Title: Geometric interpretation of Witten's d-bar equation

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Abstract: The Witten d-bar equation is a generalization of the parametrized holomorphic curve equation associated to a holomorphic function (superpotential) on a Kahler manifold X. It plays a central role in the work of Gaiotto-Moore-Witten on the "algebra of the infrared".

The talk will explain an "intrinsic" point of view on the equation as a condition on a real surface S embedded into X (i.e., not involving any parametrization of S). This is possible if S is not a holomorphic curve in the usual sense.

Geometric interpret. of Witten's $\bar\partial$-equation

① WE & Grad. flow.

Kähler

$$(X, h) \xrightarrow{W} \mathbb{C}, \qquad h = g + i\omega$$

$$\varphi : \mathbb{C} \longrightarrow X, \qquad \mathbb{C} \ni \sigma = s + it$$

$$\frac{\partial \varphi}{\partial \bar\sigma} = \mathrm{Grad}_h(W)_{\varphi(\sigma)} \quad \text{in } T_{\varphi(\sigma)}X$$

$$\frac{\partial \varphi}{\partial \bar\sigma} = \frac{\partial \varphi}{\partial s} + i\,\frac{\partial \varphi}{\partial t}$$

usual vect. field on X

antilinear

$$h_x : T_x X \to T^*_x X$$

$$\mathrm{Grad.} \mapsto dW$$

In local holom. coord. $z_1 \dots z_n$, $z_\nu = \varphi_\nu(\sigma)$

$$\frac{\partial \varphi_\nu}{\partial \bar\sigma} = \sum h^{\mu\bar\nu}(\varphi(\sigma))\, \overline{\frac{\partial W}{\partial z_\mu}(\varphi(\sigma))}$$

NB. $\mathrm{Grad}_h(W) =$ usual grad. of $\mathrm{Re}(W)$ w.r.t. g.

Compare w. usual grad. flow

$$\underset{\text{Riem}}{(M,g)} \xrightarrow{f} \mathbb{R}$$

$$\gamma : \mathbb{R} \to M \qquad \frac{d\gamma}{dt} = \mathrm{grad}_g(f)_{\gamma(t)}$$

Reasons to consider WE

Fan-Jarvis-Ruan

GMW

1) Analogy between d and $\bar{\partial}$

$H^{\bullet}(X,\mathbb{C})$	$H^{\bullet}(X,\mathcal{O})$
$\underline{\mathrm{LocSys}_G}$	$\underline{\mathrm{Bun}_G}$

Physical reason:

Jarvis-Ruan

W

(X, θ)

Physical reason: it is the GF of GF

$$M = X \qquad f = \mathrm{Re}(W)$$

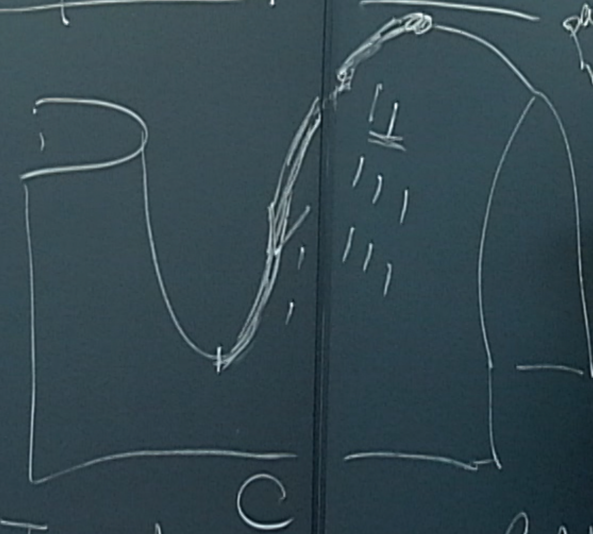
$$\mathrm{grad}_g(\mathrm{Re}\,W) = \mathrm{Ham}_\omega(\mathrm{Im}(W))$$

Grad. flow = Hamilt. flow
" extremals of action

$$S[\gamma] = \int p\,dq + \mathrm{Im}(W)\,dt$$

GF for S = 2-dim WE

Properties of GF



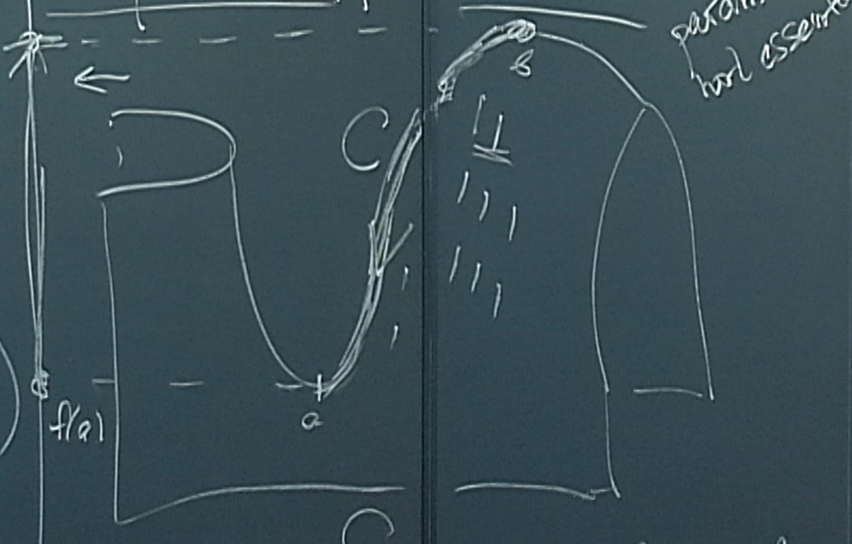
parametrization not essential

2)

1) Trajectories as embedded curves: visible.

$$T_x C = T_x\left(f^{-1}(f(x))\right)^{\perp}$$

Properties of GF



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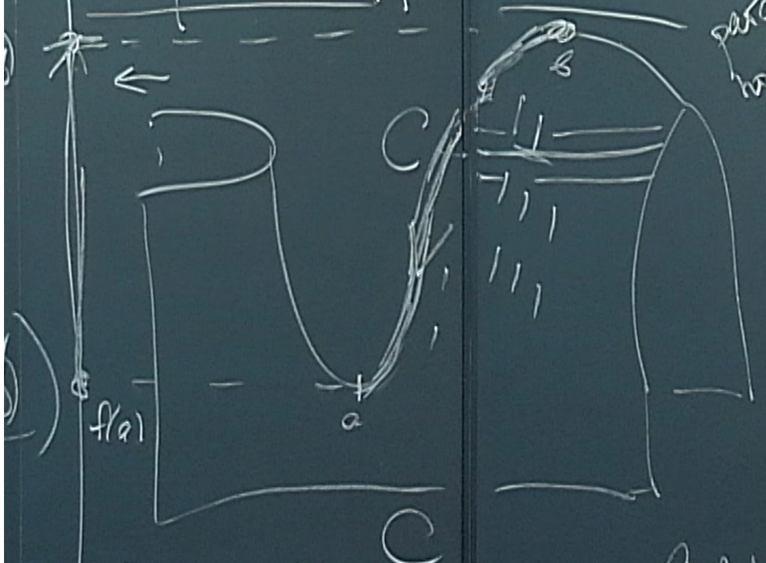
2) For C interpolating between Morse pts a, b need ∞ time $\gamma : (-\infty, +\infty) \to X$

But $C \longrightarrow (f(a), f(b))$ is 1:1

Properties of GF



paramtr t not essential

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Want: similar properties for WE

Rem. 1 In $\mathbb{C}$-case

$\Big(T_x W^{-1}(W$

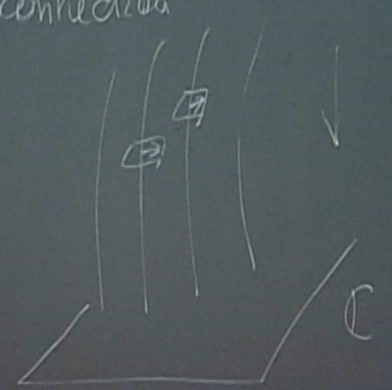
Rem. 1 In $\mathbb{C}$-case we can form

$$\left(T_x W^{-1}(W(x))\right)^{\perp} = \underline{\underline{\Pi_x}} = \mathbb{C}\cdot \mathrm{Grad}_b(W)_x$$

$\mathbb{C}$-line

Non-integrable distr.

"sympl. connection"



Suppose $\varphi : \underset{\mathbb{C}}{U} \to X$ satisfies WE

$\Lambda = \varphi(U) \subset X$ embedded surface

? Can we tell from Λ if $\exists \varphi$?

Claim. Yes, unless Λ is a holom. curve.

$V = T_x X$

$\Pi =$ $\mathbb{C}$-line

Call Λ an approx. surface to Π !!
if this holds,

1st order diff. condition

Empty when dim $X = 1$

almost empty when dim $X = 2$

Becomes more restrictive as dim X grows

NB. Not true (in general)
that $T_x\Lambda = \Pi_x \ \forall x \in \Lambda$
(then Λ holom).

But it cannot be far removed.

$\mathbb{C}\cdot(T_x\Lambda) \supset \Pi_x$

$\frac{\partial \varphi}{\partial \sigma}$

Rem. 1 In

$(T_x$

Non-inte

"sympl.

② Real surfaces in $\mathbb{C}$-mflds

$\Lambda \subset X$

real 2d $\qquad \mathbb{C}$

$x \in \Lambda$

umbilic: $T_x\Lambda$ is a $\mathbb{C}$-line

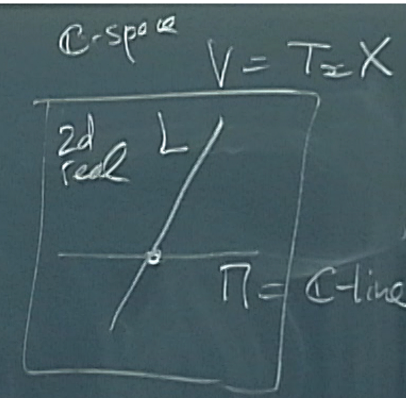
totally real: $T_x\Lambda$ 2d real but not complex

$\Lambda = \Lambda_{umb} \sqcup \Lambda_{RR}$

closed $\qquad$ open

Call (L, Π) in proximity if

$\Pi \subset \mathbb{C} \cdot L$

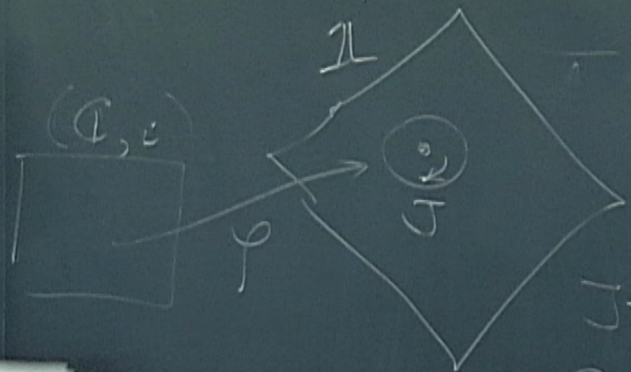


$G_R(2, V) \supset Prox(\Pi)$

$4n-4 \qquad 2n$

$n \geq 2$

Now suppose $\Lambda = \varphi(U)$ φ sol



$J = (d\varphi)_*(i)$

$\mathbb{C}$-str. on Λ

Claim Over $\Lambda_{\mathbb{RR}}$ is defined by Λ uniquely

$$G_{\mathbb{R}}(2,V) = \mathbb{P}_{\mathbb{C}}(V) \sqcup G_{\mathbb{RR}}(2,V)$$

$4n-4$ $\quad$ $2n-2$

dim count: $\Lambda = X$

$\dim_{\mathbb{C}} X = 1$ all pts umbilic

$= 2$ $\quad$ $< \infty$ umbil pts for generic compact

> 2 $\quad$ typically no umbilics.

③ Reminder on $\bar\partial$ $\qquad \bar{V} = (V, -J)$

$$V = (V, J)$$

$\mathbb{C}$-space real $J^2 = -1$

$$A : (L, J_L) \xrightarrow[\mathbb{R}\text{-linear}]{} (V, J_V)$$

$\bar\partial(A) : L \longrightarrow \bar{V}$ antilinear

$$l \longmapsto A(Jl) - J(Al)$$

$= 0$ iff A is $\mathbb{C}$-linear

Usually: $V \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes V$

$$J \qquad V_J^{1,0} \oplus V_J^{0,1}$$

$$J = +i \qquad J = -i$$

$\bar\partial$ $\{(V, J) \to (V_{\mathbb{C}}, i)\}$ is

an anti-isom $V \longrightarrow V_J^{0,1}$

Call (L

proximi

Π

$G_R(2$

$4n-$

$\mathbb{C}\otimes V$

$\oplus\, V_J^{0,1}$

$J = -i$

is

$\Rightarrow V_J^{0,1}$

<u>This gives:</u> $\mathbb{C}\mathrm{Str}(V)$

$$\Big\updownarrow$$

$$\left\{ \Pi \in G_{\mathbb{C}}(n, V_{\mathbb{C}}) \;\middle|\; \begin{array}{l} \Pi \cap V = 0 \\ \Pi \oplus \overline{\Pi} = V \end{array} \right\}$$

For $n = 2$

$$\mathbb{P}_{\mathbb{C}}(V_{\mathbb{C}}) - \mathbb{P}_{\mathbb{R}}(V)$$

$$\mathbb{C}P^1 - \mathbb{R}P^1$$

<u>Pf of claim</u>

$L = T_x\Lambda$

$$\underline{\underline{\mathbb{C}\cdot L = \mathbb{C}\otimes L}}$$

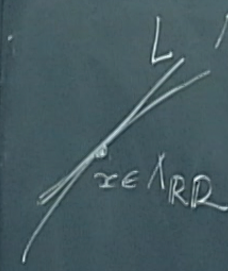
$$\Pi \qquad \Pi_x$$

$\Rightarrow \exists!\, J$ on L s.t.

$$\partial\{(L,J) \to T_x X\}$$

$$\searrow \qquad \nearrow$$

$$\Pi_x$$



Λ

$\mathbb{R}$

Call Λ admissible if $\forall x \in \Lambda$

$\mathbb{C}\cdot(T_x\Lambda) \supset \Pi_x$

$T_x\Lambda \cap \Pi_x = \begin{cases} \text{either } 0 \\ \text{or } \Pi_x \end{cases}$

Cor. An admissible Λ carries a $\mathbb{C}$-structure on $\Lambda_{\mathbb{R}\mathbb{R}}$

$\varphi : U \to X$

Parameter-free interpretation of WE

Λ admissible

x totally real

not $\mathbb{C}$-linear

$\{\mathcal{E}_x : T_x\Lambda \to T_x X$

$$\begin{tikzcd} T_x\Lambda \arrow[r, "\mathcal{E}_x"] \arrow[dr, "\overline{\partial}\mathcal{E}_x"'] & T_xX \\ & \Pi_x \arrow[u] \end{tikzcd}$$

J_x

$\mathbb{C}\cdot \mathrm{Grad}_x \in \Pi_x$

$\mathrm{Grad}_h(W)_x$

$\overline{\partial}\mathcal{E}_x\left(\varphi_{*}\left(\frac{\partial}{\partial\sigma}\right)\right) = \mathrm{Grad}_x$

define ξ_x like this:

$\xi_x = (\overline{\partial}\mathcal{E}_x)^{-1}(\mathrm{Grad}_x)$

? Can we tell $\exists\varphi$?

Claim. Yes, unless

...tation of WE

not $\mathbb{C}$-linear

$$\varepsilon_x : T_x\Lambda \longrightarrow T_xX$$

J_x

$\bar{\partial}\varepsilon_x$

$\mathbb{C}\cdot$Grad

$\in \Pi_x$

$= \mathrm{Grad}_h(W)_x$

Call Λ an totally real instanton surface if it is admissible and

ξ is a holomorphic vect. field w.r.t. $\bar{J}$,

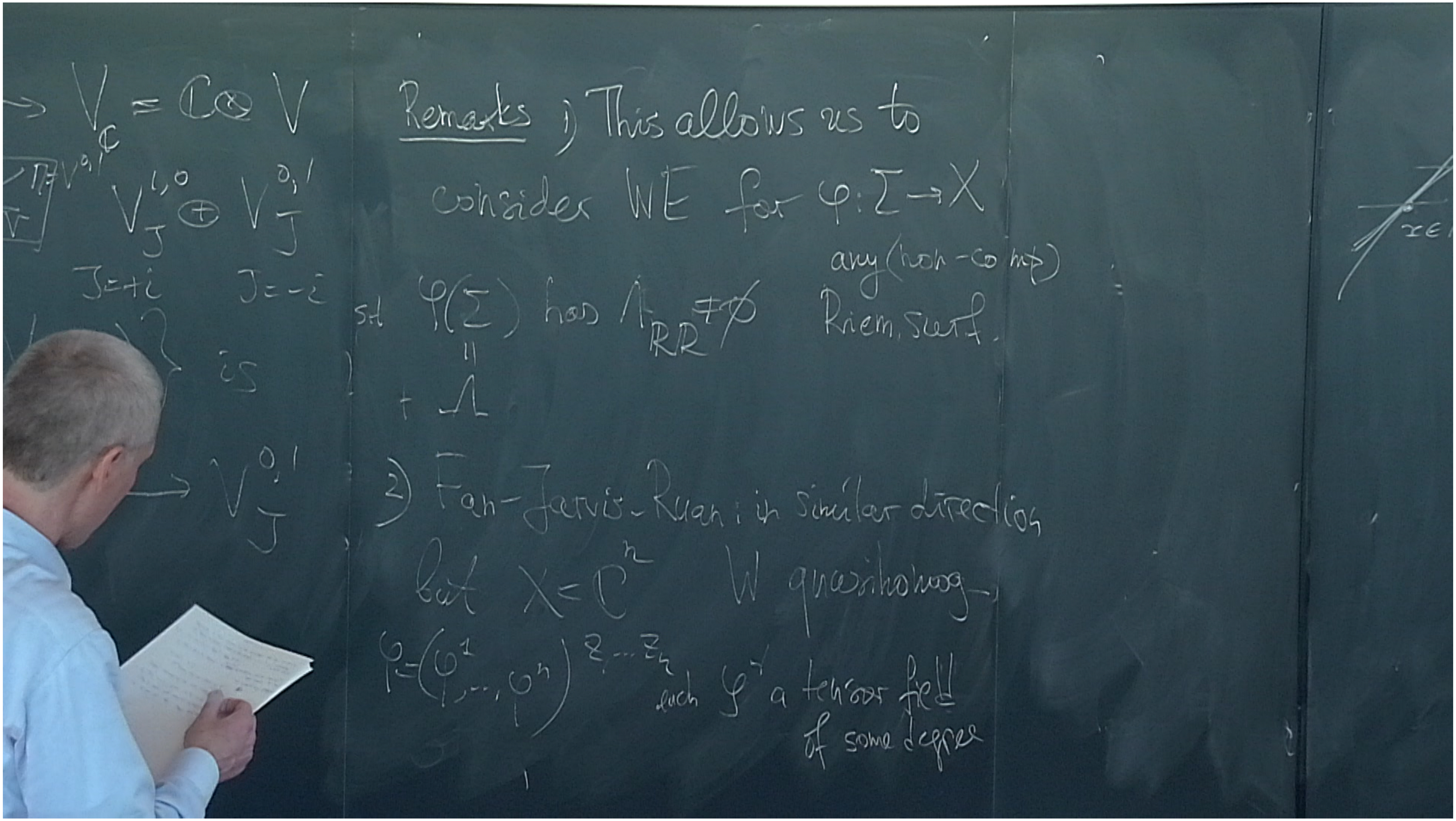
2^{nd} order condition

Th.: Λ is an instanton surf

$\Updownarrow$ $\qquad \varphi$ a solution

locally $\Lambda = \varphi(U)$

(integrate the hol. field ξ)


$V_{\mathbb{C}} = \mathbb{C} \otimes V$
$V_J^{1,0} \oplus V_J^{0,1}$
$J = +i \quad J = -i$
is
$\to V_J^{0,1}$
Remarks 1) This allows us to
consider WE for $\varphi : \Sigma \to X$
any (non-comp)
s.t. $\varphi(\Sigma)$ has $\Lambda_{RR} \neq \emptyset$ Riem. surf.
$+ \Lambda$
2) Fan-Jarvis-Ruan: in similar direction
but $X = \mathbb{C}^n$ W quasihomog.
$\varphi = (\varphi^1, \dots, \varphi^n)$ $z_1 \dots z_n$
each φ^i a tensor field
of some degree

Solutions with asymptotic boundary conditions and projection to polygon

GMW

$\gamma : \mathbb{R} \to X$

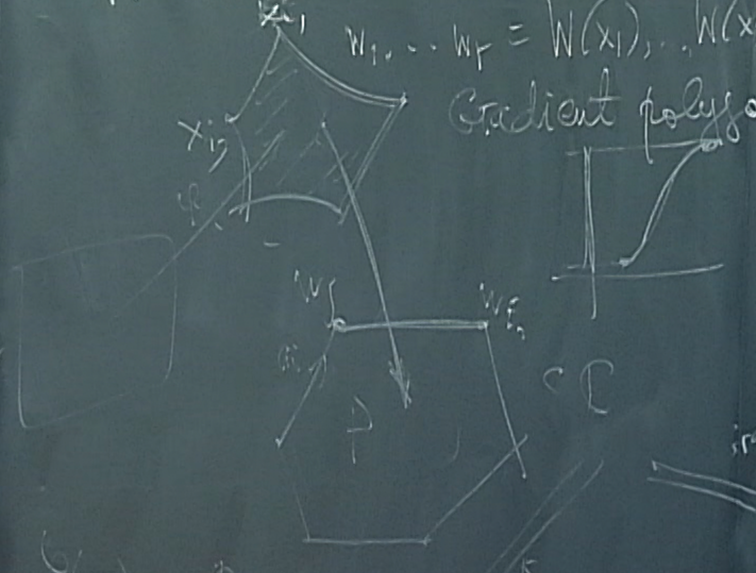
$\varphi(\sigma) = \gamma(\mathrm{Re}(a\sigma))$ satisfies WE

$\updownarrow$

γ grad. traj. for $\mathrm{Re}\left(\frac{a}{2} W\right)$

W morse $x_1 \dots x_r$

$w_1, \dots w_r = W(x_1), \dots W(x_r)$



$\varphi(\sigma) \sim \mathrm{Re}\left(\frac{a_i}{2} W\right)$

Ca

i

Th

loc

$$\psi(\sigma) = W(\varphi(\sigma)) : \mathbb{C} \to \mathbb{C}$$

$$\frac{\partial \psi}{\partial \bar\sigma} = \|dW\|^2 \in \mathbb{R}_+ \subset \mathbb{C}$$

For $\mathbb{R}$-Morse functions:

positivity argum.

Here: Rotation # argument

Claim For such solutions φ!

$\Lambda = \varphi(\mathbb{C}) \subset X$ projects by W

in a 1:1 way

unless Λ passes through

other crit. pts

Solut

GMW

γ:

$\varphi(\sigma$

$\overline{V} = (V, -J)$

$J)$

$J^2 = -1$

(V, J_V)

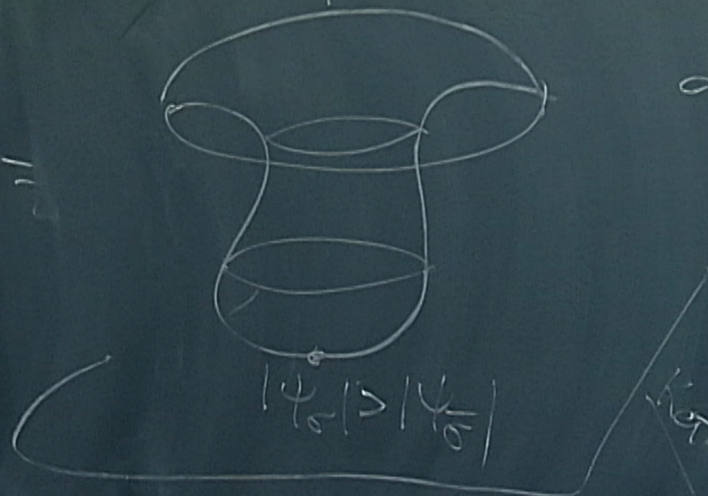
near

antilinear

$-J(A\ell)$

-linear

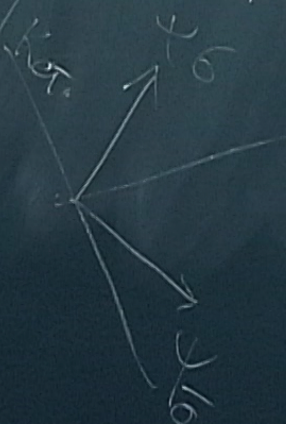
A priori ψ can have folds $\psi(\sigma) = W(\varphi(\sigma)) : \mathbb{C}$



$d\psi$ - has Ker.

$\|$

$|\psi_\sigma| = |\psi_{\bar\sigma}|$



$\dfrac{\partial \psi}{\partial \bar\sigma} = \| d$

For R-Morse fu

po

Here: Rotation