

Title: Perverse Hirzebruch y-genus of Higgs moduli spaces

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Abstract: I will discuss in the framework of the P=W conjecture, how one can conjecture formulas for the perverse Hirzebruch y-genus of Higgs moduli spaces. The form of the conjecture raises the possibility that they can be obtained as the partition function of a 2D TQFT.

Perverse Hirzebruch γ -genus of Hitchin systems

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Hitchin systems in mathematics and physics
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Diffeomorphic spaces in non-Abelian Hodge theory

- C genus g curve; fix $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$

$$\mathcal{M}_{\text{Dol}}^n := \left\{ \begin{array}{l} \text{moduli space of semistable rank } n \\ \text{degree } d \text{ Higgs bundles } (E, \phi) \\ \phi \in H^0(C, \text{End}(E) \otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}_B^n := \{A_1, B_1, \dots, A_g, B_g \in \text{GL}_n | \prod_{i=1}^g A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} \text{Id}\} // \text{PGL}_n$$

when $(d, n) = 1$ these are smooth non-compact varieties

- Non-Abelian Hodge Theorem: $\mathcal{M}_{\text{Dol}}^n \xrightarrow{\text{diff}} \mathcal{M}_B^n$
(Hitchin, Donaldson, Corlette, Simpson)
- Problem: what is Poincaré polynomial
 $P(\mathcal{M}_{\text{Dol}}^n; t) = P(\mathcal{M}_B^n; t)?$

Weight polynomials

- (Deligne 1971) proved the existence of $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X; \mathbb{Q})$ for any complex algebraic variety X
- $WH(X; q, t) = \sum \dim(W_i/W_{i-1}(H^k(X)))t^k q^{\frac{i}{2}}$ weight polynomial
- $P(X; t) = WH(X; 1, t)$ Poincaré polynomial
- $E(X; q) = q^{d_X} WH(1/q, -1)$ E-polynomial of X (for smooth X)

Theorem (Katz 2008)

If M is a smooth quasi-projective variety defined over \mathbb{Z} and

$$\#\{M(\mathbb{F}_q)\} = E(q)$$

is a polynomial in q , then $E(M; q) = E(q)$.

E-polynomials of \mathcal{M}_B^n

Theorem (Frobenius 1896, Hurwitz 1902, Freed-Quinn 1993,...)

Let $z \in G$ in a finite group G then

$$\begin{aligned} \frac{1}{|G|} \# \{a_j, b_j \in G | [a_1, b_1] \dots [a_g, b_g] = z\} &= \\ &= \sum_{\chi \in \text{Irr}(G)} \frac{|G|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(z)}{\chi(1)} \end{aligned}$$

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^n; q) \stackrel{\text{Katz}}{=} |\mathcal{M}_B^n(\mathbb{F}_q)| = \sum_{\chi \in \text{Irr}(\text{GL}_n(\mathbb{F}_q))} \frac{|\text{GL}_n(\mathbb{F}_q)|^{2g-2}(q-1)}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$$

- $\sim E(\mathcal{M}_B^{d,r}; q) = Z_{\text{GL}_n(\mathbb{F}_q)}(C, \{\xi_n^d\})$ is the partition function of 2D Chern-Simons theory with finite gauge group $\text{GL}_n(\mathbb{F}_q)$ of (Freed-Quinn 1993)

- e.g. $\frac{E(\mathcal{M}_B^2)}{(q-1)^{2g}} =$

$$(q^2 - 1)^{2g-2} + q^{2g-2}(q^2 - 1)^{2g-2} - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}$$

Weight polynomials of \mathcal{M}_B^n

Conjecture (Hausel–Villegas, 2008)

$$\sum_{\lambda} \prod \frac{(z^{2l+1}-w^{2a+1})^{2g}}{(z^{2l+2}-w^{2a})(z^{2l}-w^{2a+2})} T^{|\lambda|} = \exp \left(\sum_{n,k} \frac{WH(\mathcal{M}_B^n; w^{2k}, -(zw)^{-2k})(zw)^{d_n}}{(z^{2k}-1)(1-w^{2k})} \frac{T^{nk}}{k} \right)$$

- when $r = 2$ Conjecture \leadsto Theorem (Hausel–Villegas, 2008)

$$\begin{aligned} \frac{WH(\mathcal{M}_B^2; q, t)}{(1+qt)^{2g}} &= \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \\ &- \frac{1}{2} \frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2} \frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)} \end{aligned}$$

- Conjecture is consistent with formula for
 - $P(\mathcal{M}_{Dol}^3; t)$ of (Gothen 1994)
 - $P(\mathcal{M}_{Dol}^4; t)$ of (Garcia-Prada–Heinloth–Schmitt 2011)
 - conjectured for $P(\mathcal{M}_{Dol}^n; t)$ by (Chuang–Diaconescu–Pan 2010) via (Mozgovoy 2011) and (Mellit 2016)
 - $P(\mathcal{M}_{Dol}^n; t)$ of (Maulik–Pixton, 2016>) announced rigorous completion of the CDP picture
 - $WH(\mathcal{M}_{Dol}^n; q, t)$ by (Chuang–Diaconescu–Pan 2012) as refined Gopakumar–Vafa conj for local curve CY 3-fold

Curious Hard Lefschetz

- (Hausel-Villegas 2008) calculates

$$E(\mathcal{M}_B^n; q) = |\mathcal{M}_B^n(\mathbb{F}_q)| = \sum_{\chi \in Irr(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-2}} \frac{\chi(\xi_n^d)}{\chi(1)}$$

- we find $E(\mathcal{M}_B^n; q) = q^{d_n} E(\mathcal{M}_B^n; 1/q)$ palindromic by *Alvis-Curtis duality*

$$q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q) \text{ for dual pair } \chi, \chi' \in Irr(\mathrm{GL}_n(\mathbb{F}_q))$$

- \leadsto Curious Hard Lefschetz Conjecture (theorem when $n = 2$):

$$\begin{aligned} L^I : \quad Gr_{d_n-2I}^W H^{i-I}(\mathcal{M}_B^n) &\rightarrow Gr_{d_n+2I}^W H^{i+I}(\mathcal{M}_B^n) \\ x &\mapsto x \cup \alpha^I \end{aligned}$$

where $\alpha \in W_4 H^2(\mathcal{M}_B^n)$

- The implied functional equation on the conjectured $WH(\mathcal{M}_B^n; q, t) = (qt)^{d_n} WH(\mathcal{M}_B^n; \frac{1}{qt^2}, t)$ holds

Perverse filtration

- $f : X \rightarrow Y$ a *proper* map between complex algebraic varieties of relative dimension d
- (de Cataldo-Migliorini 2005) introduce *perverse filtration*
 $P_0 \subset \dots \subset P_i \subset \dots P_k(X) \cong H^k(X)$ from the study of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem for $Rf_*(\mathbb{Q}_X)$ into perverse sheaves
- recipe (de Cataldo-Migliorini, 2008) for perverse filtration when X smooth and Y affine:

$Y_0 \subset \dots \subset Y_i \subset \dots Y_d = Y$ s.t.l Y_i generic with $\dim(Y_i) = i$

$$P_{k-i-1}H^k(X) = \ker(H^k(X) \rightarrow H^k(f^{-1}(Y_i)))$$

- the Relative Hard Lefschetz Theorem holds:

$$\begin{array}{ccc} L^I : & Gr_{d-I}^P H^*(X) & \rightarrow & Gr_{d+I}^P H^{*+2I}(X) \\ & x & \mapsto & x \cup \alpha^I \end{array}$$

where $\alpha \in H^2(X)$ is a relative ample class

P=W conjecture

- recall Hitchin map $\chi : \begin{matrix} \mathcal{M}_{\text{Dol}}^n \\ (E, \phi) \end{matrix} \rightarrow \mathbb{A} := \bigoplus_{i=1}^n H^0(C; K^i)$
 $\chi(E, \phi) \mapsto \text{charpol}(\phi)$
- (Hitchin 1987) \rightarrow completely integrable Hamiltonian system
and *proper*

Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)

$P_k(\mathcal{M}_{\text{Dol}}^n) \cong W_{2k}(\mathcal{M}_B^n)$ under the isomorphism

$H^*(\mathcal{M}_{\text{Dol}}^n) \cong H^*(\mathcal{M}_B^n)$ from non-Abelian Hodge theory

In particular $WH(\mathcal{M}_B^n; q, t) = PH(\mathcal{M}_{\text{Dol}}^n; q, t)$

Theorem (de Cataldo-Hausel-Migliorini 2010)

$P = W$ for $n = 2$

$$\begin{aligned} \frac{PH(\mathcal{M}_{\text{Dol}}^2; q, t)}{(1 + qt)^{2g}} &= \frac{(q^2t^3 + 1)^{2g}}{(q^2t^2 - 1)(q^2t^4 - 1)} + \frac{q^{2g-2}t^{4g-4}(q^2t + 1)^{2g}}{(q^2 - 1)(q^2t^2 - 1)} - \\ &- \frac{1}{2} \frac{q^{2g-2}t^{4g-4}(qt + 1)^{2g}}{(qt^2 - 1)(q - 1)} - \frac{1}{2} \frac{q^{2g-2}t^{4g-4}(qt - 1)^{2g}}{(q + 1)(qt^2 + 1)} \end{aligned}$$

Preverse Hirzebruch y-genus

- $PH(\mathcal{M}_{\text{Dol}}^n; q, x, y) = \sum \dim(H^{l,m}(P_i/P_{i-1}(H^{l+m}(\mathcal{M}_{\text{Dol}}))))x^ly^m q^i$
perverse Hodge polynomial
 $PH(\mathcal{M}_{\text{Dol}}^n; 1, x, y)$ *Hodge polynomial*
 $PH(\mathcal{M}_{\text{Dol}}^n; 1, -1, y)$ *Hirzebruch y-genus*
 $PH(\mathcal{M}_{\text{Dol}}^n; q, -1, y)$ *perverse Hirzebruch y-genus*

Conjecture (Hausel 2004 + de Cataldo–Hausel–Migliorini 2010)

$$\sum_{\lambda} \prod \frac{(qxy)^{(2-2g)l} (1-(qxy)^lyq^{a+1})^g (1-(qxy)^lxq^{a+1})^g}{(1-(qxy)^{l+1}q^a)(1-(qxy)^lq^{a+1})} T^{|\lambda|} = \\ \exp \left(\sum_{n,k} \frac{PH(\mathcal{M}_{\text{Dol}}^n; q, -x, -y) (qxy)^{d_n}}{(1-qxy)(1-q)} \frac{T^{nk}}{k} \right)$$

- the implied conjecture on $PH(\mathcal{M}_{\text{Dol}}^n; 1, -1, y)$ proved by (Garcia-Prada, Heinloth 2013)
- (Hausel 2004) \leadsto the implied conjecture on $PH(\mathcal{M}_{\text{Dol}}^n; q, -1, y)$ is manifestly polynomial

A TQFT problem

- the upgraded formula $\frac{PH(\mathcal{M}_{Dol}^2; q, x, y)}{(1+qx)^g(1+qy)^g}$ gives

$$\frac{(q^2x^2y + 1)^g(q^2y^2x + 1)^g}{(q^2xy - 1)(q^2(xy)^2 - 1)} + \frac{(qxy)^{2g-2}(q^2x + 1)^g(q^2y + 1)^g}{(q^2 - 1)(q^2xy - 1)} - \\ - \frac{1}{2} \frac{(qxy)^{2g-2}(qx + 1)^g(qy + 1)^g}{(qxy - 1)(q - 1)} - \frac{1}{2} \frac{(qxy)^{2g-2}(qx - 1)^g(qy - 1)^g}{(q + 1)(qxy + 1)}$$

- it gives $PH(\mathcal{M}_{Dol}^2; 1, x, y)$ the Hodge polynomial and $PH(\mathcal{M}_{Dol}^2; 1, -1, y)$ the Hirzebruch y -genus of \mathcal{M}_{Dol}^2
- we get for $\frac{PH(\mathcal{M}_{Dol}^2; q, -1, y)}{(1-q)^g(1+qy)^g}$

$$\left((q^2y + 1)(q^2y^2 - 1)\right)^{g-1} + \left((qy)^2(q^2 - 1)(q^2y + 1)\right)^{g-1} - \\ - \frac{1}{2} \left((qy)^2(q - 1)(qy + 1)\right)^{g-1} - \frac{1}{2} \left((qy)^2(q - 1)(qy - 1)\right)^{g-1}$$

- Problem: is this the partition function of a TQFT on C ?