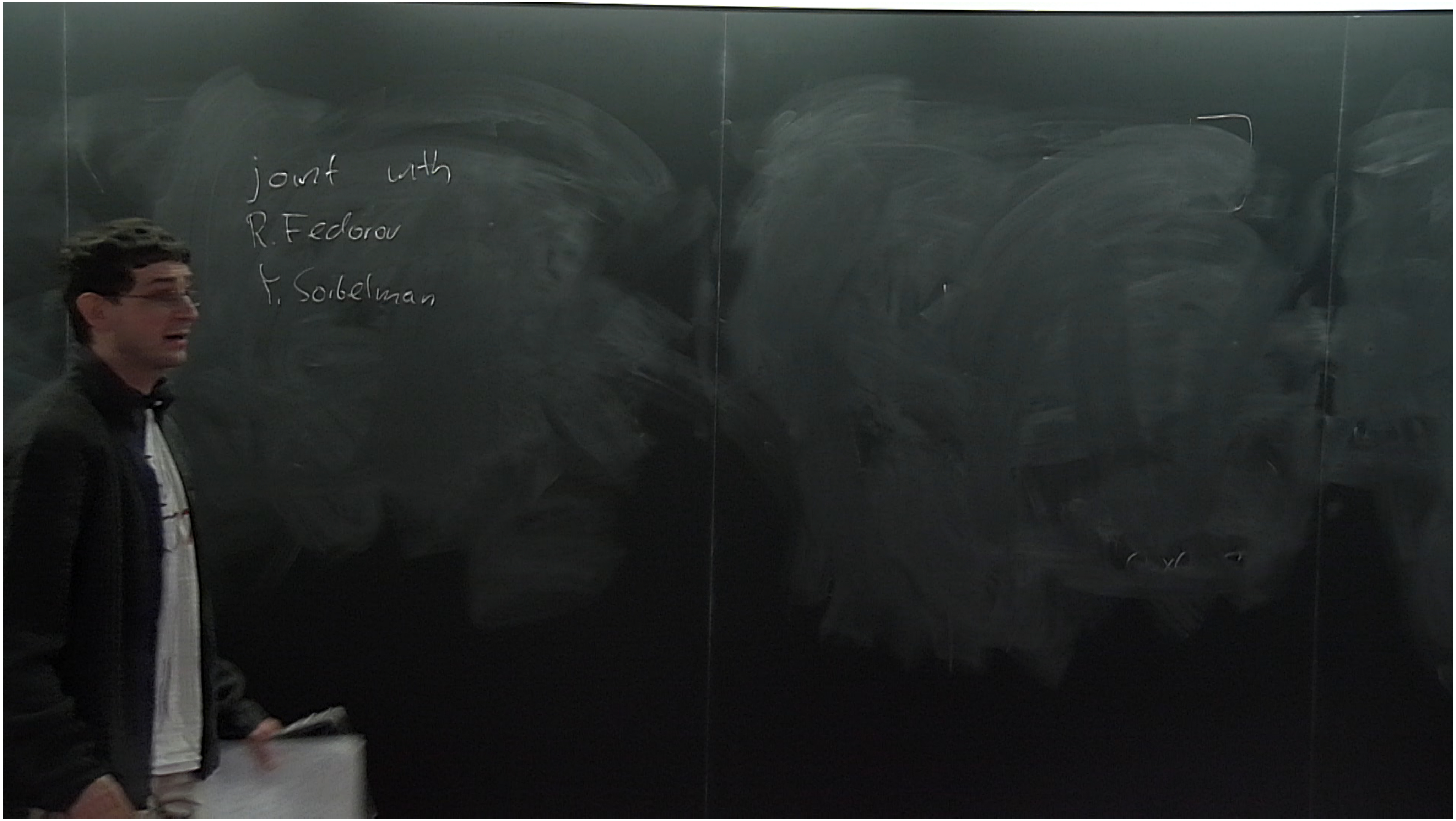
Title: Motivic Classes for Moduli of Connections

Date: Feb 14, 2017 02:00 PM

URL: http://pirsa.org/17020023

Abstract: In their paper, "On the motivic class of the stack of bundles", Behrend and Dhillon were able to derive a formula for the class of a stack of vector bundles on a curve in a completion of the K-ring of varieties. Later, Mozgovoy and Schiffmann performed a similar computation in order to obtain the number of points over a finite field in the moduli space of twisted Higgs bundles. We will briefly introduce motivic classes. Then, following Mozgovoy and Schiffmann's argument, we will outline an approach for computing motivic classes for the moduli stack of vector bundles with connections on a curve. This is a work in progress with Roman Fedorov and Yan Soibelman.


joint with
R. Fedorov
Y. Soibelman

k - field

$K_0(Var_k)$.

$[X] = [X \setminus Z] + [Z]$

$Z \subset X$ closed

$[\mathbb{A}^1] = \mathbb{L}$

$[GL_n(k)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \dots (\mathbb{L}^n - \mathbb{L}^{n-1})$

$\hat{K}_0(Var_k)$ - dimensional completion of $K_0(Var_k)[\frac{1}{\mathbb{L}}]$

For $\mathrm{Bun}_{r,d}(X)$ - rank r degree d vector bundles over a smooth projective curve of genus g X

(Behrend-Dhillon)

$$[\mathrm{Bun}_{r,d}(X)] = \mathbb{L}^{(r^2-1)(g-1)} \frac{[\mathrm{Jac}\, X]}{\mathbb{L}-1} \prod_{i=2}^{r} Z_X(\mathbb{L}^{-i})$$

$$Z_X(t) = \sum_{n=0}^{\infty} [X^{(n)}]\, t^n$$

$X^{(n)}$ - nth symmetric power of X.

Motivations:

Kontsevich - Soibelman

Mozgovoy - Schiffmann

Hausel + coauthors

$\mathrm{Conn}_r(X)$ - algebraic connections on rank r vector bundles over X

$\mathcal{M}^{ss}_{r,d}(X)$ - semistable rank r, degree d Higgs bundles on X

char $k = 0$

<u>Thm 1</u> (FSS)

$$[\mathrm{Conn}_r] = [\mathcal{M}^{ss}_{r,0}]$$

<u>Thm 2</u> (FSS)

$$\sum_{d/r = \tau} (-\mathbb{L})^{(1-g)r^2} [\mathcal{M}^{ss}_{r,d}] z^d w^r = \mathrm{Exp}\Big(\sum_{d/r=\tau} B_{r,d} z^d w^r\Big)$$

$$\sum_{r,d} B_{r,d} z^d w^r = \mathbb{L} \,\mathrm{Log}\Big[\sum_{\lambda} (-\mathbb{L})^{(g-1)\langle\lambda,\lambda\rangle} J^{mot}_{\lambda}(z) H^{mot}_{\lambda}(z) w^{|\lambda|}\Big]$$

J^{mot}_{λ}: depends on "regularized" zeta function

H^{mot}_{λ}: iterated residue of normalized zeta

Exp, Log: plethystic

Motivations:

Kontsevich - Soibelman

Mozgovoy - Schiffmann

Hausel + coauthors

$\mathrm{Conn}_r(X)$ - algebraic connections on rank r vector bundles over X

$\mathcal{M}^{ss}_{r,d}(X)$ - semistable rank r, degree d Higgs bundles on X

char $k = 0$

<u>Thm 1</u> (FSS)

$$[\mathrm{Conn}_r] = [\mathcal{M}^{ss}_{r,0}]$$

<u>Thm 2</u> (FSS)

$$\sum_{d/r=\tau} (-\mathbb{L})^{(1-g)r^2} [\mathcal{M}^{ss}_{r,d}]\, z^d w^r = \mathrm{Exp}\Big(\sum_{d/r=\tau} B_{r,d}\, z^d w^r\Big)$$

$$\sum_{r,d} B_{r,d}\, z^d w^r = \mathbb{L}\,\mathrm{Log}\Big[\sum_{\lambda} (-\mathbb{L})^{(g-1)\langle\lambda,\lambda\rangle} J^{mot}_{\lambda}(z) H^{mot}_{\lambda}(z) w^{|\lambda|}\Big]$$

J^{mot}_{λ}: depends on "regularized" zeta function

H^{mot}_{λ}: iterated residue of normalized zeta

Exp, Log } plethystic

Sketch of Proof

$$\begin{tikzcd}
\mathrm{Conn}_r \arrow[r, leftrightarrow, "①"] & \underline{\underline{M^{iso}_{r,d}}} \arrow[r, leftrightarrow, "\text{twist}"] & M^{\geq 0\, iso}_{r,d} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "③"'] & M^{\geq 0 ss}_{r,d} \arrow[r, leftrightarrow, "\text{twist}"] & M^{ss}_{r,d} \\
& & M^{\geq 0}_{r,d} \arrow[ur, leftrightarrow, "②"'] \arrow[d, leftrightarrow, "④"] & & \\
& & \mathcal{E}^{\geq 0}_{r,d} \arrow[d, leftrightarrow, "⑤"] & & \\
& & \mathcal{E}^{\geq 0\, nilp}_{r,d} \arrow[d, leftrightarrow, "⑥"] & & \\
& & \mathrm{Bun}_{r,d} & &
\end{tikzcd}$$

≥ 0 - nonnegative HN spectrum

$M^{\geq 0}_{r,d}$ ← all Higgs with ≥ 0 spectrum

$\mathcal{E}^{\geq 0}_{r,d}$ - parametrizes pairs (E,φ) E - vector bundle, $\varphi \in \mathrm{End}(E)$

$\mathcal{E}^{\geq 0\, nilp}_{r,d}$ parametrizes (E,φ) φ - nilpotent

① $M^{iso}_{r,d}$ - all direct summands of v.b. have the same slope

$[\mathrm{Conn}_r] = [M^{iso}_{r,0}]$ - thm of Weil says connections exist on precisely these bundles

cut/paste argument shows classes are the same

Sketch of Proof

$$\mathrm{Conn}_r \overset{①}{\longleftrightarrow} \underline{\underline{M^{iso}_{r,d}}} \overset{\text{twist}}{\longleftrightarrow} M^{\geq 0\, iso}_{r,d} \longleftrightarrow M^{\geq 0\, ss}_{r,d} \overset{\text{twist}}{\longleftrightarrow} M^{ss}_{r,d}$$

③ $M^{\geq 0\, iso}_{r,d} \longleftrightarrow M^{\geq 0}_{r,d}$, ② $M^{\geq 0\, ss}_{r,d} \longleftrightarrow M^{\geq 0}_{r,d}$

$M^{\geq 0}_{r,d}$ ← all Higgs with ≥ 0 spectrum

≥ 0 - nonnegative HN spectrum

④ $M^{\geq 0}_{r,d} \longleftrightarrow \mathcal{E}^{\geq 0}_{r,d}$

$\mathcal{E}^{\geq 0}_{r,d}$ - parametrizes pairs (E, φ) E - vector bundle, $\varphi \in \mathrm{End}(E)$

⑤ $\mathcal{E}^{\geq 0}_{r,d} \longleftrightarrow \mathcal{E}^{\geq 0\, nilp}_{r,d}$

$\mathcal{E}^{\geq 0\, nilp}_{r,d}$ parametrizes (E, φ) φ - nilpotent

⑥ $\mathcal{E}^{\geq 0\, nilp}_{r,d} \longleftrightarrow \mathrm{Bun}_{r,d}$

① $M^{iso}_{r,d}$ - all direct summands of v.b. have the same slope

$[\mathrm{Conn}_r] = [M^{iso}_{r,0}]$ - thm of Weil says connections exist on precisely these bundles

cut/paste argument shows classes are the same

② $$H^{\geq 0}(z,w) = 1 + \sum_{\substack{r>0 \\ d\geq 0}} \mathbb{L}^{(1-g)r^2} [M^{\geq 0}_{r,d}] w^r z^d$$

$$H^{\geq 0\, ss}_{\tau}(z,w) = \sum_{\substack{r>0 \\ d\geq 0 \\ d/r=\tau}} \mathbb{L}^{(1-g)r^2} [M^{\geq 0\, ss}_{r,d}]$$

Computation in

$H^{\geq 0}$

...ummands of
same slope

...thm of
Weil says
...nnections exist
...n precisely these bundles

...t shows
...e same

$$\sum_{\substack{r>0\\ d\ge 0}} \mathbb{L}^{(1-g)r^2} \left[M_{r,d}^{\ge 0}\right] w^r z^d$$

$$\sum_{\substack{r>0\\ d\ge 0}} \mathbb{L}^{(1-g)r^2} \left[M_{r,d}^{\ge 0, ss}\right]$$

Computation in motivic Hall algebra

$$H^{\ge 0}(z,w) = \prod_{\tau>0} H_\tau^{\ge 0}(z,w)$$

Kontsevich-Soibelman wall-crossing

③ Let

$$H_\tau^{\ge 0, iso}(z,w) = 1 + \sum_{\substack{r>0\\ d\ge 0\\ d/r=\tau}} \mathbb{L}^{(1-g)r^2} \left[M_{r,d}^{\ge 0, iso}\right] w^r z^d$$

$$H^{\ge 0}(z,w) = \prod_{\tau>0} H_\tau^{\ge 0, iso}(z,w)$$

uses decomposition
into indecomposables

(4) Cut/paste argument

$$[\mathcal{M}^{\geq 0}_{r,d}] = \mathbb{L}^{(g-1)r^2}\,[\mathcal{E}^{\geq 0}_{r,d}]$$

(5) $$\sum_{r,d} [\mathcal{E}^{\geq 0}_{r,d}]\, z^d w^r = \mathrm{Exp}\left(\mathbb{L}\,\mathrm{Log}\left(\sum_{r,d} [\mathcal{E}^{\geq 0, nil}_{r,d}]\, z^d w^r\right)\right)$$

uses decomposition into indecomposables

+ Fitting's Lemma

(6) Decompose $\mathcal{E}^{\geq 0\, nil}_{r,d}$ based on flag type for φ

$$0 \hookrightarrow \underset{r_1,d_1}{\ker\varphi} \hookrightarrow \underset{r_2,d_2}{\ker\varphi^2} \hookrightarrow \cdots \hookrightarrow \underset{r,d}{\ker\varphi^s} = E$$

char $k = 0$

Thm 1 (FSS)

$[\mathrm{Conn}_r] = [\mathcal{M}$

Thm 2 (FSS)

$\sum_{d/r=?} (-\mathbb{L})^{(1-}$

$\sum_{r,d} B$

twist

oss

$,d \leftarrow\rightarrow M^{ss}_{r,d}$

iggs with ≥ 0 spectrum

trizes pairs

) E - vector bundle

$\varphi \in \mathrm{End}(E)$

metrizes

φ - nilpotent

⑥ reduce to
case where the flag
is complete using
motivic version of Harder residue formula

direct computation
reduces this
to $Bun_{r,d}$

(eliminates torsion +
negative HN spectrum)

Computation in motivic

$H^{\geq 0}(z,w) = \prod_{\tau \geq 0} H^{\geq 0}_{\tau}($

Kontsevich-Soibelman

③ Let

$H^{\geq 0, <\infty}_{\tau}(z,w) = 1+$

$H^{\geq 0}(z,w) = \prod$

uses decompositio
into indecompos

Parabolic Bundle Case

Do

Lin

$E = (E, E_{ij})$ - parabolic bundle $D = x_1 + \dots + x_k$

$x_i \in X$ - points on curve

$E_{x_i} = E_{i0} \supset \dots \supset E_{i w_i}$

Parabolic Connections:

$$\nabla : E \to E \otimes \Omega^1_X(D)$$

$$(\mathrm{Res}_{x_i} \nabla - \zeta_{ij})(E_{ij-1}) \subset E_{ij}$$

ζ_{ij} - fixed

$\mathrm{Conn}_{\zeta, \alpha, D}(X)$ -

parabolic connections

where D is fixed

$\zeta = (\zeta_{ij})$

$\alpha = (\alpha_0, \alpha_{ij})$

$rk = \alpha_0$

$\dim E_{ij} = \alpha_{ij}$

$(\sum B_{n,d} z^{q_w}$

$\dots(-1)^{\langle \lambda, \lambda \rangle}$

$J_\lambda^{mot}(z)$

depends on "regularized zeta functio[n]"

≥ 0

$\longleftrightarrow M^{\geq 0\, ss}_{r,d} \longleftrightarrow M^{ss}_{r,d}$

twist

≥ 0

② all Higgs with ≥ 0 spectrum

parametrizes pairs

E - vector bundle

$\varphi \in End(E)$

parametrizes

φ - nilpotent

Thm $[Conn_{\xi,d,D}] = [M^{ss}_{\alpha,0,Im\,\xi,D}]$

divisor

degree

dimensions of flags

The steps ②-⑤ also work in parabolic case

Computation in motivic Hall

$H^{\geq 0}(z,w) = \prod_{\tau \geq 0} H^{\geq 0}_{\tau}(z,w)$

Kontsevich-Soibelman wall

③ Let

$H^{\geq 0, ss}_{\tau}(z,w) = 1 + \sum_{\substack{r>0 \\ d\geq 0 \\ d/r=\tau}}$

$H^{\geq 0}(z,w) = \prod_{\tau \geq 0} H^{\geq 0}_{\tau}$

uses decomposition into indecomposables

$$Z_X(t) = \sum_{n=0}^{\infty} [X^{(n)}] t^n$$

$\tilde{Z}_X(t) = z^{1-g} Z_X(t)$ - normalized version

$$Z^*_X(\mathbb{L}^{-u} t^v) = \begin{cases} Z_X(\mathbb{L}^{-u} t^v & \text{if } v>0 \\ \operatorname{res}_{z=\mathbb{L}^{-1}} Z_X & \text{if } (u,v)=(1,0) \\ \operatorname{res}_{z=1} Z_X & \text{if } (u,v)=(u,0) \end{cases}$$

$$J_\lambda^{mot}(z) = \prod_{s\in\lambda} Z_X\left(\mathbb{L}^{-1-\ell(s)} z^{a(s)}\right)$$

$\ell(s)$ - leg length

$a(s)$ - arm length

$$L^{mot}(z_n, \dots, z_1) = \frac{1}{\prod_{i<j} \tilde{Z}_X\left(\frac{z_i}{z_j}\right)} \sum_{\sigma\in S_n} \left(\prod_{i<j} \tilde{Z}_X\left(\frac{z_i}{z_j}\right) \frac{1}{\prod_{i<n}\left(1-\mathbb{L}\frac{z_{i+1}}{z_i}\right)} \cdot \frac{1}{1-z_1} \right)$$

dim.

For λ

For $\lambda = 1^{r_1} 2^{r_2} \dots t^{r_t}$ $\sum_i r_i = n$ $r_{<i} = \sum_{k<i} r_k$

let res_λ be the iterated residue with respect to

$\frac{z_n}{z_{n-1}} = \mathbb{L}^{-1}, \frac{z_{n-1}}{z_{n-2}} = \mathbb{L}^{-1} \dots, \frac{z_{2+r<t}}{z_{1+r<t}} = \mathbb{L}^{-1}$

$\frac{z_{r_1}}{z_{r_1-1}} = \mathbb{L}^{-1}, \frac{z_{r_1-1}}{z_{r_1-2}} = \mathbb{L}^{-1}, \dots \frac{z_2}{z_1} = \mathbb{L}^{-1}$

dim.

$\tilde{H}_\lambda^{mot}(z_{1+r<t}, z_{1+r<t-1}, \dots, z_1) = res_\lambda L^{mot}(z_n, \dots, z_1)$

$H_\lambda^{mot}(z) = \tilde{H}_\lambda^{mot}(z^t \mathbb{L}^{-r<t}, \dots, z^1 \mathbb{L}^{-r<i}, \dots, z)$

zeta

Sketch of Proof

$\mathrm{Conn}_r \overset{①}{\longleftrightarrow} \mathcal{M}^{iso}_{r,d} \overset{\text{twist}}{\longleftrightarrow} \mathcal{M}^{\geq 0}_{r}$

③

≥ 0 - nonnegative HN spectrum

Bun