

Title: Symplectic geometry related to G/U and 'Sicilian theories'

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Abstract: We construct an action of the Weyl group on the affine closure of the cotangent bundle on G/U . The construction involves Hamiltonian reduction with respect to the 'universal centralizer' and an interesting Lagrangian variety, the Miura variety. A closely related construction produces symplectic manifolds which play a role in 'Sicilian theories' and whose existence was conjectured by Moore and Tachikawa. Some of these constructions may be reinterpreted, via the Geometric Satake, in terms of the affine grassmannian.

joint D. Kazhdan.

G complex reductive $SL_n(\mathbb{C})$

U a max. unipotent subgr-p.

$$\text{e.g. } U = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$W := \text{Weyl gr-p.}$

$\mathcal{D}(G/U)$ algebraic diff. ops on G/U .

Thm (Gelfand-Graev 1960's).

$\exists$ natural action of W

on $\mathcal{D}(G/U)$ by algebra automorphisms.

E.g. $G =$

kazhdan.

ductive $SL_n(\mathbb{C})$

unipotent subgr-p.

$\begin{pmatrix} * & \\ & 1 \end{pmatrix}$

gr-p.

$\mathcal{D}(G/U)$ algebraic diff. ops on G/U .

Thm (Gelfand-Graev 1960's)

$\exists$ natural action of W

on $\mathcal{D}(G/U)$ by algebra automorphisms.

E.g. $G = SL_2$, $U = \begin{pmatrix} 1 & + \\ 0 & 1 \end{pmatrix}$ = stabilizer of element in $\mathbb{C}^2$.

$$G/U = \mathbb{C}^2 \setminus 0.$$

$$\mathcal{D}(G/U) = \mathcal{D}(\mathbb{C}^2)$$

$W = \mathbb{Z}/2$ acts on $\mathcal{D}(\mathbb{C}^2)$ by

$$\begin{aligned} \text{Fourier } x &\mapsto \frac{\partial}{\partial x} \\ y &\mapsto -\frac{\partial}{\partial y} \end{aligned}$$

Moore-Tachikawa (2011).

TQFT with values in holomorphic symplectic m-ds.

Moment map

$$\mathcal{M}(\Sigma) \longrightarrow \prod_{\text{punctures}} \mathfrak{g}^*$$

$$\mu^*: \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma') \rightarrow \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$$



Higgs
branch

$\mathcal{M}(\Sigma)$

Gluing axiom:

$\mathcal{M}(\Sigma)$ holomorphic symplectic.
with a hamiltonian action
of $\prod \mathfrak{g}$
punctures.



$$\Sigma \times \Sigma'$$

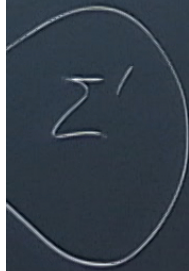
$\rightarrow \prod g^*$
punches.

$$\mu \times \mu: \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma') \rightarrow g^* \times g^* \rightarrow g^*$$

Axiom $\mathcal{M}(\Sigma \times \Sigma') \simeq (\mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma')) // G$

$$:= \left\{ m \in \mathcal{M}(\Sigma) \mid \begin{array}{l} m' \in \mathcal{M}(\Sigma') \\ \mu(m) = \mu(m') \end{array} \right\} / G.$$

axiom;



Boils down

$$\mathcal{M}_n = \mathcal{M}(S^2, n \text{ marked pts}), n=1,2,\dots$$

$\mathcal{M}_1$

should satisfy

$$(\mathcal{M}_m \times \mathcal{M}_n) // G \cong \mathcal{M}_{m+n-2}.$$

Should have a natural action of
the symplectic group $G_n \cong \mathrm{Sp}(n)$

$$\{u^i\} / G.$$

n -marked
pts), $n=1,2,\dots$

M_Λ = principal Nahm
pole.

$$/// G \cong M_{m+n-2}.$$

natural action of

G on M_Λ

$\mathcal{Q}^\Psi(G/U) := \text{quantization of } \mathbb{C}[M_\Lambda]$

operators which act on $f \sim s$.

$$f: G \rightarrow \mathbb{C}$$

$$f(gu) = \Psi(u)f(g) \quad \forall u \in U.$$

Fix nondegenerate character
 $\Psi: U \rightarrow \mathbb{C}^\times$

e.g.

$$\begin{pmatrix} 1 & u_1 & & \\ & 1 & & \\ & & \ddots & \\ 0 & & & u_{n-1} & \\ & & & & 1 \end{pmatrix} \xrightarrow{\text{exp}} \begin{pmatrix} 1 & u_1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \xrightarrow{\text{exp}} \begin{pmatrix} 1 & u_1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$\rho := \text{quantization of } \mathbb{C}[u_1]$.

which act on f.s.

$$\rho \rightarrow \mathbb{C}$$

$$\rho(u) = \psi(u) f(g) \quad \forall u \in U,$$

$$\begin{pmatrix} 1 & u_1 & + \\ & 1 & \\ 0 & & u_{n-1} \\ & & 1 \end{pmatrix}$$

$$\rightarrow \exp(c_1 u_1 + \dots + c_{n-1} u_{n-1})$$

for any fixed $c_1, \dots, c_{n-1}$ s.t.
 $c_i \neq 0 \quad \forall i.$

$$U \subset B = \text{Borel in } G$$

$$\mathfrak{u} = \text{Lie } U$$

$$\mathfrak{b} = \text{Lie } B$$

$\mathcal{D}(G/U)$ quantization of f-s on

$$T^*(G/U) = G \times_U \mathfrak{b}$$

$$\mathcal{D}^{\Psi}(G/U) = \text{--- " ---}$$

$$T^{\Psi}(G/U) = G \times_U (\mathfrak{u} + \mathfrak{b})$$

$\mathcal{D}$
Thm
In
on

intertwiner of f-s on

$$T^*(G/U) = G \times_U \mathfrak{b}$$

— " —

$$T^*(G/U) = G \times_U (\mathfrak{p} + \mathfrak{b})$$

Quantum Ham. reduction,

$$A_1 \xleftarrow{\quad} \mathbb{Z} \xrightarrow{\quad} A_2$$

associative algebras

alg. homs.

$$(A_1 \otimes_{\mathbb{Z}} A_2)^{\mathbb{Z}} :=$$

$$= \left\{ b \in A_1 \otimes_{\mathbb{Z}} A_2 \mid zb = bz \ \forall z \in \mathbb{Z} \right\}$$

Claim $(A_1 \otimes_{\mathbb{Z}} A_2)^{\mathbb{Z}}$ inherits
an alg. structure.

E.g. $G = SL_2$, U

$$G/U = \mathbb{C}^2 \setminus \{0\}$$

$$\mathcal{D}(G/U) = \mathcal{D}(\mathbb{C}^2 \setminus \{0\})$$

$$W = \mathbb{Z}/2 \text{ acts on } \mathcal{D}(\mathbb{C}^2)$$

Fourier x

is linear
element
in $\mathbb{C}$.

U of enveloping alg.

Z = Center of U of

$$\begin{aligned} Z \hookrightarrow U_{\text{of}} &\longrightarrow \mathcal{D}(G/U) \\ &\longrightarrow \mathcal{D}^{\psi}(G/U). \end{aligned}$$

T max. torus in G .

$\mathfrak{t} = \text{Lie } T$ Cartan.

$$Z \simeq (U\mathfrak{t})^W \hookrightarrow U\mathfrak{t} = \text{Synt} \hookrightarrow \mathcal{D}(T).$$

↑ Harish-Chandra

Thm 1. $\exists$ alg. isom

$$\mathcal{D}(G/U) = \left(\mathcal{D}^{\psi}(G/U) \otimes \mathcal{D}(T) \right)^Z_Z$$

The GG-action is LHS $\longleftrightarrow$ W-action on $\mathcal{D}(T)$

$\mu \times \mu: \mathcal{M}(\Sigma$

Axiom $\mathcal{M}(\mathfrak{g}$

$\mathcal{D}^\Psi := \mathcal{D}^\Psi(G/U)$ = quantization of princ. value.

Quantization of $\mathcal{M}_n := \left(\mathcal{D}^\Psi \otimes_{\mathbb{Z}} \mathcal{D}^\Psi \otimes_{\mathbb{Z}} \mathcal{D}^\Psi \dots \otimes_{\mathbb{Z}} \mathcal{D}^\Psi \right)$

$$= \left\{ \begin{array}{c} u_1 \otimes u_2 \dots u_n \\ \hline i \quad i+1 \\ u_0 \dots \mathbb{Z} [u_i \otimes u_{i+1}] = (u_i \otimes u_{i+1}) \mathbb{Z} \end{array} \right.$$

$$\mathbb{Z}^{n \times n} x_1 \dots x_n \mathbb{Z}$$

$$\mathcal{M}_n = \text{prin}$$

$$\mathcal{D}^\Psi(G/U) :=$$

operators which
 $f \in \mathbb{C} \rightarrow$
 $f(\mathcal{D}^\Psi) =$

Geometry

$$A_{\mathfrak{g}} = \{ (g, x) \in G \times \mathfrak{g}^* \mid \text{Ad}_g(x) = x \}$$

$\downarrow$
 $\mathfrak{g}^*$

centralizer group scheme.

Let G act hamiltonianly on M

moment map $M \rightarrow \mathfrak{g}^*$

Have an A -action.

$$\text{regular } \pi: \mathfrak{g}_r \subset \mathfrak{g}^* \rightarrow \mathfrak{g}^*/G = \mathfrak{t}/W =: \mathfrak{t}/W$$

Kostant: $\pi: \mathfrak{g}_r \rightarrow \mathfrak{t}/W$ smooth
each fiber is t -orbit.

$A_{\mathfrak{g}_r}$ descends to

a commutative flat
gr-p scheme $A \rightarrow \mathfrak{t}/W$.

(A is connected if G has no
center)

t/w smooth
 is t-orbit.

$$L^*A = \Gamma^* \circ \Gamma$$

Classical version of M_n .

$$\cong (U^t)^W$$

Harish-Ch

$$\text{Ker}_r = \left(\begin{array}{c} A \times A \times \dots \times A \\ \circ \quad \circ \quad \quad \quad \circ \end{array} \xrightarrow{\text{mult.}} A \right)$$

Thm

$$= \{ u_1 \otimes u_2 \dots \underbrace{\dots}_{i \quad i+1} \dots u_n \}$$

$$u_1 \otimes \dots \otimes u_i \otimes u_{i+1} = (u_i \otimes u_{i+1}) \otimes \dots$$

$$M_n := M_1^{\otimes n} / \text{Ker}_n = (M_1^{\otimes n} \otimes M_1^{\otimes n} \otimes \dots \otimes M_1^{\otimes n})$$

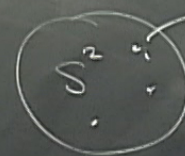
$\mathcal{D}^\vee := \mathcal{D}^\vee(G/U) = \text{quantization of princ. Walm. pre.}$

Quantization of $\mathcal{M}_n := \left(\mathcal{D}^\vee \otimes_{\mathbb{Z}} \mathcal{D}^\vee \otimes_{\mathbb{Z}} \mathcal{D}^\vee \dots \otimes_{\mathbb{Z}} \mathcal{D}^\vee \right)$

$$= \left\{ \begin{array}{c} u_1 \otimes u_2 \dots u_n \\ \underbrace{\quad}_{i=1}^{i=n} \\ u \otimes \dots \otimes u_{i+1} = (u_i \otimes u_{i+1}) \otimes \dots \end{array} \right.$$

$$\mathcal{M}_n := \mathcal{M}_n^n // \text{Ker}_n = (\mathcal{M}_1 \times_{\mathcal{O}} \mathcal{M}^{\times} \times_{\mathcal{O}} \dots \times_{\mathcal{O}} \mathcal{M}_1)$$

Decorated version. $([MT])$



$p_i: \mathfrak{sl}_2 \rightarrow \mathfrak{g}$

Let p

$$\mathcal{M}(S^2, p_1, \dots, p_n)$$

$$= \{ u_1 \oplus u_2 \dots u_n \}$$

$$u \otimes \dots \otimes u_i \otimes u_{i+1} = (u_i \otimes u_{i+1}) \otimes \dots$$

$$\mathcal{M}(S, p_1, \dots, p_n)$$

$$\mathcal{M}_1^n // \text{Ker}_n = (\mathcal{M}_1 \times \sigma \mathcal{M} \times \sigma \dots \times \mathcal{M}_1)$$

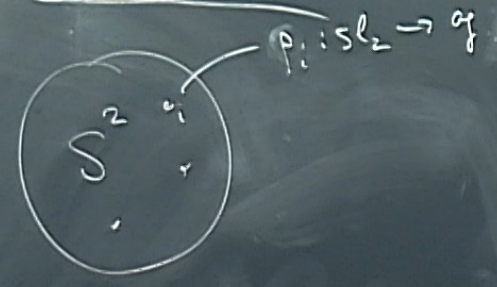
$$\supset \mathcal{M}(p', p) \xrightarrow{A} \sigma$$

princ. Wdh pre.

$$\bigoplus_{\mathbb{Z}} \mathcal{D}^4 \otimes \mathcal{D}^4 \dots \bigoplus_{\mathbb{Z}} \mathcal{D}^4$$

u_{i+1}
 $\otimes u_{i+1} = (u_i \otimes u_{i+1}) \otimes$
 $\dots \otimes u_1$
 $(p) \xrightarrow{A} \alpha$

Decorated version. $([MT])$



$$\mathcal{M}(S^2, p_1, \dots, p_n)$$

$$= (\mathcal{M}(p_1, p) \times \mathcal{M}(p_2, p) \times \dots \times \mathcal{M}(p_n, p)) // \text{Ker.}$$

Let $p, p': \text{sl}_2 \rightarrow \mathfrak{g}$.

$\mathcal{M}(p', p) =$ sol-s of the
 Nahm eq - n
 $(0, 1)$
 with pole p' at 0.
 p at 1!
 Bielanski

Let $p := \text{trivial sl}_2$.

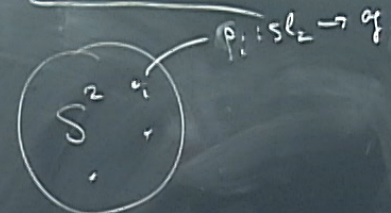
$\mathcal{D}^\Psi := \mathcal{D}^\Psi(G/U) = \text{quantization of princ. Mahan pre.}$

$$\text{Quantization of } \mathcal{M}_n := \left(\mathcal{D}^\Psi \otimes_{\mathbb{Z}} \mathcal{D}^\Psi \otimes_{\mathbb{Z}} \mathcal{D}^\Psi \dots \otimes_{\mathbb{Z}} \mathcal{D}^\Psi \right)$$

$$= \left\{ u_1 \otimes u_2 \otimes \dots \otimes u_n \right. \\ \left. u_1 \otimes \dots \otimes u_i \otimes u_{i+1} = (u_i \otimes u_{i+1}) \otimes \dots \right\}$$

$$\mathcal{M}_n := \mathcal{M}_1 // \text{Ker}_n = (\mathcal{M}_1^x \otimes \mathcal{M}_1^x \otimes \dots \otimes \mathcal{M}_1^x) \\ \supset \mathcal{M}(p', p) \xrightarrow{A} \sigma$$

Decorated version. $([MT])$



$$\mathcal{M}(S^2, p_1, \dots, p_n)$$

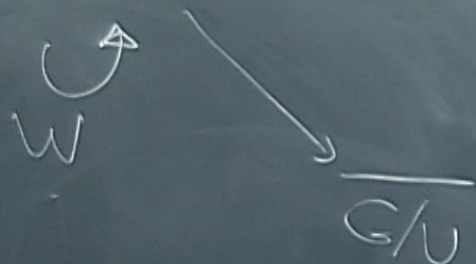
$$= (\mathcal{M}(p_1, p) \times \mathcal{M}(p_2, p) \times \dots \times \mathcal{M}(p_n, p)) // \text{Ker}_n.$$

$$T^*(G/U) \leftrightarrow T^*(G/U).$$

$$\text{Let } A = T^* \alpha$$

$$\simeq (U \cdot t)^W \hookrightarrow$$

Harish-Chandra



$$\text{Vol} \in \wedge^3 \mathbb{C}^3$$

Thm 1

$$T^*(SL_3/U)$$

$$= \left\{ \begin{array}{l} v_1, v_2, v_3 \in \mathbb{C}^3 \\ u_{12}, u_{23}, u_{13} \in \wedge^2 \mathbb{C}^3 \\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \end{array} \right\}$$

$$\begin{array}{l} v_1 \wedge u_{12} = 0 \\ v_2 \wedge u_{12} = 0. \end{array}$$

$$v_1 \wedge u_{23} = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdot \text{Vol.}$$

$$u_{23} \quad v_2 \wedge v_3$$

$$\mathcal{Q}(G/U)$$

Conj. $T^*(G/U)$ has symplectic singularity.

The