Title: A mathematical definition of 3d indices

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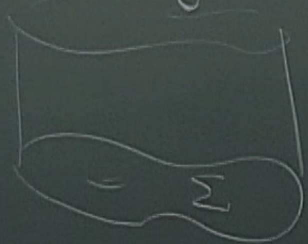
Abstract: 3d field theories with N=2 supersymmetry play a special role in the evolving web of connections between geometry and physics originating in the 6d (2,0) theory. Specifically, these 3d theories are associated to 3-manifolds M, and their vacuum structure captures the geometry of local systems on M. (Sometimes M arises as a cobordism between two surfaces C, C', in which case the 3d theories encode some functorial relation between the geometry of Hitchin systems on C and C'.) I would like to explain some of the mathematics of 3d N=2 theories. In particular, I would like to explain how Hilbert spaces in these theories arise as Dolbeault cohomology of certain moduli spaces of bundles. One application is a homological interpretation of the "pentagon relation" relating flips of triangulation on a surface.

Hilbert space of 3d $\mathcal{N}=2$ thy's

- SUSY g.s.
-



$\mathcal{H}(\Sigma)$

today: $\Sigma = D$

Why?

3d $\mathcal{N}=2$ dualities

e.g. particle - vortex

Usually studied w/ indices

part^n f^n on $\left\{ \begin{matrix} S^2 \times S^1 \\ D \times S^1 \end{matrix} \right\}$

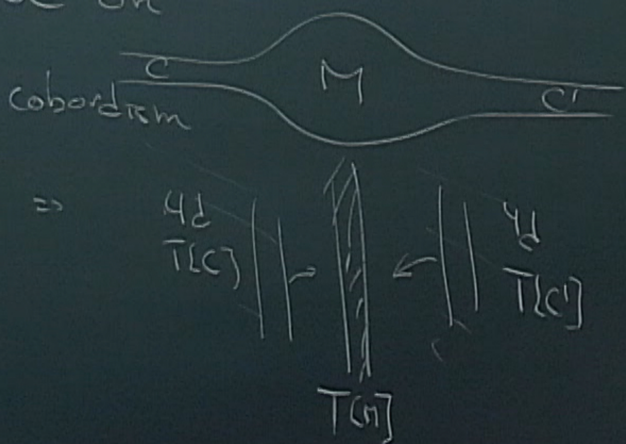
data of thy $\xrightarrow{\text{geometry}}$ functions build part^n f^n

Class S story

6d (2,0) thy on C (surface) $\rightsquigarrow$ 4d $\mathcal{N}=2$

6d on M 3-manifold $\rightsquigarrow$ 3d $\mathcal{N}=2$

6d on



big goal: categorify interfaces

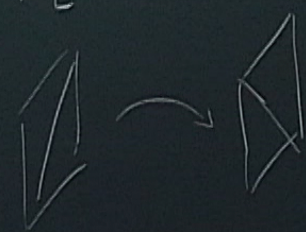
$\mathcal{H}(T[M])$

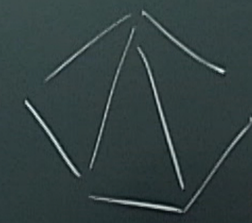
as bimodule

for $\otimes$ category of line ops (...es)

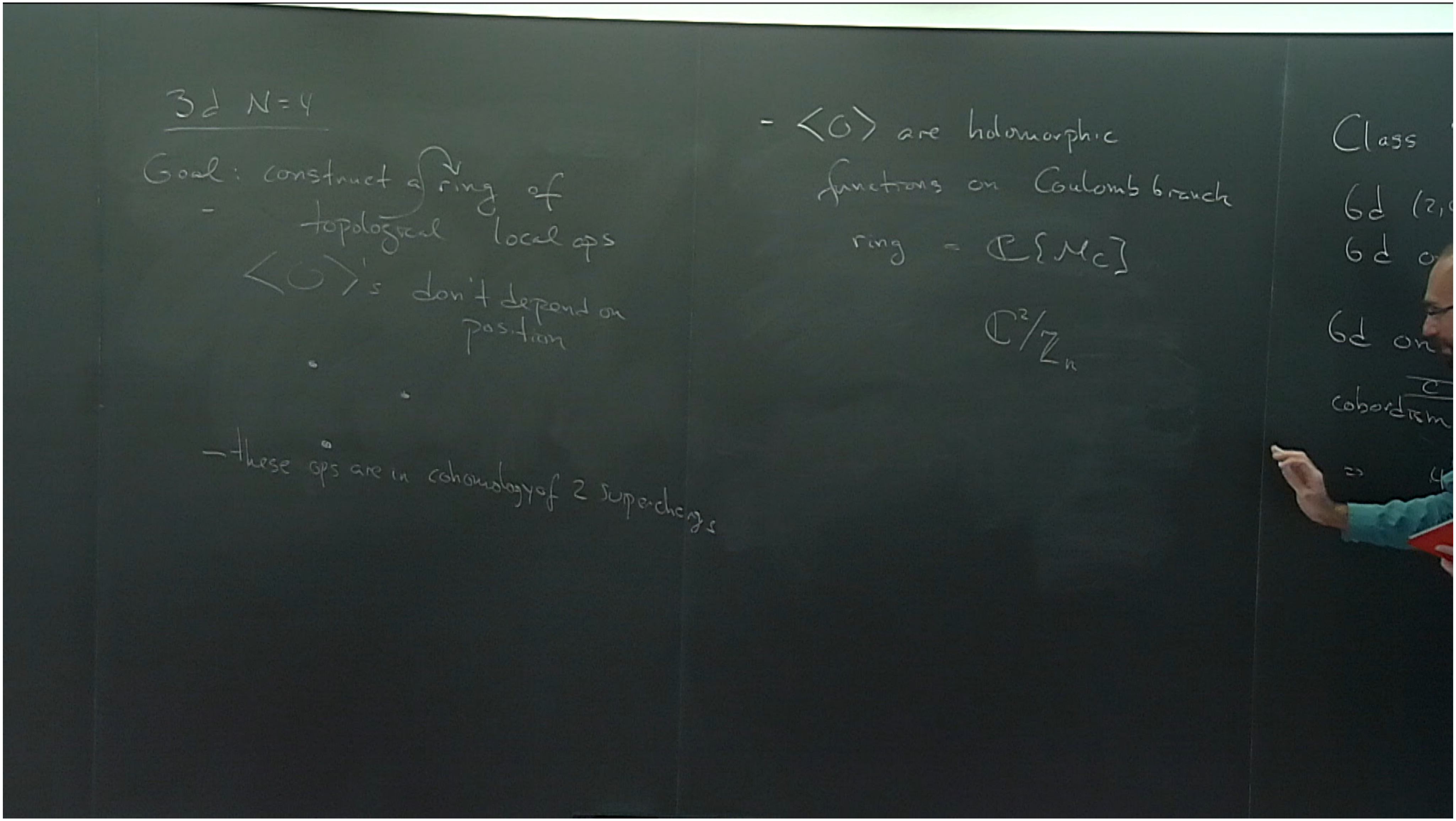
Simplest 3d $\mathcal{N}=2$ thy

$T[\triangle] =$ free chiral





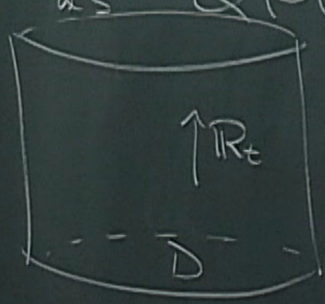

pentagon
(2-3 move)
SQED ↔ XYZ


3d $\mathcal{N}=4$
Goal: construct a ring of
topological local ops
<O>'s don't depend on position
– these ops are in cohomology of 2 supercharges
– <O> are holomorphic functions on Coulomb branch
ring = $\mathbb{C}[\mathcal{M}_C]$
$\mathbb{C}^2/\mathbb{Z}_n$
Class
6d
6d
6d on
cobordism

ranch

Useful: rewrite 3d $\mathcal{N}=4$ thy

as QM



$\mathbb{R}_t \times \mathbb{C}$

$\times D$

target $= \{$ all field configs on $D \}$

3d thy labelled by

G cpt gp

$R \oplus R^*$ quat^c rep

$= \{ (E, X, Y) : E$ $G_{\mathbb{C}}$ bundle on D, X, Y sections of $R \oplus R^*$ bundle $\}$

SQED

T

3d thy labelled by

G cpt gp

$R \oplus R^*$ quat^c rep

d

$$= \left\{ (E, X, Y) : E \ G_{\mathbb{C}} \text{ bundle on } D, \ X, Y \text{ sections of } (R \oplus R^*) \text{ bundle} \otimes K^{1/2} \right\} /$$

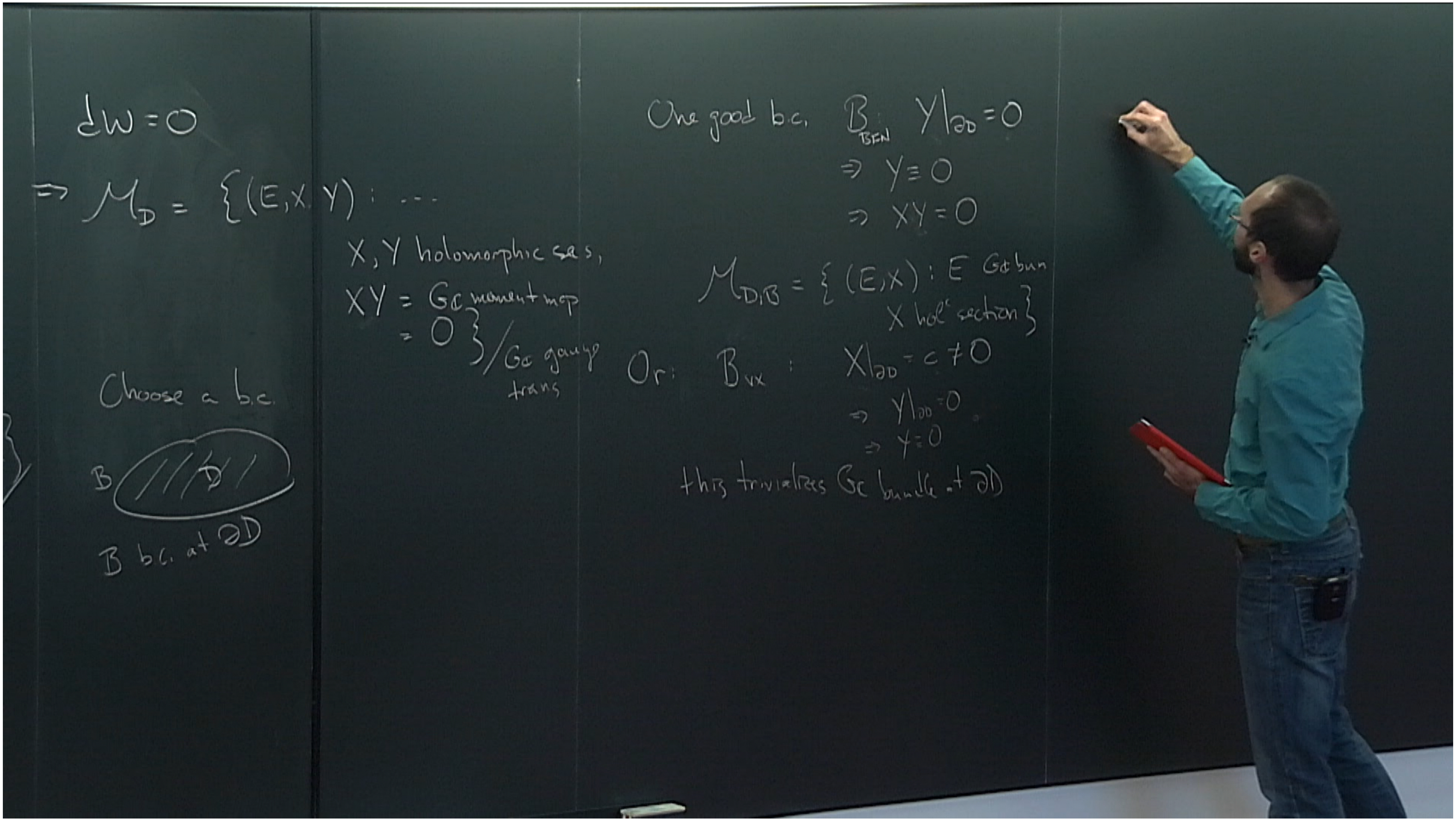
$$W = \int_D \langle Y, D_{\bar{z}} X \rangle$$

$$dW = 0$$

$$\Rightarrow \mathcal{M}_D = \{ (E, X, Y) : \ldots$$

X, Y holomorphic secs,

$XY = G_{\mathbb{C}}$ moment map $= 0 \} / G_{\mathbb{C}}$ gauge trans


dW = 0
⇒ M_D = {(E, X, Y) : . . .
X, Y holomorphic secs,
XY = G_C moment map
= 0 } / G_C gauge trans
Choose a b.c.
B
D
B b.c. at ∂D
One good b.c. B_BFN Y|∂D = 0
⇒ Y ≡ 0
⇒ XY = 0
M_{D,B} = { (E, X) : E G_C bun
X hol^c section }
Or: B_vx : X|∂D = c ≠ 0
⇒ Y|∂D = 0
⇒ Y = 0
this trivializes G_C bundle at ∂D

$|_{\partial D} = 0$

$= 0$

$E\ G_{\mathbb{C}}$ bun

c section $\}$

$\neq 0$

$= 0$

at ∂D

$$\mathcal{M}_{D \times x} = \Big\{ (E, X) \;\Big|\; E\ G_{\mathbb{C}}\text{ bundle},\ X \text{ hol}^c,\ X \to c \text{ at } \partial D \ \ (\text{up to } G_{\mathbb{C}} \text{ gauge trans}) \Big\}$$

$$= \Big\{ (E, X) \;\Big|\; E\ G_{\mathbb{C}}\text{ bundle on } \mathbb{P}^1,\ X \text{ hol}^c \to c \text{ at } \infty \Big\}$$

$$= \coprod_{\text{degree } d} \mathcal{M}^d_{D \times x}$$

Either case:

$$\mathcal{H} = H^\bullet_{dR}(\mathcal{M}_{D,B})$$

BFN $= H^\bullet_{G_{\mathcal{K}}}(p)$

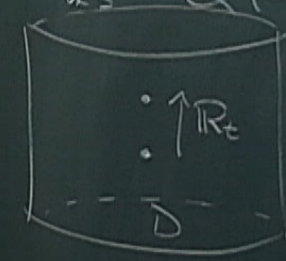
Vx $= \bigoplus_d H^\bullet(\mathcal{M}^d_{Vx})$

- $\langle \mathcal{O} \rangle$ are holomorphic functions on Coulomb branch ring $= \mathbb{C}[\mathcal{M}_C]$

ops: $D \rightsquigarrow$ $\mathbb{C}^2/\mathbb{Z}_n$

$D \cup_{D^*} D$

Useful: rewrite as QM



3d $\mathcal{N}=2$

data:
- G cpt gp
- R unitary rep
- $W \in \mathrm{Hom}_{G_{\mathbb{C}}}(R, \mathbb{C})$ hol^c superpot^l
- $k \in H^4(BG)$

rewrite as QM

$$\text{target} = \{ (E, X, \gamma) \mid E \ G_{\mathbb{C}}\text{ bundle}, \ X, \gamma \text{ sections of } R \oplus R^*[-1] \}$$

<u>two</u> superpotentials

impose
$$\left\{ \begin{array}{l} dW(X) = 0 \\ D_{\bar z} X = D_{\bar z} \gamma = 0 \end{array} \right\}$$

$G_{\mathbb{C}}$ bundle

$\{R \oplus R^*_{[-1]}\}$

So $\mathcal{M}_D = \{(E, X, Y) \mid dW = 0$
X, Y hol$^c\}$

Choose b.c. B

Lagrangian in shifted symp^c space

B $D_{-1,2}$

$$\mathcal{H} = H^{\circ}_{\bar\partial}(\mathcal{M}_D, B)$$

E.g.

$T[\Delta]$ free chiral

$G = id$

$R = \mathbb{C}$

$W = 0$

X, Y secs of $\mathbb{C} \oplus \mathbb{C}_{[-1]}$ bundle on D

$\mathcal{M}_D = \{$ polys $X(z), Y(z)\}$

$G \circlearrowright$

$U(1)_e$ flavor

$(X, Y) \to (e^{i\theta} X, e^{-i\theta} Y)$

$U(1)_J$ loop rotation $z \to e^{i\theta} z$

$$B: \quad \gamma\big|_{\partial D} = 0$$

$\ldots, \gamma$ secs of $\mathbb{C} \oplus \mathbb{C}_{(-1)}$

bundle on D

$$\mathcal{M}_{D,B} = \{\text{polys } X(z)\} = \mathbb{C}[z] = \bigoplus_{n \geq 0} \mathbb{C}\langle z^n \rangle$$

$X(z), \gamma(z)\}$

$$\mathcal{H} = H_{\bar\partial}(\mathcal{M}_{D,B})$$

$$\mathrm{Tr}_{\mathcal{H}}\, \underset{\text{hom}}{(-1)^F}\, q^{J/2}\, x^e = \frac{1}{1-x}\,\frac{1}{1-qx}\,\frac{1}{1-q^2x}\cdots = \frac{1}{(x,q)_\infty}$$

$\text{on} \ \to e^{i\theta} z$

Dual: $G = U(1)$

$R = \mathbb{C}$

$W = 0$

$\widetilde{\mathcal{M}}_D = \{ \mathbb{C}^* \text{ bundles on } D$

$X(z),\ Y(z)$ hol' sections$\}$

$\mathcal{O}(d) \quad \mathcal{O}(-d)$

$\widetilde{B}$: $X|_{\partial D} = c \neq 0$

$$\widetilde{\mathcal{M}}_{D,B} = \coprod_{d \geq 0} \underbrace{\mathcal{M}^d_{D,B}}_{\{\text{polys } X(z) \text{ of degree } d\}}$$

$$\mathrm{Tr}_{\widetilde{\mathcal{H}}} = \sum_d x^d \frac{1}{(1-q)(1-q^2)\cdots(1-q^d)} = \sum_d \frac{x^d}{(q)_d}$$

3d $\mathcal{N}=2$

data: G cpt g

R unit

$W \in$ Ho

hol

$k \in H^4($