Title: The Hitchin system, past and present

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Abstract: The talk will be a survey of Higgs bundles, their moduli spaces and the associated fibration structure from a historical, and personal, point of view.

The Hitchin system, past and present

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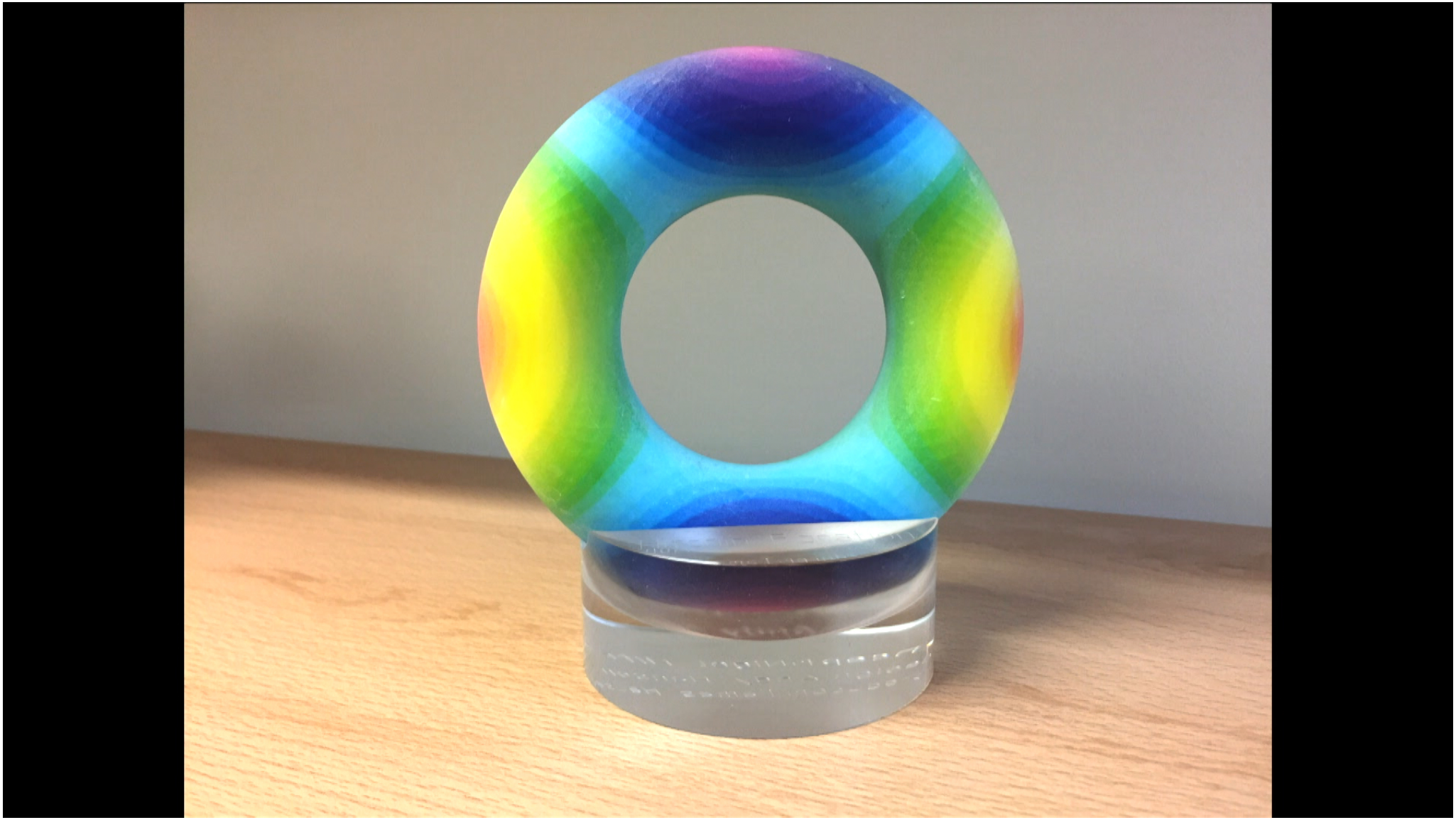
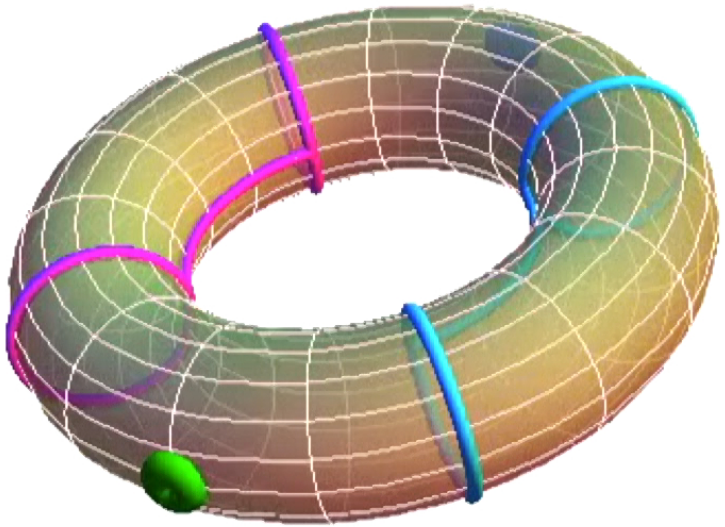
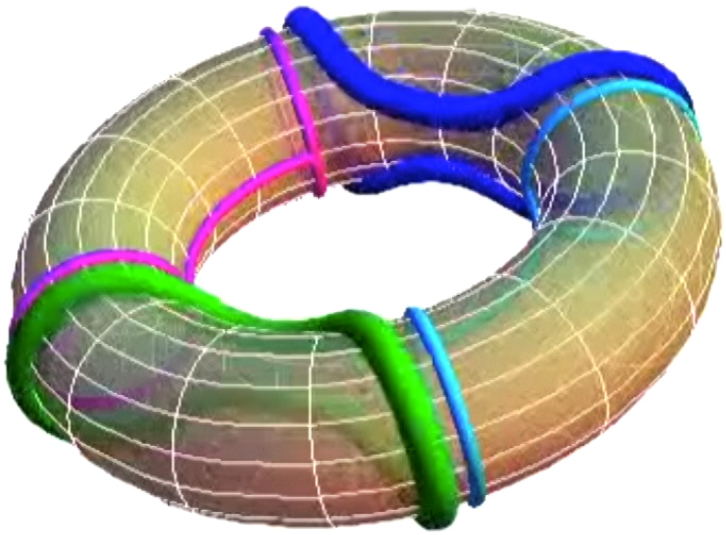
4.2 A toy model $\mathcal{M}_{toy}$

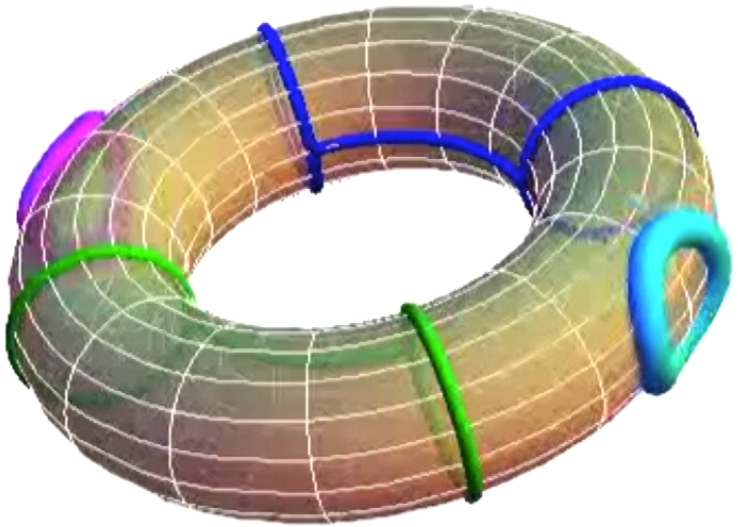
Unfortunately, even when $g = 2$ the moduli space $\mathcal{M}$ is already 6 dimensional, too big to serve as an instructive example. We rather choose $\mathcal{M}_{toy}$, the moduli space[1] of stable parabolic Higgs bundles on $\mathbb{P}^1$, with four marked points, in order to show how our later constructions work. We choose this example because it is a complex surface, and can be constructed explicitly.

Tamas Hausel PhD thesis (1998)









- elliptic curve E
- involution $u \mapsto -u$, 4 fixed points u_1, u_2, u_3, u_4
- $\sigma : E \times \mathbf{C} \to E \times \mathbf{C}$, $\sigma(u, z) = (-u, -z)$

  fixed points $(u_i, 0)$

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  fixed points $(u_i, 0)$
- quotient by σ: identify pairs of points
- blow up the four singularities
- symplectic form $dz \wedge du$

- $h : M \to \mathbf{C}$

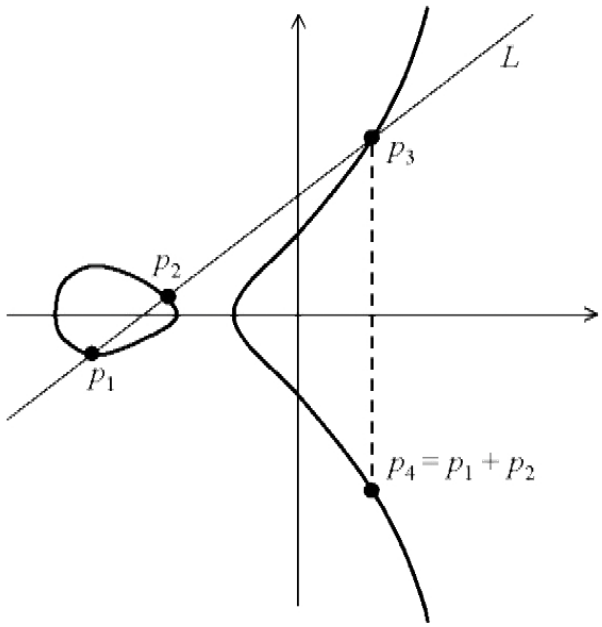
  $h(u, z) = z^2$

- $h^{-1}(b) \cong E$ if $b \neq 0$

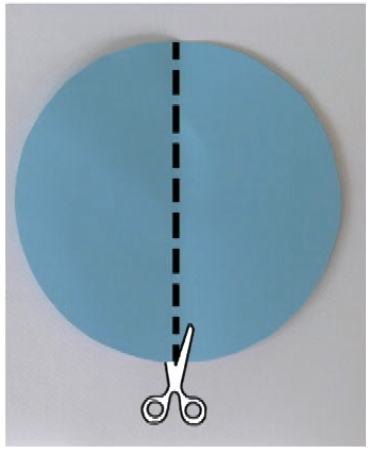
- $h^{-1}(0) = E/\pm 1 = \mathbf{P}^1$

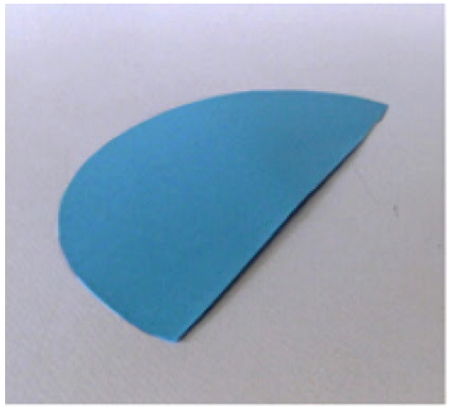
  + 4 more copies of $\mathbf{P}^1$

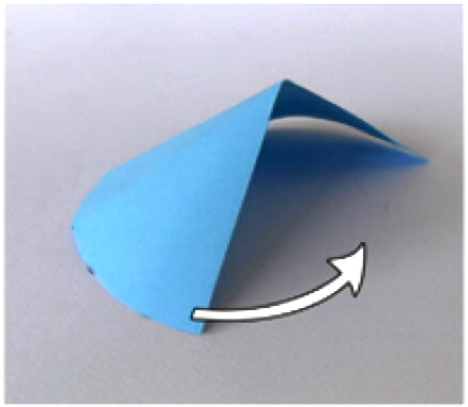
- real points of a torus E: two circles



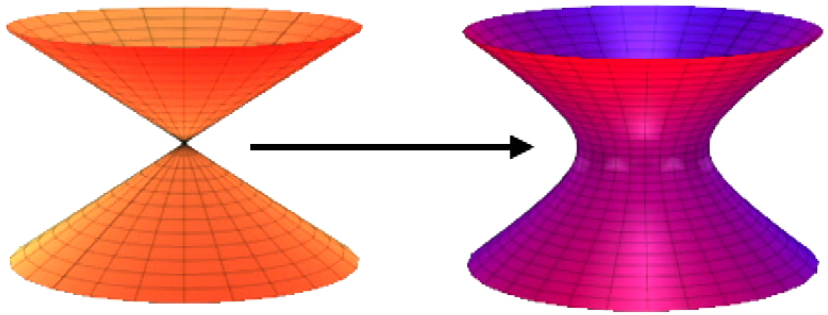
- identify x with $-x$

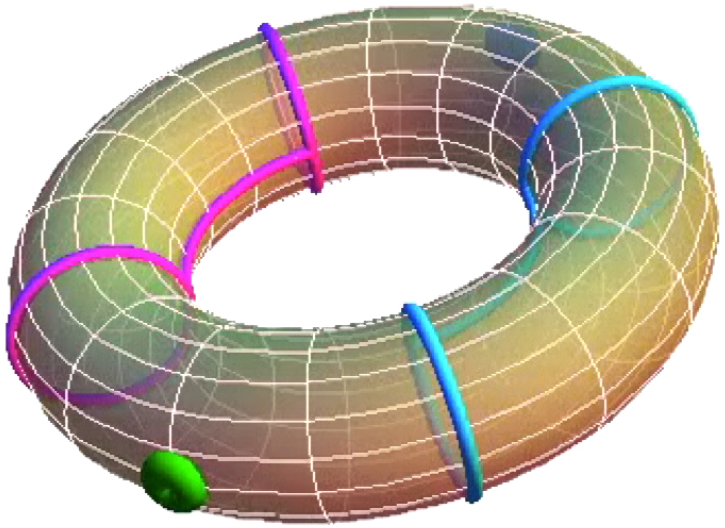






- blow up singularity to a circle:





- Riemann surface Σ
- Vector bundle E + hermitian metric
- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}\, E \otimes K)$
- equations $F_A + [\Phi, \Phi^*] = 0$

HIGGS BUNDLES

1977.... INSTANTONS

- anti-self-dual Yang-Mills, invariant under x_4-translation
- $\phi \sim A_4$
- boundary conditions $\int_{\mathbf{R}^3} \|F\|^2 + \|\nabla\phi\|^2 < \infty$

N.J.Hitchin *On the construction of monopoles*, Comm.Math.Phys. **83** (1982) 579–602.

- E.Corrigan & P.Goddard: *Construction of instanton and monopole solutions and reciprocity*, Ann. of Phys. **154** (1984) 253–279.

- ADHM construction of instantons:

  4 dimensions $\Leftrightarrow$ 0 dimensions (matrices)

- Nahm construction of monopoles:

  3 dimensions $\Leftrightarrow$ 1 dimensions (ODE)

- 2 dimensions $\Leftrightarrow$ 2 dimensions??

term have acquired special values. One can obtain first order equations which imply the field equations, and these are deduced from eqs. (6):

$$F_{ij}^{a} = \pm e\, \epsilon_{ij} \epsilon^{abc} \psi^{b} \phi^{c} \ ,$$

$$D_i \psi^{a} \pm \epsilon_{ij} D_j \phi^{a} = 0 \ . \qquad (11)$$

Unfortunately, in this case the model is not interesting because the action is always zero.

- connection on $\mathbf{R}^2$
- $A_i(x)$ + Higgs fields ϕ_1, ϕ_2, $n \times n$ skew Hermitian
- curvature $F_{ij} = [\nabla_i, \nabla_j]$
- equations

  $F_{12} + [\phi_1, \phi_2] = 0 = [\nabla_1, \phi_2] + [\nabla_2, \phi_1] = [\nabla_1, \phi_1] + [\nabla_2, \phi_2]$

- $[\nabla_1 + i\nabla_2, \phi_1 + i\phi_2] = 0$ Cauchy-Riemann
- $z = x_1 + ix_2 \in \mathbf{C}$
- $\Phi = (\phi_1 + i\phi_2)d(x_1 + ix_2)$
- $F + [\Phi, \Phi^*] = 0$ conformally invariant

  $\Rightarrow$ define on any Riemann surface

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  $\Rightarrow$ define on any Riemann surface

1983 .... HYPERKÄHLER QUOTIENTS

- manifold $I, J, K : T \to T$
- $I^2 = J^2 = K^2 = IJK = -1$
- metric g
- Hermitian 2 -forms

  $\omega_1(X, Y) = g(IX, Y) \quad \omega_2(X, Y) = g(JX, Y) \quad \omega_3(X, Y) = g(KX, Y)$
- g is hyperkähler if $d\omega_1 = 0 = d\omega_2 = 0 = d\omega_3$

- (M, ω) symplectic manifold, G-action preserving ω
- $x \in \mathfrak{g}$ defines vector field X
- $\mathcal{L}_X \omega = 0 \Rightarrow i_X \omega = d\mu$ moment map
- $\mu : M \to \mathfrak{g}^*$
- $\mu^{-1}(0)/G$ is symplectic

- group action G on M, preserving $\omega_1, \omega_2, \omega_3$
- $x \in \mathfrak{g}$ defines vector field X
- $\mathcal{L}_X \omega_i = 0 \Rightarrow i_X \omega_i = d\mu_i$
- $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$
- $\mu^{-1}(0)/G$ is hyperkähler

N.J.Hitchin, A.Karlhede, U.Lindström & M.Roček *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535–589.

M.F.Atiyah & R.Bott, *Yang-Mills and bundles over algebraic curves*, Proc. Indian Acad. Sci. Math. Sci. **90** (1981) 11-20

- Σ compact Riemann surface
- $\mathcal{A}$ = space of connections on a principal $G = U(n)$ bundle
- $\nabla - \nabla' \in \Omega^1(\mathfrak{g})$ flat affine space
- complex $I : \Omega^1(\mathfrak{g}) \to \Omega^1(\mathfrak{g})$
- Kähler $\omega(\dot{A}, \dot{B}) = \int_\Sigma \mathrm{tr}(\dot{A}\dot{B})$

- $\mathcal{G}$ = group of gauge transformations
- acts on $\mathcal{A}$
- $L(\mathcal{G}) = \Omega^0(\mathfrak{g}), L(\mathcal{G})^* = \Omega^2(\mathfrak{g})$
- $\mu : \mathcal{A} \to \Omega^2(\mathfrak{g})$ = curvature
- $\mu^{-1}(0)/\mathcal{G}$ = moduli space of flat connections

- $\mathcal{A}$ complex: $\Omega^1(\mathfrak{g}) \cong \Omega^{0,1}(\mathfrak{g}^c)$
- $\nabla^{0,1} = \bar{\partial}$-operator
- quotient is Kähler = moduli space of stable bundles

S.K.Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geom. **18** (1983) 269-277

- $\mathcal{A}$ complex: $\Omega^1(\mathfrak{g}) \cong \Omega^{0,1}(\mathfrak{g}^c)$
- $\nabla^{0,1} = \bar{\partial}$-operator
- $T^*\mathcal{A}$ complex symplectic manifold $= \mathcal{A} \times \Omega^{1,0}(\mathfrak{g}^c)$
- $(\bar{\partial}_A, \Phi) \in T^*\mathcal{A}$
- $\mathcal{A}$ flat Kähler $\Rightarrow T^*\mathcal{A}$ hyperkähler

- $\omega_2 + i\omega_3$ holomorphic symplectic structure
- moment map $\bar{\partial}_A \Phi$
- ω_1 Kähler structure
- moment map $F_A + [\Phi, \Phi^*]$
- hyperkähler quotient = solutions to equations modulo gauge equivalence.

EXISTENCE

- Equations $\bar{\partial}_A \Phi = 0 = F_A + [\Phi, \Phi^*]$
- $\Phi = 0$ flat $U(n)$-connection
- $\pi_1(\Sigma) \to U(n)$
- $A_i, B_i \in U(n), [A_1, B_1][A_2, B_2] \dots [A_g, B_g] = 1$
- implicit function theorem $\Rightarrow$ solution for small Φ

N.J.Hitchin, *Metrics on moduli spaces*, The Lefschetz centennial conference, (Mexico City, 1984), 157-178, Contemp. Math., **58** AMS Providence, RI, 1986

- (E, Φ) stable if for each Φ-invariant subbundle $V \subset E$,
- .... $\deg V / \operatorname{rk} V < \deg E / \operatorname{rk} E$
- **THM:** A stable Higgs bundle admits a unique Hermitian metric for which the equations hold.

N.J.Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** (1987) 59-126

C.T.Simpson *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867-918

THE CHARACTERISTIC POLYNOMIAL

- characteristic polynomial $\det(x - \Phi) = x^n + a_1 x^{n-1} + \ldots + a_n$
- $a_m \in H^0(\Sigma, K^m)$
- invariant subbundle $\Rightarrow$ factorizes
- $\det(x - \Phi) = 0$ algebraic curve in $T^*\Sigma$, spectral curve S
- S irreducible $\Rightarrow$ (E, Φ) stable

- **THM:** (Uhlenbeck). If $A_n \in L^2_1$ is a sequence of connections such that $F_n \in L^2$ is bounded, then there are unitary gauge transformations g_n such that $g_n(A_n)$ has a convergent subsequence.

- $\|F_n + [\Phi_n, \Phi_n^*]\|_2 < m$

- $\bar{\partial}_A \Phi = 0$: Weitzenböck formula

$$\|\partial_A \Phi\|_2^2 = \langle [F, \Phi], \Phi \rangle_2 + (2g-2)\|\Phi\|_2^2$$

$$\Rightarrow \|[\Phi_n, \Phi_n^*]\|_2^2 \leq c_1 + c_2 \|\Phi_n\|_2^2$$

- A complex $n \times n$ matrix
- $[A, A^*] = 0 \Rightarrow$ diagonalized by unitary matrix
- $|A| = 1 \Rightarrow$ characteristic polynomial and $|[A, A^*]|$ do not vanish simultaneously
- bounds on $\det(x - A)$ and $[A, A^*] \Rightarrow$ bounds on A

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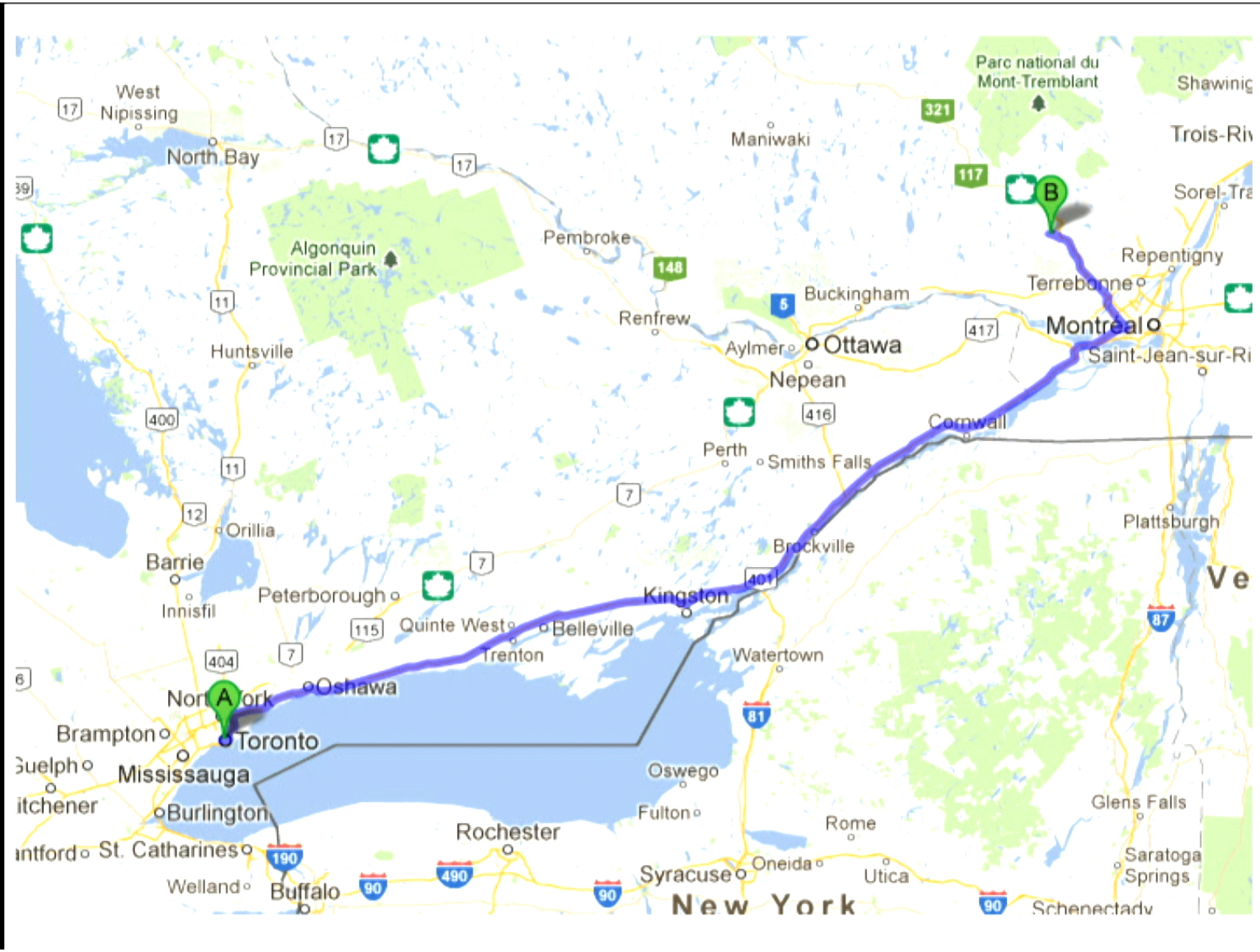
- $F_n + [\Phi_n, \Phi_n^*] = 0$ + bound on characteristic polynomial

  $\Rightarrow$ convergence

- moduli space $\mathcal{M}$

- $p : \mathcal{M} \to \bigoplus_{m=1}^{n} H^0(\Sigma, K^m)$ proper map

THE INTEGRABLE SYSTEM


West Nipissing
North Bay
Algonquin Provincial Park
Maniwaki
Parc national du Mont-Tremblant
Shawinig
Trois-Riv
Sorel-Tra
Pembroke
Repentigny
Terrebonne
Buckingham
Renfrew
Montréal
Aylmer
Ottawa
Saint-Jean-sur-Ri
Huntsville
Nepean
Cornwall
Perth
Smiths Falls
Plattsburgh
Orillia
Brockville
Barrie
Peterborough
Kingston
Innisfil
Quinte West
Belleville
Trenton
Watertown
Oshawa
Brampton
Toronto
Mississauga
Oswego
Guelph
Burlington
Fulton
Rochester
Rome
Glens Falls
St. Catharines
Syracuse
Oneida
Utica
Saratoga Springs
Welland
Buffalo
New York
Schenectady
A
B

Christieville, St Sauveur, Québec

- moduli space $\mathcal{M}$ complex symplectic $\omega_c = \omega_2 + i\omega_3$

- $$p : \mathcal{M} \to \bigoplus_{m=1}^{n} H^0(\Sigma, K^m)$$

  $\{p_i, p_j\} = 0$ Poisson commute

- generic fibre a complex torus

EXAMPLE: $G^c = SL(2,C)$ $g = 2$

B van Geemen and E Previato : *On the Hitchin system*, Duke Math. J. **85** (1996) 659–683.

K Gawędzki and P Tran-Ngoc-Bich: *Self-duality of the* SL_2 *Hitchin integrable system at genus* 2, Comm. Math. Phys. **196** (1998) 641–670.

- $\Sigma : y^2 = \prod_1^6 (x - x_i)$
- (semi) stable rank 2 bundle E, $\Lambda^2 E \cong \mathcal{O}$
- moduli space: $\mathbf{P}^3$
- $T^*\mathbf{P}^3 \subset \mathcal{M}$

- $T^*\mathbf{P}(V) = \{(p,q) \in V \times V^* : p \neq 0, \langle p,q\rangle = 0\}/\mathbf{C}^*$

- **integrable system**

$$H(p,q) = -\frac{1}{128\pi^2}\sum_{i\neq j}\frac{\langle\sigma(ij)p,q\rangle^2}{(x-x_i)(x-x_j)}dx^2$$

  - $\sigma(ij)$ action of a finite Heisenberg group

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- coefficients of $\det(x - \Phi)$: basis for invariant polynomials on $GL(n, \mathbf{C})$
- simple Lie group G: take basis
- : Lie group G, Riemann surface $\Sigma \Rightarrow$ integrable system
- torus fibres are abelian varieties

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- Φ generically breaks the symmetry from G to maximal torus

MIRROR SYMMETRY

- symplectic geometry: A-model
- complex geometry: B-model
- + Calabi-Yau

SYZ MIRROR SYMMETRY

- special Lagrangian fibration
- mirror = dual fibration
- $\mathcal{M}(G)$ is mirror to $\mathcal{M}({}^{L}G)$

R.Donagi & T.Pantev, *Langlands duality for Hitchin systems*, Invent. math. **189** (2012), 653–735.

- torus T dimension n
- dual torus $=$ flat unitary line bundles on T
- $\hat{T} = \mathrm{Hom}(\pi_1(T), U(1)) = H^1(T, \mathbf{R})/H^1(T, \mathbf{Z})$
- T abelian variety so is $\hat{T}$

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- abelian variety $T \subset \mathsf{P}^N$
- restrict hyperplane bundle H to T
- translation $T_a(x) = x + a$
- $(T_a^* H) H^{-1}$ degree zero line bundle on T
- $T \mapsto \widehat{T}$ finite kernel

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CHARACTER VARIETIES

- $\bar{\partial}_A \Phi = 0 \qquad F_A + [\Phi, \Phi^*] = 0$
- $\Rightarrow \nabla + \Phi + \Phi^*$ flat connection
- moduli space $\mathrm{Hom}(\pi_1(\Sigma), G^c)/G^c \cong \mathcal{M}$
- (complex structure J, Higgs bundle complex structure I)

- G^r = split real form of G^c

  e.g. $SL(n, \mathbf{R}) \subset SL(n, \mathbf{C})$

- $\mathrm{Hom}(\pi_1(\Sigma), G^r)/G^r$ intersects a generic torus fibre $h^{-1}(b)$ in points x such that $2x = 0$

L.Schaposnik, *Spectral data for G-Higgs bundles*, DPhil thesis (2013)

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- topological invariants of the G^r bundle partition these points

- e.g. $SO(n) \subset SL(n, \mathbf{R})$ Stiefel-Whitney class w_2

- $Sp(2n, \mathbf{R}) \subset Sp(2n, \mathbf{C})$ split real form
- maximal compact $U(n)$: Chern class $c_1 \in H^2(\Sigma, \mathbf{Z}) = \mathbf{Z}$
- spectral curve $S : x^{2n} + a_1 x^{2n-2} + \ldots + a_n = 0$ has involution $x \mapsto -x$

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- spectral curve $S : x^{2n} + a_1 x^{2n-2} + \ldots + a_n = 0$ has involution $x \mapsto -x$
- generic fibre $=$ line bundles L on S such that $\sigma^* L \cong L^*$

  $= P(S, \bar{S})$ where $(\bar{S} = S/\sigma)$
- order 2 points: L^2 trivial, so $L \cong L^*$

- at a fixed point $z \in S$ of σ action on L_z is ± 1
- no of $+1$ points $\sim$ Chern class

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- no of $+1$ points $\sim$ Chern class
- Langlands dual of $Sp(2n, \mathbf{C})$ is $SO(2n+1, \mathbf{C})$
- split real form $SO(n+1, n)$
- maximal compact $S(O(n+1) \times O(n))$: no Chern class

- dual of $P(S,\bar{S}) \cong P(S,\bar{S})/\pi^* H^1(\bar{S}, \mathbf{Z}_2)$
- corresponding partition of points of order 2
- $\Rightarrow$ integer invariant of a flat $SO(n+1,n)$ connection

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B.Collier, *Finite order automorphisms of Higgs bundles*, PhD thesis (2016)