Title: BPS algebras and twisted character varieties

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Abstract: In this talk I will explain how a perverse filtration on the Kontsevich-Soibelman cohomological Hall algebra enables us to define the Lie algebra of BPS states associated to a smooth algebra with potential. I will then explain what this means for character varieties, and in particular, how to build the "genus g Kac-Moody Lie algebra" out of the cohomology of representations of the fundamental group of a surface.

BPS algebras and character varieties

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Character varieties and stacks

Throughout, let Σ_g be a genus g Riemann surface.

The (smooth) character variety

$X^1_{g,n} = \{A_1, \ldots, A_g, B_1, \ldots, B_g \in \mathrm{GL}_n(\mathbb{C}) \mid \prod_{s=1}^{g}(A_s, B_s) = \zeta_n \mathrm{Id}_{n\times n}\}$.
This *smooth* variety is acted on by $\mathrm{GL}_n(\mathbb{C})$ via simultaneous conjugation, with free induced $\mathrm{PGL}_n(\mathbb{C})$ action.

$$\therefore \text{The character variety } \mathrm{Rep}^1_n(\Sigma_g) := X^1_{g,n}/\,\mathrm{PGL}_n(\mathbb{C}) \text{ is } \underline{\text{smooth}}$$

$$\mathrm{H}(\mathrm{Rep}^1_n(\Sigma_g), \mathbb{Q}) \cong \mathrm{H}^{BM}(\mathrm{Rep}^1_n(\Sigma_g), \mathbb{Q}) \otimes \mathrm{H}_c(\mathbb{A}^{\mathrm{Rep}^1_n(\Sigma_g)}, \mathbb{Q}).$$

The character stack

Let $X^{(0)}_{g,n} = \{A_1, \ldots, A_g, B_1, \ldots, B_g \in \mathrm{GL}_n(\mathbb{C}) \mid \prod_{s=1}^{g}(A_s, B_s) = \mathrm{Id}_{n\times n}\}$.

$$\text{The character stack } \mathfrak{Rep}_n(\Sigma_g) := X_{g,n}/\,\mathrm{GL}_n(\mathbb{C}) \text{ is } \underline{\text{singular and stacky}}$$

We study $\mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q}) \cong \mathrm{H}_{c,\mathrm{GL}_n}(X_{g,n}, \mathbb{Q})^\vee$ $(\not\cong \mathrm{H}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q}))$.

Characteristic functions and plethystic exponentials

Definition (Characteristic function)

Let $V = \bigoplus_{n\in\mathbb{N}} V_n$ be a direct sum of elements of appropriately bounded elements of $\mathcal{D}(\mathrm{MHS})$. Then

$$\chi_q(V) := \sum_{n\in\mathbb{N}} \sum_{i,k\in\mathbb{Z}} (-1)^i \dim(\mathrm{Gr}_k^W(\mathrm{H}^i(V_n)))q^{k/2}x^n \in \mathbb{Z}((q))[[x]]$$

Definition (Plethystic exponential)

$$\mathrm{Exp}\colon x\mathbb{Z}((q^{1/2}))[[x]] \to \mathbb{Z}((q^{1/2}))[[x]]$$

is the unique operation satisfying $\mathrm{Exp}(\chi_q(V)) = \chi_q(\mathrm{Sym}(V))$ for strictly positively graded V.

Hausel-Villegas Theorem :
$\mathrm{Exp}\left(\chi_q\left(\bigoplus_{n\geq 1}\mathrm{H}\left(\mathrm{Rep}_n^1(\Sigma_g),\mathbb{Q}\right)(1-q)^{-1}\right)\right) =$
$\chi_q\left(\bigoplus_{n\geq 0}\mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g),\mathbb{Q})\right)$

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Hausel-Villegas Theorem(s):
$\mathrm{Exp}\left(\chi_q\left(\bigoplus_{n\geq 1} \mathrm{H}\left(\mathfrak{Rep}^1_n(\Sigma_g), \mathbb{Q}\right)\right)\right) = \chi_q\left(\bigoplus_{n\geq 0} \mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q})\right)$

Noncommutative DT theory

Ingredients

- Let A be a "nc smooth" algebra. For example $\mathbb{C}\langle x_1^{\pm 1}, \ldots, x_m^{\pm 1}, \omega_1, \ldots, \omega_n\rangle$, or $\mathbb{C}Q$ (for Q a quiver).
- Let $W \in A/[A,A]_{vect}$. This gives us a function $\mathrm{Tr}(W)$ on each stack $\mathfrak{Rep}_n(A)$ defined by $\rho \mapsto \mathrm{Tr}(\rho(W))$.

Definition (BPS invariants)

$$\chi_q\left(\bigoplus_{n\in\mathbb{N}} \mathrm{H}\left(\mathfrak{Rep}_n(A), \varphi_{\mathrm{Tr}(W)}\right)\right) = \mathrm{Exp}\left(\sum_{n\geqslant 1} \Omega_{A,W,n}(q^{1/2})\, x^n \frac{q^{1/2}}{(1-q)}\right)$$

where $\Omega_{A,W,n}(q^{1/2}) \in \mathbb{Z}[q^{\pm 1/2}]$.

It is easy to show that there are unique $\Omega_{A,W,n}(q^{1/2}) \in \mathbb{Z}((q^{1/2}))$ satisfying the defining equation. The statement that these invariants are Laurent polynomials is the "integrality conjecture" proved in this form by Kontsevich and Soibelman.

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Motivating question: can we "categorify" this theorem/definition?

Vanishing cycles

Let $f\colon X \to \mathbb{C}$ be a regular function on a complex manifold. Let $X_{<0} := f^{-1}(\mathbb{R}_{<0})$, $X_0 = f^{-1}(0)$. Then define $\psi_f, \phi_f\colon \mathcal{D}_c(X) \to \mathcal{D}_c(X)$ by

$$\psi_f \mathcal{F} = ((X_{<0} \to X)_*(X_{<0} \to X)^* \mathcal{F})_{X_0}$$
$$\phi_f \mathcal{F} = \mathrm{cone}(\mathcal{F}_{X_0} \to \psi_f \mathcal{F})[-1]$$

If X is smooth, then $\varphi_f := \phi_f \mathbb{Q}_X \otimes \mathrm{H}_c\left(\mathbb{A}^1, \mathbb{Q}\right)^{\otimes(-\dim(X)/2)}$ is a (M)MHM detecting singularities of f.

Example (Nontrivial monodromy)

$X = \mathbb{A}^1$, $f(x) = x^{d+1}$. Then
$\phi_f \mathbb{Q}_{\mathbb{A}^1} = \mathrm{cone}(\mathbb{Q}_0 \xrightarrow{a \mapsto (a,\ldots,a)} \mathbb{Q}_0^{\oplus(d+1)})[-1] = \mathbb{Q}_0^{\oplus d}[-1]$

Example (No monodromy)

$X = \mathbb{A}^2$, $f(x,y) = xy$. Then from exact sequence
$\mathrm{H}^1(X_0, \mathbb{Q}) \to \mathrm{H}^1(X, \psi_f) \to \mathrm{H}^2(X, \phi_f) \to \mathrm{H}^2(X_0, \mathbb{Q}) = 0$ we deduce
$\phi_f \mathbb{Q}_{\mathbb{A}^2} \cong \mathbb{Q}_0 \otimes \mathrm{H}^{BM}(\mathbb{A}^1, \mathbb{Q})$.

Importing character varieties into DT theory

Ingredients for recovering character stack cohomology

- $A_g = \mathbb{C}\langle a_1^{\pm 1}, \ldots, a_g^{\pm 1}, b_1^{\pm 1}, \ldots, b_g^{\pm 1}, \omega\rangle$ is our nc smooth algebra.
- $W_g = \omega(1 - \prod_{s=1}^{g}(a_s, b_s))$

Forgetful map $\mathfrak{Rep}_n(A_g) \to \mathfrak{Rep}_n(\mathbb{C}\langle a_1^{\pm 1}, \ldots, a_g^{\pm 1}, b_1^{\pm 1}, \ldots, b_g^{\pm 1}\rangle)$ is a vector bundle projection, and $\mathrm{Tr}(W_g)$ is linear in the fibres. Therefore

$$\mathrm{H}^{BM}(\mathfrak{Rep}(F_{2g+1}), \phi_{\mathrm{Tr}(W)}\mathbb{Q}) \cong \mathrm{H}^{BM}\left(Z\left(1 - \prod_{s=1}^{g}(\rho(a_s), \rho(b_s))\right), \mathbb{Q}\right) \otimes \mathbb{L}^{n^2}$$

$$\cong \mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q}) \otimes \mathbb{L}^{n^2}$$

Linking the two subjects

$\bigoplus_n \mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q}) \cong \mathrm{Sym}\left(\bigoplus_{n\geqslant 1} \mathrm{H}^{BM}\left(\mathfrak{Rep}_n^1(\Sigma_g), \mathbb{Q}\right)\right)$, would imply the "categorified" integrality conjecture for the pair (A_g, W_g), and that $\mathrm{H}(\mathrm{Rep}_n^1(\Sigma_g), \mathbb{Q})$ are the BPS spaces for this pair.

Approximation by proper maps

Definition

Let $p\colon \mathcal{X} \to Y$ be a map from an Artin stack to a finite type scheme. Then p is *approximated by proper maps* if $\forall N \geqslant 0$ we have commutative

$$\begin{array}{ccc} X_N & \xrightarrow{q_N} & \mathcal{X} \\ & {\scriptstyle p_N}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

such that

1. p_N is a proper map of schemes.
2. q_N is smooth.
3. $\mathcal{H}^n(p_*\mathbb{Q}_{\mathcal{X}}) \to \mathcal{H}^n(p_{N,*}\mathbb{Q}_{X_N})$ is an isomorphism for $n \leqslant N$.

- Say p is APM, and $\mathcal{X}$ is smooth. Then $p_*\mathbb{Q}_{\mathcal{X}}$ is pure.
- Say also $f \in \Gamma(Y)$. Then $\mathrm{H}(X, \varphi_{fp})$ carries a natural perverse filtration.

Hidden properness for $\mathfrak{Rep}(A)$

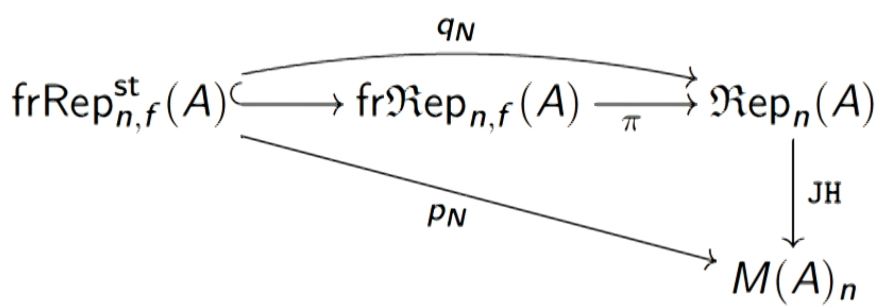
Theorem

Let A be a finitely generated algebra. Let $M(A)_n$ be the coarse moduli space of n-dimensional A-modules, parameterising semisimple A-modules. Then $\mathfrak{Rep}_n(A) \xrightarrow{\mathtt{JH}} M(A)_n$ is approximated by proper maps.

Sketch proof: Let $\mathrm{fr}\mathfrak{Rep}_{n,f}(A)$ be the moduli stack of pairs

$$(\rho \text{ an } n\text{-dimensional } A\text{-module}, \iota \in \mathrm{Hom}_{\mathrm{vect}}(\mathbb{C}^f, \rho)) \quad .$$

Let $\mathrm{frRep}^{\mathrm{st}}_{n,f}(A) \subset \mathrm{fr}\mathfrak{Rep}_{n,f}(A)$ be pairs such that $A \cdot \mathrm{im}(\iota) = \rho$. Consider



1): p_N is GIT quotient map. 2): q_N is composition of smooth maps.
3): Compactified Totaro construction $\mathrm{H}^{\leqslant N}(X/G) \cong \mathrm{H}^{\leqslant N}((X \times U)/G)$.

The relative cohomological integrality theorem

Let A be a nc smooth algebra. Let $M(A)$ be the coarse moduli space of all f.d. semisimple A-modules. Direct sum map $\oplus\colon M(A)\times M(A)\to M(A)$. For $\mathcal{F},\mathcal{G}\in \mathrm{MHM}(M(A))$ define $\mathcal{F}\boxplus_{\oplus}\mathcal{G}=\oplus_*(\mathcal{F}\boxtimes\mathcal{G})$. Then

$$
\begin{aligned}
[\mathrm{JH}_*\,\mathbb{Q}_{\mathfrak{Rep}(A)}] &= \mathrm{Exp}([\mathrm{IC}_{M(A)^{\mathrm{st}}}(\mathbb{Q})\otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}]) && \text{Meinhardt-Reineke}\\
\mathrm{JH}_*\,\mathbb{Q}_{\mathfrak{Rep}(A)} &\cong \mathrm{Sym}\left(\mathrm{IC}_{M(A)^{\mathrm{st}}}(\mathbb{Q})\otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right) && \text{by APM+purity}\\
\mathrm{JH}_*\,\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathfrak{Rep}(A)} &\cong \phi_{\mathrm{Tr}(W)}\,\mathrm{JH}_*\,\mathbb{Q}_{\mathfrak{Rep}(A)} && \text{by APM}\\
&\cong \mathrm{Sym}\left(\phi_{\mathrm{Tr}(W)}\,\mathrm{IC}_{M(A)^{\mathrm{st}}}(\mathbb{Q})\otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right)
\end{aligned}
$$

Taking hypercohomology:

Theorem (D-Meinhardt)

$$\bigoplus_{n\geqslant 0}\mathrm{H}(\mathfrak{Rep}_n(A),\varphi_{\mathrm{Tr}(W)})\cong \mathrm{Sym}\left(\bigoplus_{n\geqslant 1}\mathrm{H}\left(M(A)_n,\mathcal{BPS}_{A,W,n}[u]u^{1/2}\right)\right)$$

where $\mathcal{BPS}_{A,W,n}:=\phi_{\mathrm{Tr}(W)}\,\mathrm{IC}_{M(A)^{\mathrm{st}}_n}(\mathbb{Q})$.

An aside: The BPS sheaf and (B,B,B)-branes

The theorem says that the cohomological BPS invariant $\Omega_{A,W,n}$ for the pair (A, W) is given by taking hypercohomology of the "BPS sheaf" $\mathcal{BPS}_{Q,W,n} := \phi_{\mathrm{Tr}(W)} \mathrm{IC}_{M(A)_n^{\mathrm{st}}}(\mathbb{Q})$.

Example

Let $g = 1$. Then $A_1 = \mathbb{C}\langle a^{\pm 1}, b^{\pm 1}, \omega\rangle$, $W_1 = \omega(1 - aba^{-1}b^{-1})$. $\mathrm{Supp}(\varphi_{\mathrm{Tr}(W)}) = \mathfrak{Rep}(\mathbb{C}[a^{\pm 1}, b^{\pm 1}, \omega])$, and the relevant part of $M(A_1)_n$ is the coarse moduli space of length n sheaves on $\mathcal{C} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}$. Let $\Delta_n \colon \mathcal{C} \to M(A_1)_n$ be the inclusion of the small diagonal. Then $\mathcal{BPS}_{A_1,W_1,n} = \Delta_{n,*} \mathrm{IC}_{\mathcal{C}}(\mathbb{Q})$.

This enables us to calculate $\mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_1), \mathbb{Q})$.
In general, $(M(\mathbb{C}[\pi_1(\Sigma_g)][y])_n \to M(\mathbb{C}[\pi_1(\Sigma_g)])_n)_! \mathcal{BPS}_{A_g,W_g,n}$ is pure.

Conjecture

The supports of these direct images are (B,B,B) branes (i.e. complex varieties on the Higgs side of nonabelian Hodge theory).

The DT cohomological Hall algebra

The standard correspondence diagram

Let A be a nc smooth algebra. Let $n', n'' \in \mathbb{N}$ and let $n = n' + n''$. Consider the diagram

$$\mathfrak{Rep}_{n'}(A) \times \mathfrak{Rep}_{n''}(A) \xleftarrow{\pi_1 \times \pi_3} \mathfrak{Rep}_{n',n''}(A) \xrightarrow{\pi_2} \mathfrak{Rep}_n(A)$$

where π_i is the map taking a short exact sequence of A-modules to its ith term.

- (KS) Pullback+pushforward gives us a product $\mathrm{H}(\mathfrak{Rep}_{n'}(A), \varphi_{\mathrm{Tr}(W)}) \otimes \mathrm{H}(\mathfrak{Rep}_{n''}(A), \varphi_{\mathrm{Tr}(W)}) \to \mathrm{H}(\mathfrak{Rep}_n(A), \varphi_{\mathrm{Tr}(W)})$ on $\bigoplus \mathrm{H}(\mathfrak{Rep}_n(A), \varphi_{\mathrm{Tr}(W)})$
- (D) Pushforward+pullback gives a (localised) coproduct going the other way.
- (DM) Product and coproduct are compatible with the perverse filtration.

The perverse associated graded algebra

The graded mixed Hodge structure $\mathcal{H}_{A,W} := \bigoplus_n \mathrm{H}(\mathfrak{Rep}(A)_n, \varphi_{\mathrm{Tr}(W)})$ is a localised bialgebra, with operations respecting the perverse filtration.

Primitives

In $\mathrm{Gr}_P(\mathcal{H}_{A,W=0}) =: \mathcal{P}_{A,0}$, the composition
$\mathrm{H}(M(A)_m, \mathrm{IC}_{M(A)_n^{\mathrm{st}}}(\mathbb{Q})[u]u^{1/2}) \xrightarrow{\iota} \mathcal{P}_{A,0,n} \xrightarrow{\Delta} \bigoplus_{n'+n''=n} \mathcal{P}_{A,0,n'} \otimes \mathcal{P}_{A,0,n''}$
is given by $\iota \otimes \mathrm{id} + \mathrm{id} \otimes \iota$. I.e. $\mathcal{BPS}_{A,W=0,n}[u]u^{1/2}$ is primitive in $\mathrm{Gr}_P(\mathcal{H}_{A,W=0})$.

1. $U(\bigoplus_{n\geqslant 1} \mathcal{BPS}_{A,W=0,n}[u]u^{1/2}) \to \mathrm{Gr}_P(\mathcal{H}_{A,W=0})$ is an injection, hence an isomorphism.
2. Again for support reasons, the Lie algebra structure on $\mathcal{BPS}_{A,0}[u]u^{1/2}$ considered as a space of primitives in $\mathrm{Gr}_P(\mathcal{H}_{A,W=0})$, is zero.
3. I.e. $\mathrm{Gr}_P(\mathcal{H}_{A,W=0})$ is the Hopf algebra $\mathrm{Sym}(\bigoplus_{n\geqslant 1} \mathcal{BPS}_{A,W=0,n}[u]u^{1/2})$.
4. Applying $\phi_{\mathrm{Tr}(W)}$ to everything in sight, $\mathrm{Gr}_P(\mathcal{H}_{A,W})$ is the Hopf algebra $\mathrm{Sym}(\bigoplus_{n\geqslant 1} \mathcal{BPS}_{A,W,n}[u]u^{1/2})$ for general (A, W).

The BPS Lie algebra

We now have everything we need to define the BPS Lie algebra:

The underlying mixed Hodge structure of $\mathfrak{g}_{A,W}$

Note that

$$P_1\mathcal{H}_{A,W} = P_1 \operatorname{Sym}\left(\bigoplus_{n\geqslant 0} \mathrm{H}(M(A)_n, \phi_{\mathrm{Tr}(W)} \mathrm{IC}_{M(A)}(\mathbb{Q}))[u]u^{1/2}\right)$$
$$= \bigoplus_{n\geqslant 1} \mathrm{H}(M(A)_n, \phi_{\mathrm{Tr}(W)} \mathrm{IC}_{M(A)}(\mathbb{Q})) \otimes \mathbb{L}^{1/2}$$
$$= \mathcal{BPS}_{A,W} \otimes \mathbb{L}^{1/2} =: \mathfrak{g}_{A,W}$$

The Lie bracket

If $[\bullet, \bullet]$ is the commutator in $\mathcal{H}_{A,W}$, then a priori $[\mathfrak{g}_{A,W}, \mathfrak{g}_{A,W}] \subset P_2\mathcal{H}_{A,W}$. But since the perverse associated graded is commutative, $[\mathfrak{g}_{A,W}, \mathfrak{g}_{A,W}] \subset P_1\mathcal{H}_{A,W} = \mathfrak{g}_{A,W}$.

A friendly face

Let Q be a quiver. Form $\tilde{Q}$ by

1. For all arrows a in Q, add an arrow a^* going the other way.
2. For all vertices i in Q, add a one-cycle ω_i based at i.

Let $A_Q = \mathbb{C}\tilde{Q}$, $W_Q = \sum_{a \text{ an arrow of } Q}[a, a^*]\sum_{i \text{ a vertex of } Q} \omega_i$.
Then by the general recipe above we obtain a Lie algebra structure on $\mathfrak{g}_{A_Q,W_Q} = P_1(\mathcal{H}_{A_Q,W_Q})$. $\chi_q(\mathfrak{g}_{A_Q,W_Q}) = \sum_\gamma a_\gamma(q)x^\gamma$.

Theorem

Let Q_{re} be the full subquiver of Q containing those vertices without 1-cycles. Then $\mathrm{H}^0(\mathfrak{g}_{A_Q,W_Q}) = \mathfrak{g}^+_{Q_{re}}$.

Conjecturally, $\mathcal{H}_{A_Q,W_Q}$ is the strictly positive piece of the Maulik-Okounkov Yangian.

The genus g Kac–Moody Lie algebra

Back to Σ_g

We have a pair A_g, W_g such that $\mathcal{H}_{A_g,W_g} \cong \bigoplus_{n\geqslant 0} \mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q})$. We deduce that this cohomology has the structure of some sort of Yangian-like quantum group, and in particular there is a "genus g Kac Moody Lie algebra"

$$\mathfrak{g}_{\Sigma_g} := \mathcal{BPS}_{A_g,W_g} \otimes \mathbb{L}^{1/2} \subset \bigoplus_{n\geqslant 0} \mathrm{H}^{BM}(\mathfrak{Rep}_n(\Sigma_g), \mathbb{Q}).$$

Conjecture

There is a natural isomorphism $\mathfrak{g}_{\Sigma_g} \cong \mathrm{H}(\mathrm{Rep}^1_n(\Sigma_g), \mathbb{Q})$.

- Applying χ_q to the form of the PBW isomorphism implied by the conjecture, we get the Hausel–Villegas equality.
- Confirmed in genus 1