

Title: Higgs sheaves on a curve and Cohomological Hall algebras

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Abstract: Cohomological Hall algebras associated with preprojective algebras of quivers play a preeminent role in geometric representation theory and mathematical physics. In the present talk, I will introduce and describe CoHAs associated with the stack of Higgs sheaves on a smooth projective curve. Moreover, I will address the connections with representation theory and gauge theory. (This is a joint work with Olivier Schiffmann.)

Higgs sheaves on a curve and Hall algebras

Goal of the talk

X = smooth proj. curve/ $\mathbb{A}^1$

Q. What is the CoHA associated with X ?

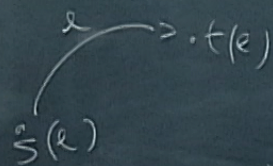
Attention

CoHA = Cohom. Hall algebras à la Schiffmann-Vasserot
= specific example of Kontsevich-Segalman Hall algebra.

Notation

$Q = \text{quiver}$, $Q_0 = \{\text{vertices}\}$, $Q_1 = \{\text{oriented arcs}\}$

$e \in Q_1$,



$\vec{v} = (v_i)_{i \in Q_0}$, $V_i = \mathbb{C}^{v_i}$

$$\text{Rep}(Q, \vec{v}) = \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)})$$

space of reps of Q with fixed
dim. vector $\vec{v}$

Thm (Schiffmann-Vasserot)

$\exists$ an associative algebra structure $\hat{=}$ Hall on

$$\bigoplus_{n \geq 2} H_n^T \left(T^* \left[\frac{\text{Rep}(\mathcal{G}, \vec{v})}{\prod_{i \in \mathcal{Q}_0} GL(V_i)} \right] \right) = \bigoplus_{\vec{v}} H_n^T \left(\left[\frac{\mu_{\mathcal{G}, \vec{v}}^{-1}(0)}{GL(\vec{v})} \right] \right) = \bigoplus_{\vec{v}} H_n^T \left(\frac{T_X \in GL(\vec{v})}{X} \right)$$

$$\frac{S_x}{\hbar}$$

$$\mathcal{Q} = \mathbb{C}$$

$$T = \mathbb{C}^* \times \mathbb{C}^*$$

$$\mu_{G,n}^{-1}(0) = \left\{ (A, B) \in \mathcal{M}_H(n)^{\oplus 2} \mid [A, B] = 0 \right\}$$

... algebra structure $\simeq$ Hall on

$$\bigoplus_{i \in \mathbb{N}} H_*^T \left(T^* \left[\text{Rep}(\mathcal{Q}, \vec{\nu}) / \prod_{i \in \mathcal{Q}_0} GL(V_i) \right] \right) = \bigoplus_{\vec{\nu}} H_*^T \left(\left[\mu_{\vec{\mathcal{Q}}, \vec{\nu}}^{-1}(0) / GL(\vec{\nu}) \right] \right)$$

$\parallel$
 $GL(\vec{\nu})$

$$T \subset (V^*)^{\mathcal{Q}} \times \mathbb{A}^*$$

Q. Is this thm true if we replace $\left[\text{Rep}(Q, \overline{V}) / \text{GL}(V) \right]$ by $\text{Coh}(X)$?

$[B] = 0$ $\left\{ \begin{array}{l} \text{Coh}(X) = \text{algebraic stack of coh. sheaves on } X \end{array} \right.$

Q. Is this thm true if we replace $\left[\text{Rep}(Q, \overline{\mathbb{F}}) / \text{GL}(\overline{\mathbb{F}}) \right]$ by $\text{Coh}(X)$?

$\text{Coh}(X)$ = algebraic stack of coh. sheaves on X

$\alpha = (r, d)$ $(r, d \in \mathbb{Z}, r \geq 0, \text{ if } r = 0 \Rightarrow d \geq 0)$

$\text{Coh}_\alpha(X) =$ "
of rk r and degree d

Higgs sheaves on a curve and Hall algebras

$$T^* \text{Coh}_\alpha(X) = \text{Spec} \text{Sym} H^0(\mathcal{L}_{\text{Coh}_\alpha(X)}^\vee)$$

= stack $\text{Higgs}_\alpha(X)$ of Higgs sheaves

$(E, f: E \rightarrow \Omega_X \otimes E)$ on X , where E is
a coh. sheaf on X of rk r and degree d

Thm (S-Schiffman)

$\exists$ ass. algebra structure $\rightarrow$ Hall on

$$\bigoplus_{\alpha} H_{*}^{\mathcal{Q}^*}(\text{Higgs}_{\alpha}(X))$$

is Here $\mathcal{Q}^*$ acts by mult. on the Higgs field.

Thm (S-Schiffman)

$\exists$ ass. algebra structure $\rightarrow$ Hall on

$$\text{CotHA}_X := \bigoplus_{\alpha} H_X^{\mathbb{Q}^*}(\text{Higgs}_{\alpha}(X))$$

is Here $\mathbb{Q}^*$ acts by mult. on the Higgs field.

$= \{ \text{vertices} \}$, $Q_1 = \{ \text{oriented arcs} \}$

$\cdot t(e)$

v_i

$$(\vec{v}) = \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)})$$

of reps of Q with fixed
for $\vec{v}$

Thm (Schiffmann-Vasserot)

$\exists$ an associative algebra structure $\hat{H}$ on

$$\text{CoHA}_Q := \bigoplus_{\vec{v}} H_*^T \left(T^* \left[\text{Rep}(Q, \vec{v}) / \prod_{i \in Q_0} \text{GL}(v_i) \right] \right) = \bigoplus_{\vec{v}} H_*$$

$\text{GL}(\vec{v})$

$$T = (L^*)^Q \times \mathbb{C}^*$$

$$\text{Higgs}_\alpha^{\text{nil}}(X) = \{ (E, f) \in \text{Higgs}_\alpha(X) \mid f \text{ is nilpotent} \}$$

$$\text{CoHA}_X^{\text{nil}} = \bigoplus_\alpha H_*^{C^*}(\text{Higgs}_\alpha^{\text{nil}}(X)) \subset \text{CoHA}_X$$

$$= \bigoplus_{\alpha} H_X^T$$

Conj

$$\mathrm{CoHA}_X^{\mathrm{nil}} \otimes_{R_C^*} K_C^* \simeq \mathrm{CoHA}_X \otimes_{R_C^*} K_C^*$$

$$R_C^* = H_C^*(pt), \quad K_C^* = \mathrm{Frac}(R_C^*)$$

nilpotently

A_X

$$\mathrm{CoHA}_X^{\mathrm{zero}} = \bigoplus_d H_*^{(d)}(\mathrm{Higgs}_{(0,d)}(X))$$

$$\mathrm{CoHA}_X^{\mathrm{nil}, \mathrm{zero}} = \bigoplus_d H_*^{(d)}(\mathrm{Higgs}_{(0,d)}^{\mathrm{nil}}(X))$$

Thm (Mumford-Schiffmann)

$\exists$ an action of $\mathrm{CoHA}_X^{\mathrm{zero}}$

on $\bigoplus_n H_*^{\mathrm{c}^*}(\mathrm{Hilb}^n(T^*X))$

Thm (Mumford-Schiffmann)

$\exists$ an action of $\mathrm{CoHA}_X^{\mathrm{zero}}$ on $\bigoplus_n H_*^{c,*}(\mathrm{Hilb}^n(T^*X))$

$\parallel \text{---} \mathrm{CoHA}_X^{h_1, \mathrm{zero}} \text{---} \parallel$

Idea of the proof

1) BNR correspondence

$\text{Higgs}_{(0,d)}(X) \simeq \text{Stack Coh}_{(0,d)}(T^*X)$ of 0-dim
coh sheaves on T^*X of length d .

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1) BNR correspondence $\text{Higgs}_{(0,d)}(X) \simeq \text{Stack Coh}_{(0,d)}(T^*X)$ of 0-dim
coh sheaves on T^*X of length d .

2)

$$\begin{array}{ccc}
 & \text{Hilb}^{[d,n]}(T^*X) & \\
 \swarrow p_{d,n} & & \searrow q_{d,n} \\
 \text{Coh}_{(0,d)}(T^*X) \times \text{Hilb}^{d+n}(T^*X) & & \text{Hilb}^n(T^*X)
 \end{array}$$

Thm (Minkowski-Schiffmann)

an action of $\text{CoHA}_X^{\text{zero}}$

$$I_{d+n} \in \text{Hilb}^{d+n}(T^*X)$$

$$I_n \in \text{Hilb}^n(T^*X)$$

$$T_d \in \text{Coh}_{(0,d)}(T^*X)$$

2 of the proof

BNR correspondence $\text{Higgs}_{(0,d)}(X) \simeq \text{stack Coh}_{(0,d)}(T^*X)$ of 0-dim
coh sheaves on T^*X of length d

$$\text{Hilb}^{[d,n]}(T^*X)$$

$q_{d,n}$

$p_{d,n}$

$$a_{d,n} = (q_{d,n})_* \circ (p_{d,n})^*$$

$$\text{Coh}_{(0,d)}(T^*X) \times \text{Hilb}^{d+n}(T^*X)$$

$$\text{Hilb}^n(T^*X)$$

$$p_{d,n}: (*) \mapsto (\tau_d, I_{n+d})$$

$$q_{d,n}: (*) \mapsto I_n$$

of the proof

BNR correspondence

$\text{Higgs}_{(0,d)}(X) \simeq \text{stack Coh}_{(0,d)}(T^*X)$ of 0 -dim
coh sheaves on T^*X of length d

Hecke correspondence $\text{Hilb}^{[d,n]}(T^*X)$

$$a_{d,n} = (q_{d,n})_* \circ (p_{d,n})^*$$

$$\text{Coh}_{(0,d)}(T^*X) \times \text{Hilb}^{d+n}(T^*X)$$

$$\text{Hilb}^n(T^*X)$$

$$p_{d,n}: (x) \mapsto (\tau_d, I_{n+d})$$

$$q_{d,n}: (x) \mapsto I_n$$

Rem

$$1) S = \mathbb{P}(\mathcal{O}_X \oplus \Omega_X)$$

$\mathcal{L}_\infty$ = compactifying divisor

$$\mathcal{M}_{S, \mathcal{L}_\infty}(\tau, c_1, \Delta) = \text{moduli}$$

stack of framed sheaves

$$(E, \phi_E: E|_{\mathcal{L}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathcal{L}_\infty}^{\oplus r})$$

$$\underline{\text{Expect}} \text{CoHA}_X^{\text{zero}}, \text{CoHA}_X^{\text{nil, zero}}$$

$$\bigoplus H_*^{\mathbb{C}^* \times (\mathbb{C}^*)^r}(\mathcal{M}_{S, \mathcal{L}_\infty}(\tau, c_1, \Delta))$$

$$\underline{\text{Thm}} (S\text{-Schiffmann})$$

$\exists$ ass. algebra structure $\Rightarrow$ is

$$\text{CoHA}_X := \bigoplus_n H_*^{\mathbb{C}^*}(\text{Higgs}_X(X))$$

Here $\mathbb{C}^*$ acts by mult. on the Higgs

$$SV: \text{CoHA}_{\bullet, \bullet} \hookrightarrow \bigoplus_n H_x^{(C^*)^2 \times (C^*)^n} (M(r, n))$$

$$\Rightarrow \exists \text{Vir} \hookrightarrow \bigoplus_n H_x^{(C^*)^2 \times (C^*)^n} (M(r, n))$$

Higgs_a^{hil}

CoHA_x^{hil}

Conj

CoHA

$$x(C^*)^r \left(\mathcal{M}_{\Sigma, \infty}(\sigma, c, \Delta) \right)$$

$$\mathcal{M}(1,n) = \text{Hilb}^n(\mathbb{C}^2)$$

$$\text{SV: } \text{CoHA}_\Delta \hookrightarrow \bigoplus_n H_x^{(\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^n}(\mathcal{M}(1,n))$$

$$\Rightarrow \exists \text{Vir} \hookrightarrow \bigoplus_n H_x^{(\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^n}(\mathcal{M}(2,n))$$

$$\text{Higgs}_\alpha^{\text{nil}}(X) = \{ \dots \}$$

$$\text{CoHA}_X^{\text{nil}} = \bigoplus_\alpha \dots$$

conj

$$\text{CoHA}_X^{\text{nil}} \otimes R_{\mathbb{C}^\times}$$

$$R_{\mathbb{C}^\times}$$

$$\mathcal{M}_{\text{Sla}}(\sigma, c, \Delta)$$

Thm (S-Schiffmann)

Consider

$$\mathrm{CoHA}_{\mathrm{pt}}^{\mathrm{equiv}} = \bigoplus_{\alpha} H_{\ast}^{(\mathbb{C}^{\times})^2} \left(\mathrm{Higgs}_{\alpha}(\mathbb{P}^1) \right)$$

$$\mathrm{CoHA}_{\mathrm{pt}}^{\mathrm{equiv}, \mathrm{nil}} = \text{---} \parallel \left(\text{---} \overset{\mathrm{nil}}{\parallel} \text{---} \right)$$

These algebras act on

$$\bigoplus_{\vec{v}} H_{\ast}^{(\mathbb{C}^{\times})^2} \times (\mathbb{C}^{\times})^{\sum v_i} \left(M_2(\vec{v}, \vec{w}) \right)$$

$M(\vec{v}, \vec{w})$ = moduli space of framed sheaves

$$(\mathcal{E}, \phi_{\mathcal{E}} : \mathcal{E}|_{\tilde{\mathcal{L}}_{\infty}} \xrightarrow{\sim} \mathcal{F}_{\infty}^{\vec{w}}) \quad \text{where}$$

$\mathcal{E}$ is a torsion-free sheaf on $\mathcal{X}_2 = T^* \mathbb{P}^1 \cup \tilde{\mathcal{L}}_{\infty}$
 with $\Sigma \in K_0(\mathcal{X}_2)$ fixed by $\vec{v}, \vec{w}$. $\tilde{\mathcal{L}}_{\infty} = [\mathbb{P}^1 / \mathbb{Z}_2]$

$$\mathcal{F}_{\infty}^{\vec{w}} = \bigoplus_{i=1}^2 (\mathcal{O}_{\mathbb{P}^1} \otimes \chi_i)^{\oplus w_i} \quad \chi_i = \text{irr rep of } \mathbb{Z}_2$$

= Nakajima quiver variety
 ass with $A_1^{(1)}$ with a non-
 dominant stability cond

Idea of the proof.

1. BNR corr.

$Higgs^{h^1}(P') \simeq \text{stack Coh}_{P'}(T^{\times}P')$
of coh. sheaves $\mathcal{F}$ on $T^{\times}P'$
s.t. $\text{supp}(\mathcal{F})_{\text{red}} \subset \text{zero-section}$
 $\simeq P' \subset T^{\times}P'$

$$\text{CoHA}_X^{\text{zero}} =$$

$$\text{CoHA}_X^{h^1, \text{zero}} =$$

Hecke correspondence:

$$\mathcal{H}(\vec{v}, \vec{v}', \vec{w}) = \left\{ 0 \rightarrow \mathcal{E}_{\vec{v}, \vec{w}} \rightarrow \mathcal{E}_{\vec{v}', \vec{w}} \rightarrow \mathcal{T} \rightarrow 0 \right\}$$

$$\mathcal{T} \in \text{Coh}_{\mathbb{P}^1}(\mathcal{T}^* \mathbb{P}^1)$$

$\mathcal{T}^* \mathbb{P}^1$
- section
 $\subset \mathcal{T}^* \mathbb{P}^1$

Thm (Mumford-Schiffmann)

$\exists$ an action of $\text{Coh}_{\mathbb{A}^1_X}$

$$\text{---} \parallel \text{---} \text{Coh}_{\mathbb{A}^1_X}^{n_1, n_2}$$

$$\text{Hilb}^{[d, n]}(\mathcal{T}^* X)$$

T^*P^1

$\mathbb{C}^2 / \mathbb{Z}_2$



$$\text{Coh}_0(T^*P^1) =$$

$$= \left\{ \mathcal{F} \in \text{Coh}(T^*P^1) \mid \text{Supp}(\mathcal{F}) = \emptyset \right\}$$

$$\subseteq \text{Coh}_{P^1}(T^*P^1)$$

$$\text{Coh}_\infty(T^*P^1) = \text{---} //$$

$$\text{Supp}(\mathcal{F}) = \emptyset$$

$$\mathbb{C} \text{Coh}_{|p|}(\tau^*|p|) \subset \text{Higgs}^{\text{nil}}(|p|)$$

2 copies of

$$\Rightarrow \sqrt{\text{Coh}}_{|p|} \oplus$$

$$\subset \text{Coh}^{\text{nil}}_{|p|}$$

$$\mathbb{C} \text{ Coh}_{|p|}(\mathbb{T}^{\times}|p|) \subset \text{Higgs}^{\text{nil}}(|p|)$$

2 copies of

$$\Rightarrow \mathbb{C} \text{ Coh}_{|p|} \oplus$$

$$\subset \text{Coh}_{|p|}^{\text{nil}}$$

2 copies of

$$(\hat{g}(1))^{+}$$

Coh (S-Schiffmann)

$$\text{CotHA}_{\text{ip}}^{\text{equiv}} \supset \text{subalgebra} \simeq \left(\widehat{\mathfrak{gl}}_2 \right)^+$$

Inm(Schiffmann)

$\exists A_{g,r,d}$ st $\forall X_{/\mathbb{F}_q} = \text{smooth proj curve}$

$A_{g,r,d}(\mathcal{O}_X) = \# \{ \mathcal{F} \text{ abs incomp v.b on } X_{/\mathbb{F}_q} \}$

$\mathcal{C}_X = \{ \text{Weil numbers of } X_{/\mathbb{F}_q} \}$

If $(r,d)=1$, then

$$\sum_{n=0}^{\infty} (-1)^n \dim H_c^n(\text{Higgs}_{(r,d)}^{\text{st}}(X)) t^n = t^{2(1+(g-1)r^2)} A_{g,r,d}(t, -1/t)$$

Conj

$\text{Cot} A_X$ contains the positive part of a Yangian
of a $\mathbb{Z}^{q+1}$ -graded Lie algebra $\mathfrak{g}_X$ s.t.

$$\text{grdim } \mathfrak{g}_X[r, d] = A_{g, r, d}$$

$\mathcal{E}_X$ $X = \mathbb{P}^1$

$$A_{g, r, d} = \begin{cases} 1 & r=1 \\ 0 & r>1 \\ q+1 & r=0 \end{cases}$$

$\mathcal{E}_X$
 $X = \text{elliptic curve}$

$\sum x$

$f = \text{elliptic}$

we

$$A_{r,d} = 1 - (\alpha + \overline{\alpha}) + q$$

S.I. $\supset \Gamma \subset \Gamma_{\text{red}}$

$\supset \Gamma' \subset \Gamma^{\text{AP}}$

$\Gamma) + 9 \rightarrow X = \text{elliptic}$

$\Rightarrow ?? \text{ } \omega_{HA_X}$

$\supset \left(\hat{\hat{\hat{g}}}(1) \right)^+ ??$

$$X_K \xrightarrow{\pi_K} \mathbb{C}^2 / \mathbb{Z}_K \quad \omega \cdot (x, y) = (\omega x, \omega^{-1} y)$$



$$\pi_K^{-1}(0) = D_1 \cup \dots \cup D_{K-1}$$

$$\text{Coh}_\infty(T^{x/p^1}) = \text{---} \text{---} \text{---}$$

$$\text{Supp}(F) = \infty$$

$$\text{Coh}_{p^1}(T^{x/p^1}) = \text{Higgs}$$

2 copies of

$$\Rightarrow \text{Coh}(F)$$

$$2 \text{ copies of } (\hat{g}(1))^\dagger$$

$$X_k \xrightarrow{\pi_k} \mathbb{C}^2 / \mathbb{Z}_k \quad \omega \cdot (x, y) = (\omega x, \omega^{-1} y)$$

$$\pi_k^{-1}(0) = D_1 \cup \dots \cup D_{k-1}$$

$$\bigotimes_{i=1}^{k-1} \text{Cot} A_{D_i}^{\text{equ}} \quad \bigvee (g|_k)^+$$

$$\text{Coh}_\infty(T^{x/p^1}) = \text{---} \text{||||}$$

$$\text{Supp}(F) = \infty$$

$$\text{Coh}_{p^1}(T^{x/p^1}) = \text{Higgs}^{\text{hol}}$$

$$\Rightarrow \text{Cot} A_{D_i}$$

$$2 \text{ copies of } \bigvee (g|_1)^+$$