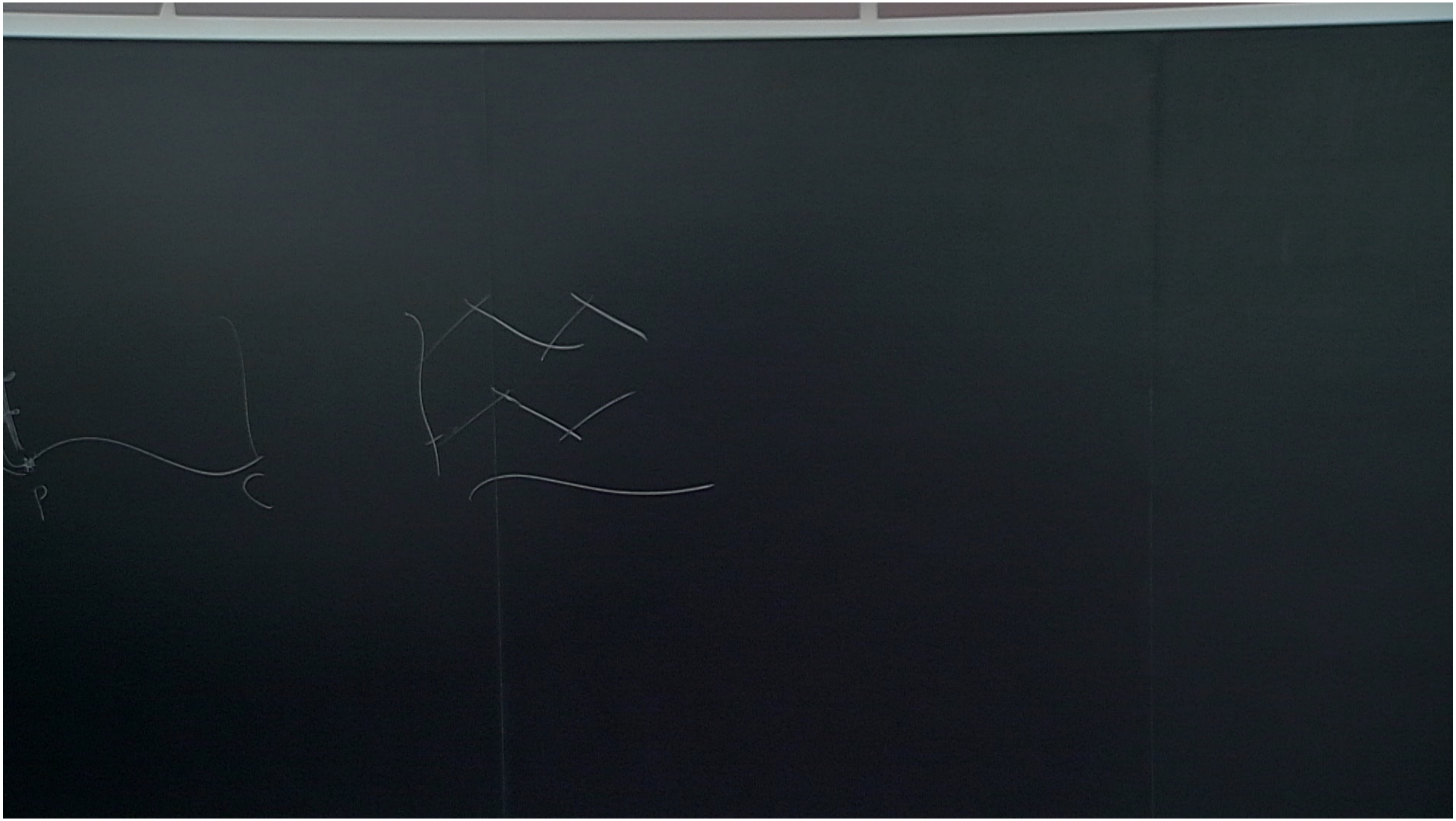
$$\mathfrak{M}^{ss}_{\beta}(S_{\underline{\xi}}; \underline{m}, c) \simeq \mathfrak{H}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, c + r(g-1))$$

$$\beta(\Sigma_{\underline{m}}) = n \sum_{i=1}^{\ell} m_i \alpha_i,$$

where g is the genus of C and $r = \sum_{i=1}^{\ell} m_i$.

- holds for any $g \geq 0$ and any number of singular points.

Home Page
Title Page
Page 14 of 24
Go Back
Full Screen
Close
Quit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4. Calabi-Yau threefold and refined invariants

- $Y_{\underline{\xi}} = S_{\underline{\xi}} \times \mathbb{A}^1$.

$$\mathfrak{M}^s_\beta(S_{\underline{\xi}}; \underline{m}, c) \simeq \mathfrak{H}^s_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, c + r(g-1)) \times \mathbb{A}^1$$

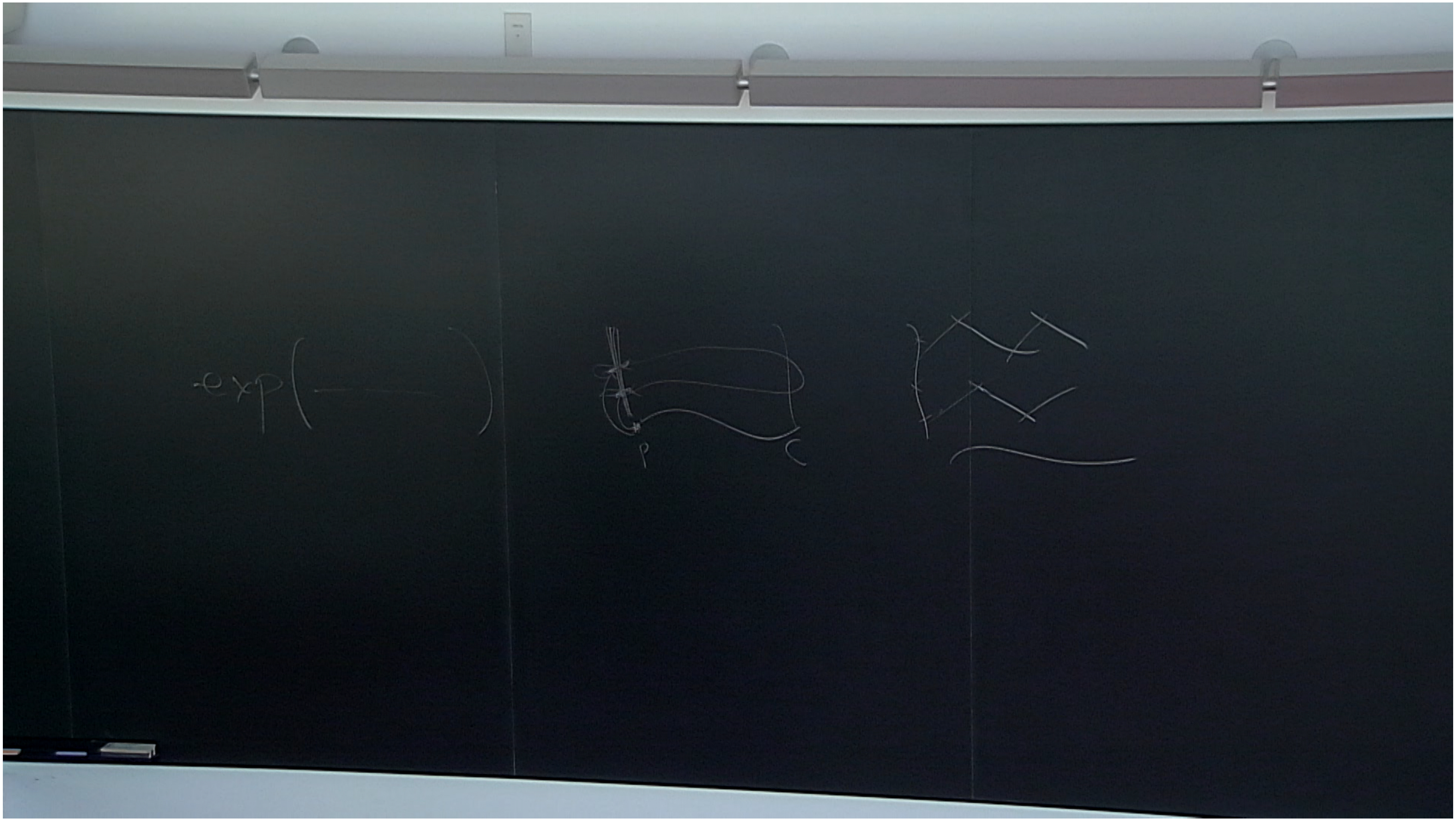
- Stable pair theory [Pandharipande, Thomas]

$$PT_{Y_{\underline{\xi}}}(\underline{m}, c) = \text{virtual number of pairs}$$

$$\mathcal{O}_{Y_{\underline{\xi}}} \xrightarrow{s} F, \quad s \text{ generically surjective}$$

$$\mathrm{ch}_1(F) = \Sigma_{\underline{m}}, \quad \chi(F) = c$$

$$F \text{ compact support.}$$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- Refined stable pair theory
  [Kontsevich and Soibleman]

$$PT_{Y_{\underline{\xi}}}(\underline{m}, c; y) = \text{virtual Poincare polynomial of moduli space of such pairs.}$$

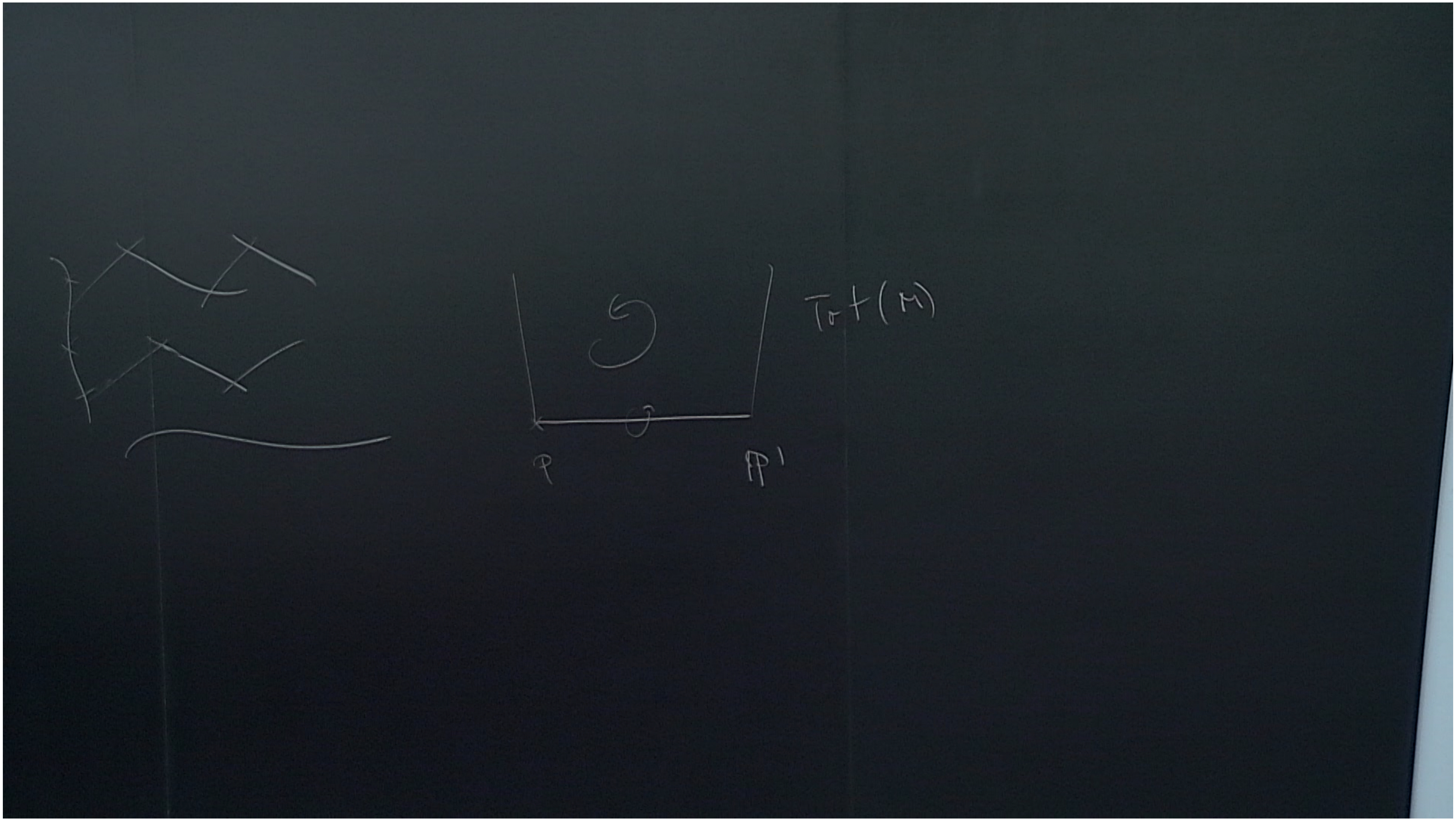
- Generating function

$$Z_{Y_{\underline{\xi}}}(q, Q_1, \ldots, Q_\ell, y) =$$

$$1 + \sum_{\substack{c,\underline{m} \\ \underline{m}\neq(0,\ldots,0)}} q^c \prod_{i=1}^{\ell} Q_i^{m_i} PT_{Y_{\underline{\xi}}}(m_1, \ldots, m_\ell, c; y).$$

- Goal: compute the generating function.

16 of 24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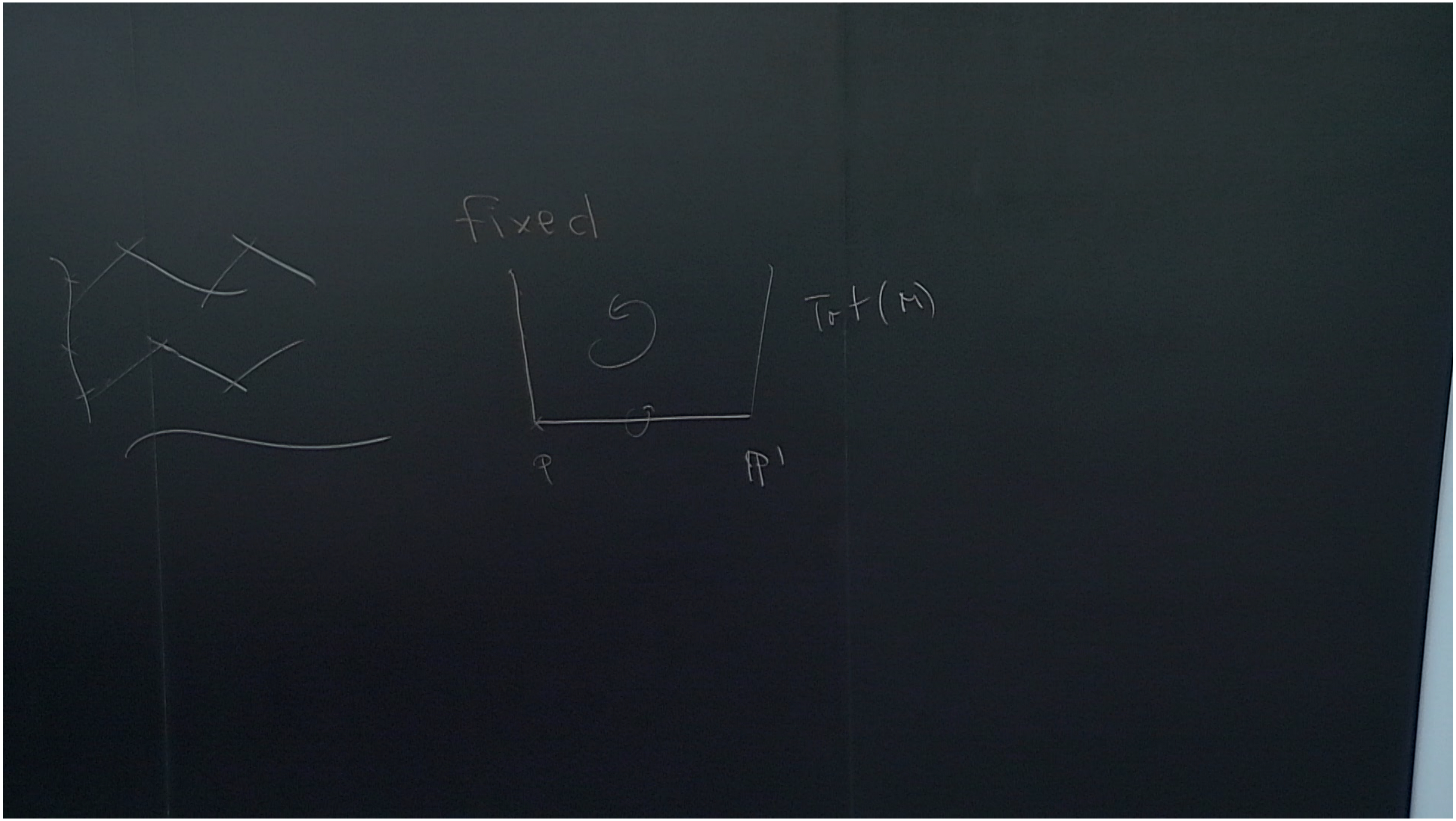
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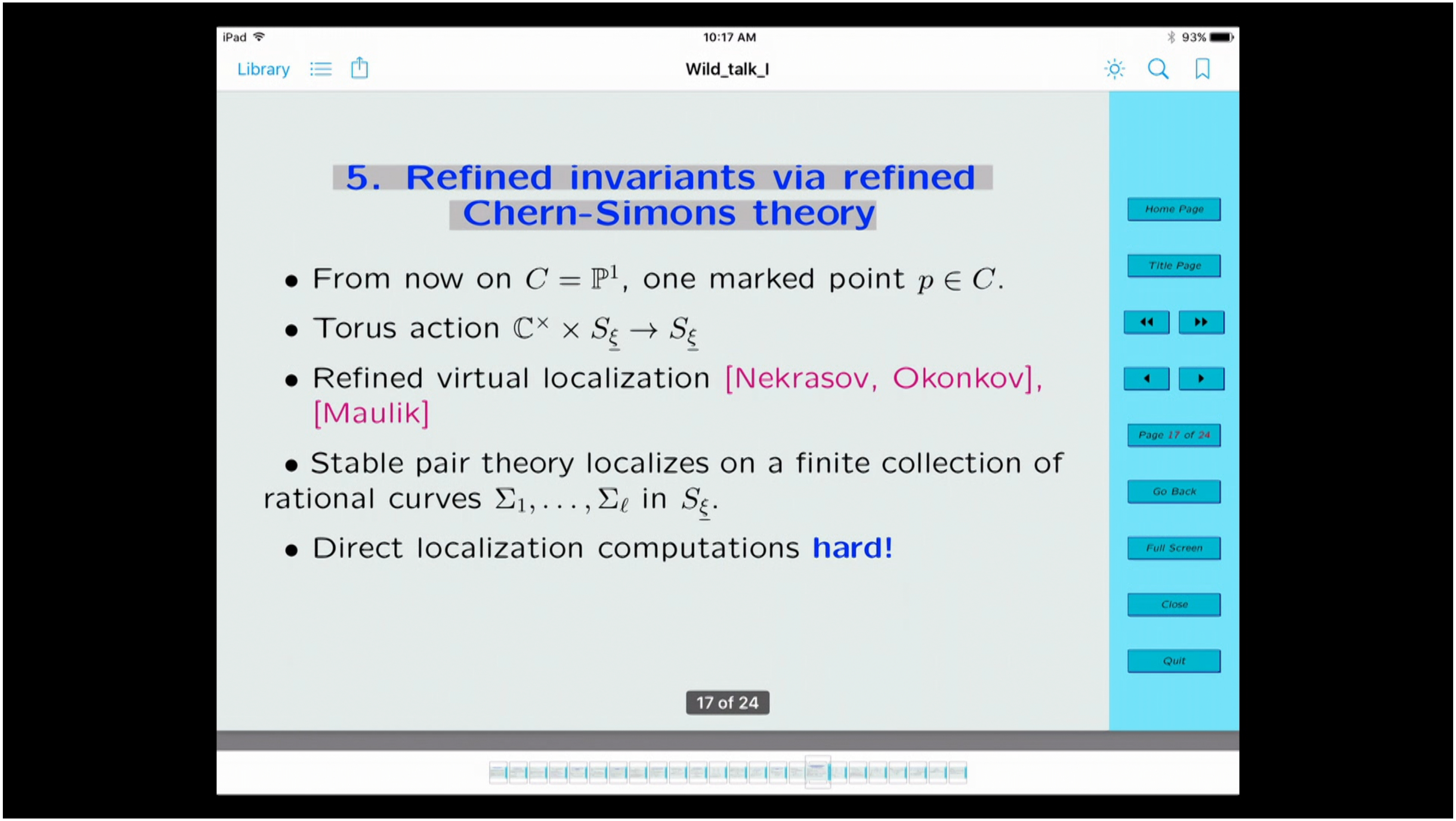
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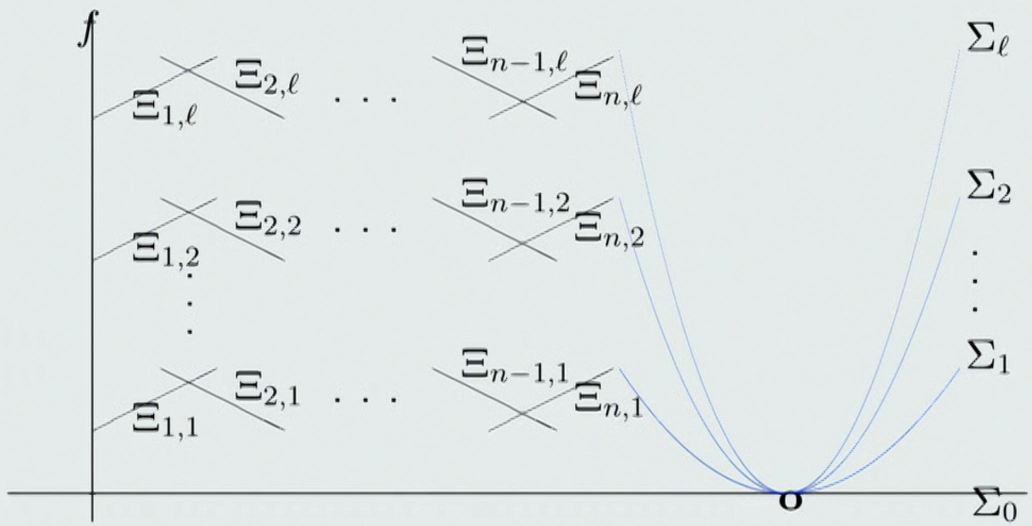
5. Refined invariants via refined Chern-Simons theory
• From now on $C = \mathbb{P}^1$, one marked point $p \in C$.
• Torus action $\mathbb{C}^\times \times S_{\underline{\xi}} \to S_{\underline{\xi}}$
• Refined virtual localization [Nekrasov, Okonkov], [Maulik]
• Stable pair theory localizes on a finite collection of rational curves $\Sigma_1, \ldots, \Sigma_\ell$ in $S_{\underline{\xi}}$.
• Direct localization computations hard!
17 of 24
Home Page
Title Page
Page 17 of 24
Go Back
Full Screen
Close
Quit



- local equation at $\mathbf{o}$

$$\prod_{i=1}^{\ell}(y - \lambda_i x^{n-2}) = 0.$$

18 of 24

- Oblomkov-Shende framework:

$$\begin{gathered}\text{Refined invariants of singular plane curve}\\ \Sigma_1 + \cdots + \Sigma_\ell \\ \updownarrow \\ \text{Refined invariants of} (\ell, (n-2)\ell)\text{−torus links}\end{gathered}$$

- One needs the colored refined variant of the conjecture formulated by [D,Hua,Soibelman].

- Colored refined invariants of torus links can be obtained from refined Chern-Simons theory [Aganagic, Shakirov], [Shakirov], also using some large N duality tricks.

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Conjecture 1

The refined stable pair theory of Y is given by

$$Z_{Y_{\underline{\xi}}}(q, Q_1, \ldots, Q_\ell, y) = \sum_{\mu_1,\ldots,\mu_\ell} \left(\widetilde{W}^{(n-2)}_{\mu_1,\ldots,\mu_\ell}(s,t) \right.$$

$$\left. \prod_{i=1}^{\ell} \left[\left(ts^{-1}Q_i\right)^{|\mu_i|/2} f_{\mu_i}(s,t)^{n-1} P_{\mu_i^t}(t,s;\underline{s}) \right] \right) \Bigg|_{s=qy,\ t=qy^{-1}}$$

where

$$\widetilde{W}^{(n-2)}_{\mu_1,\ldots,\mu_\ell}(s,t) =$$

$$\sum_{\lambda_1,\ldots,\lambda_{\ell-1}} N^{\lambda_{\ell-1}}_{\mu_\ell,\lambda_{\ell-2}} N^{\lambda_{\ell-2}}_{\mu_{\ell-1},\lambda_{\ell-3}} \cdots N^{\lambda_2}_{\mu_3,\lambda_1} N^{\lambda_1}_{\mu_2,\mu_1} f_{\lambda_{\ell-1}}(s,t)^{2-n} P_{\lambda_{\ell-1}}(s,t;\underline{t}).$$

Home Page

Title Page

Page 20 of 24

Go Back

Full Screen

Close

Quit

20 of 24

- $P_\lambda(t,s;\mathsf{x})$, $\mathsf{x}=(x_1,x_2,\ldots)$, are the (t,s)-Macdonald polynomials
- $N^\sigma_{\nu,\lambda}$ are the (s,t)-Littlewood-Richardson coefficients

$$P_\nu(t,s;\mathsf{x})P_\lambda(t,s;\mathsf{x})=\sum_\sigma N^\sigma_{\nu,\lambda}P_\sigma(t,s;\mathsf{x}).$$

- $f_\lambda(s,t)$ are refined framing factors,

$$f_\lambda(s,t)=\prod_{\square\in\lambda}s^{a(\square)}t^{-l(\square)},$$

- $\underline{t}=(t^{1/2},t^{3/2},\ldots)$, $\underline{s}=(s^{1/2},s^{3/2},\ldots)$.

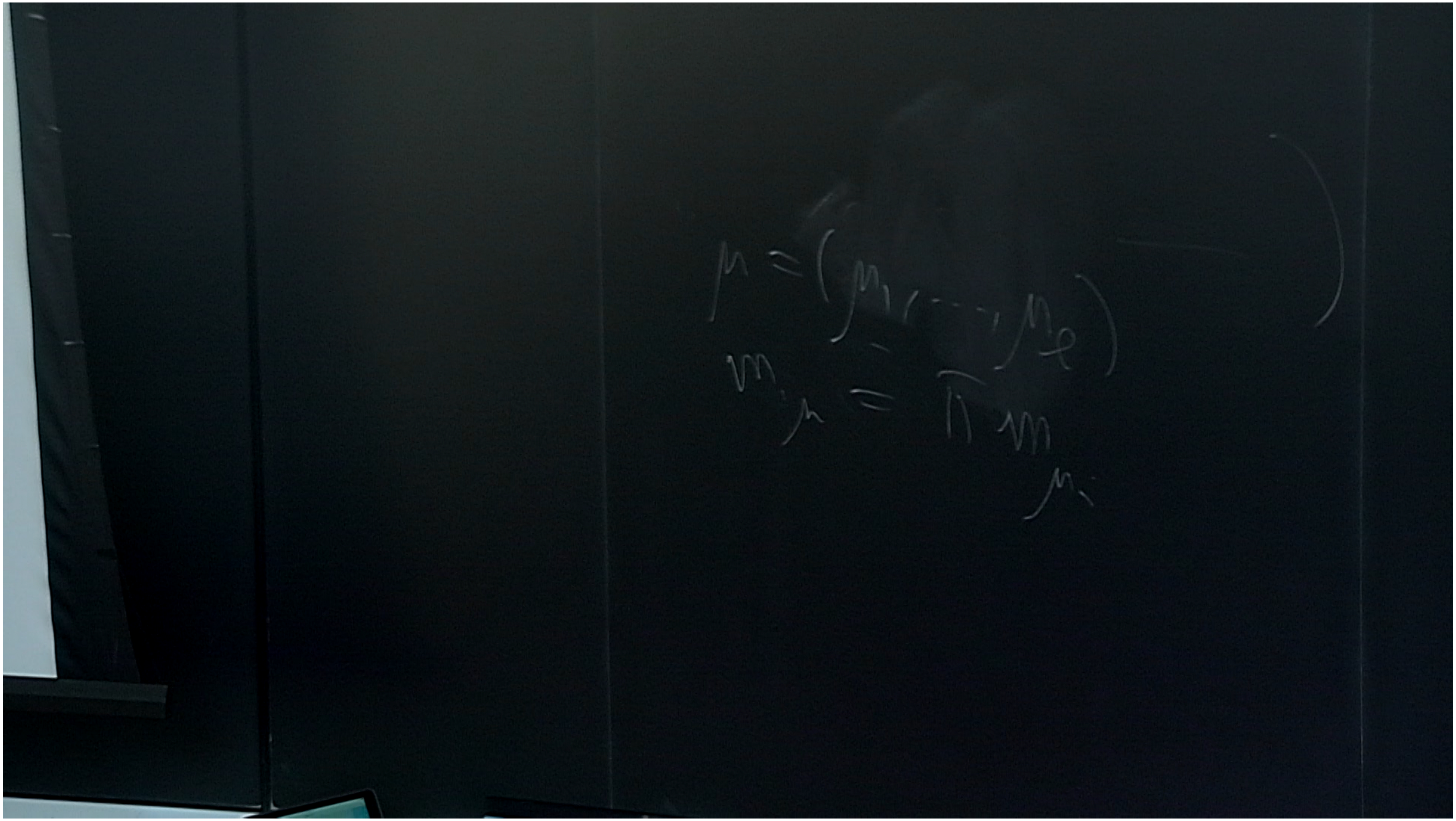
Home Page
Title Page
Page 21 of 24
Go Back
Full Screen
Close
Quit

Conjecture 2: refined Gopakumar-Vafa expansion

$$Z_{Y_{\underline{\xi}}}(q, Q_1, \ldots, Q_\ell, y) = \exp\left(-\sum_{k \geq 1} \sum_{\mu} \frac{m_\mu(Q_1^k, \ldots, Q_\ell^k, 0, \ldots)}{k} \frac{y^{-kr}(qy^{-1})^{kd_{\mu,n}/2} P_{\mu,n}((qy)^{-k}, -y^k))}{(1-(qy)^{-k})(1-(qy^{-1})^k)} \right)$$

- $m_\mu(x_1, \ldots)$ monomial symmetric functions.
- $P_{\mu,n}(u,v)$ perverse Poincaré polynomial of $\mathcal{H}^s_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ for generic $\underline{\alpha}$, μ partition

$$r = m_1 + \cdots + m_\ell.$$


$\mu = (\mu_1, \ldots, \mu_\ell)$
$m_\mu = \prod m_{\mu_i}$

Conjecture 3

For $C = \mathbb{P}^1$ with one marked point,

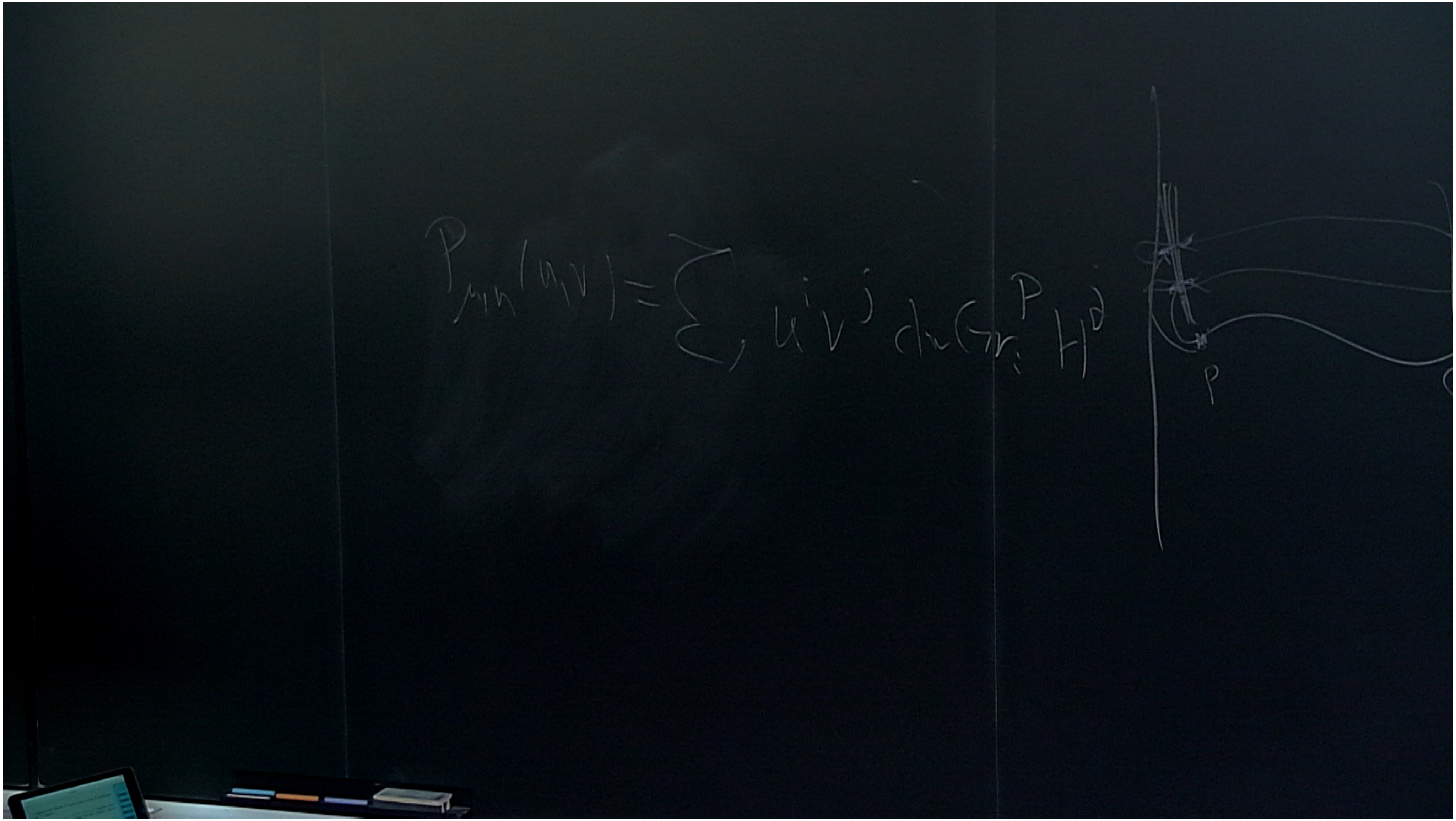
$$WP(\mathsf{Q}, \mathsf{R}; u, v) = P_{\mu,n}(u, v)$$

where μ is the partition of multiplicities of eigenvalues of R (assumed generic).

- agrees with HMW in many rank 2 and 3 examples with R regular

- For $\mu = (2,1)$, $n = \{5,6\}$, $P_{\mu,n}(1,v)$ agrees with Poincare polynomial of Higgs bundle moduli space computed by localization.

- Can one prove the $v = 1$ specialization by counting rational points on wild character varieties?


$P_{m,n}(u,v) = \sum_{i,j} u^i v^j \dim(Gr_i^P H^j)$
P