Title: Nahm transformation for parabolic harmonic bundles on the projective line with regular residues

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Abstract: I will define a generalization of the classical Laplace transform for D-modules on the projective line to parabolic harmonic bundles with finitely many logarithmic singularities with regular residues and one irregular singularity, and show some of its properties. The construction involves on the analytic side L2-cohomology, and it has algebraic de Rham and Dolbeault interpretations using certain elementary modifications of complexes. We establish stationary phase formulas, in patricular a transformation rule for the parabolic weights. In the regular semi-simple case we show that the transformation is a hyper-Kaehler isometry.

Nahm transform for parabolic harmonic bundles on the Riemann sphere with regular residues

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NOTATIONS

$G = \mathrm{Gl}_r(\mathbf{C})$
$\mathbf{P}^1$: the Riemann sphere $\mathbf{C} \cup \{\infty\}$
$U \subset \mathbf{P}^1$: an analytic open set
$P = \{z_0 = \infty, z_1, \ldots, z_n\}$: a finite set of distinct points in $\mathbf{P}^1$
$\mathcal{O}$: sheaf of holomorphic functions on $\mathbf{P}^1$
Ω^k: sheaf of smooth k-forms on $\mathbf{P}^1$
K: sheaf of holomorphic 1-forms on $\mathbf{P}^1$
V: smooth vector bundle over $\mathbf{P}^1$
$E, \mathcal{E}$: holomorphic vector bundles with underlying smooth vector bundle V

Motivation

Nahm's transformation is an integral transform for solutions of the dimensionally reduced Yang–Mills (ASD) equations

$$*F_A = -F_A$$

for a connection A in a Hermitian vector bundle over $(\mathbf{R}^4, g_{Eucl})$.
It maps invariant solutions with respect to some additive subgroup $\Lambda \subset \mathbf{R}^4$ to invariant solutions with respect to the dual subgroup $\Lambda^* \subset (\mathbf{R}^4)^*$.
$\mathbf{R}^2$-invariant solutions $\rightsquigarrow$ Hitchin's equations over $\mathbf{R}^2 = \mathbf{C}$.
Possible application: Katz' middle convolution algorithm.
Related work: push-forward of parabolic structure from the Dolbeault point of view by R. Donagi–T. Pantev–C. Simpson ('16), using C. Sabbah's and T. Mochizuki's work on pure wild twistor $\mathcal{D}$-modules.

Examples

- YM-instantons over $\mathbf{R}^4 \leftrightarrow$ representations of a quiver (ADHM-transform)
- Instantons on a torus $T^4 \leftrightarrow$ instantons on the dual torus $(T^4)^*$ (P. Braam, P. van Baal)
- Monopole equations over $\mathbf{R}^3 \leftrightarrow$ Nahm's equations on an interval (N. Hitchin, S. Donaldson, H. Nakajima,...)
- Calorons $\leftrightarrow$ Nahm's equations on a circle (W. Nahm)
- doubly-periodic instantons $\leftrightarrow$ singular solutions of Hithin's equations (M. Jardim, O. Biquard, T. Mochizuki)
- Spatially periodic instantons $\leftrightarrow$ singular monopoles (B. Charbonneau)
- periodic monopoles $\leftrightarrow$ Hitchin's equations on the cylinder (S. Cherkis, A. Kapustin)
- YM-instantons over a multi-Taub–NUT space $\leftrightarrow$ Cherkis' bow varieties

Decomposition of flat connections

Over any $U \subset \mathbf{P}^1 \setminus P$ open let

- D be a flat connection in V,
- and h a Hermitian metric on V.

Consider the decomposition

$$D = D^+ + \Phi$$

of D into h-unitary and self-adjoint parts respectively.

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$$D = D^+ + \Phi$$

of D into h-unitary and self-adjoint parts respectively.
Decompose these parts further according to their bidegree:

$$\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$$
$$D^+ = \partial^+ + \bar{\partial}^+$$
$$\Phi = \theta + \theta^*.$$

Harmonic bundles

Then, (D, h) is said to satisfy Hitchin's equations if

$$\bar{\partial}^{+}\theta = 0.$$

Notation

We denote the holomorphic vector bundle $(V, \bar{\partial}^{+})$ by $\mathcal{E}$. We also set

$$\bar{\partial}^{+} = \bar{\partial}^{\mathcal{E}}.$$

We call such a triple (V, D, h) a harmonic bundle over U.

Parabolic structures

A quasi-parabolic structure of $\mathcal{E}$ at $z_i \in P$:

$$\{0\} = F_i^{l_i} \subset F_i^{l_i - 1} \subset \cdots \subset F_i^1 \subset F_i^0 = V|_{z_i}$$

for some $1 \leq l_i \leq r$.

A parabolic structure of $\mathcal{E}$ at $z_i \in P$: a quasi-parabolic structure and real numbers

$$1 > \alpha_i^{l_i - 1} > \cdots > \alpha_i^0 \geq 0$$

called parabolic weights.

Compatible logarithmic singularities

Consider harmonic bundles such that

$$\theta \in H^0(\mathcal{E}nd(\mathcal{E}) \otimes K(2 \cdot z_0 + z_1 + \cdots + z_n)).$$

For all $1 \leq i \leq n$ we require that $\mathrm{res}_{z=z_i}(\theta)$ respects the filtration $F_i^\bullet$. We derive graded vector spaces

$$\mathrm{Gr}_i^j = \mathrm{Gr}_{F_i}^j = F_i^j / F_i^{j+1}$$

and linear maps

$$\mathrm{Gr}^j \, \mathrm{res}_{z=z_i}(\theta) \in \mathrm{End}(\mathrm{Gr}_i^j).$$

Consider the decomposition

$$\mathrm{Gr}^j \, \mathrm{res}_{z=z_i}(\theta) = S_i^j + N_i^j$$

into semi-simple and nilpotent components.

Weight filtration

There exists an increasing filtration

$$0 = W^j_{i,-r} \subseteq W^j_{i,1-r} \subseteq \cdots \subseteq W^j_{i,r} = \mathrm{Gr}^j_{F_i}$$

satisfying

- for all $k \in \mathbf{Z}$ the endomorphism N^j_i maps $W^j_{i,k}$ into $W^j_{i,k-2}$
- for all $k \in \mathbf{N}$ we have an isomorphism

$$(N^j_i)^k = N^j_i \circ \cdots \circ N^j_i : \mathrm{Gr}^{W^j_i}_k \mathrm{Gr}^j_{F_i} \cong \mathrm{Gr}^{W^j_i}_{-k} \mathrm{Gr}^j_{F_i}.$$

Compatible bases and Hermitian metrics

Let $z_i \in U_i$ be a small neighborhood. We say that a trivialization $\{\mathbf{e}_i^s\}_{s=1}^r$ of $\mathcal{E}$ over U_i is compatible with the quasi-parabolic structure if for all j we have

$$F_i^j = \mathbf{C}\left\langle \mathbf{e}_i^1(p_i), \ldots, \mathbf{e}_i^{\dim F_i^j}(p_i) \right\rangle.$$

For $1 \leq s \leq r$ we let $j(s)$ stand for the largest $j \in \{0, \ldots, l_i - 1\}$ such that $\mathbf{e}_i^s \in F_i^j$ and $k(s)$ stand for the smallest $k \in \mathbf{Z}$ such that $\mathbf{e}_i^s \in W_{i,k}^{j(s)}$.
A Hermitian metric h in $\mathcal{E}$ over U_i is said to be compatible with θ if and only if with respect to some compatible $\{\mathbf{e}_i^s\}_{s=1}^r$ it is mutually bounded with the diagonal metric

$$h_0 = \mathrm{diag}(|z - z_i|^{2\alpha_i^{j(s)}}(-\log|z - z_i|)^{k(s)})_{s=1}^r.$$

Main assumption

Assumption

- *if* $\alpha_i^0 = 0$ *then the nilpotent part* N_i^0 *acts trivially on the generalized 0-eigenspace of* $\mathrm{Gr}^0 \operatorname{res}_{z=z_i}(\theta)$;
- *for* $\alpha_i^j > 0$ *the kernel of* $\mathrm{Gr}^0 \operatorname{res}_{z=z_i}(\theta)$ *vanishes:*

$$\ker \mathrm{Gr}^j \operatorname{res}_{z_i}(\theta) = 0.$$

Remark

Necessary for our construction at this stage.
Ongoing joint work with Takuro Mochizuki: lift these restrictions.

Local form at z_0

Near $z_0 = \infty$ we assume that with respect to some compatible $\{\mathbf{e}_0^s\}_{s=1}^r$ the Higgs field has the form

$$\theta = \frac{A}{2}\mathrm{d}z + B\frac{\mathrm{d}z}{z} + \text{lower order terms},$$

for some diagonal $A \in \mathfrak{gl}_r(\mathbf{C})$ and arbitrary $B \in \mathfrak{gl}_r(\mathbf{C})$, preserving the filtration $F_0^\bullet$.

Let us denote by $\mathfrak{h}$ the Lie-algebra of the centralizer of A in $\mathrm{Gl}_r(\mathbf{C})$. Then we may assume that $B \in \mathfrak{h}$.

A Hermitian metric h in $\mathcal{E}$ over U_0 is said to be compatible with θ if it is mutually bounded with the diagonal metric

$$h_0 = \mathrm{diag}(|z|^{-2\alpha_0^j(s)}(\log |z|)^{k(s)})_{s=1}^r.$$

Assumption

$\ker \mathrm{Gr}^j \operatorname{res}_{z_0}(\theta) = 0.$

Moduli spaces

Fix

- $r, z_1, \ldots, z_n$,
- eigenvalues of A and their multiplicities,
- dimensions of $\mathrm{Gr}^j_{F_i}$,
- parabolic weights α^j_i such that $\sum_{i,j} \alpha^j_i \dim \mathrm{Gr}^j_{F_i} \in \mathbf{Z}$,
- regular coadjoint orbits $\mathcal{O}^j_i \subset \mathrm{Gr}^j_{F_i}$.

Simpson ('90), Biquard–Boalch ('04): Solutions of Hitchin's equations with the above asymptotics and such that

$$\mathrm{Gr}^j \operatorname{res}_{z_i}(\theta) \in \mathcal{O}^j_i$$

form a complete hyper-Kähler moduli space

$$\mathcal{M}(\mathbf{P}^1, r, \{z_i\}, A, \{\dim \mathrm{Gr}^j_{F_i}\}, \{\alpha^j_i\}, \{\mathcal{O}^j_i\}).$$

Aim: to describe some isometries between such moduli spaces corresponding to various data.

Exponential twist

For $\zeta \in \widehat{\mathbf{C}} \setminus \widehat{P}$ we consider the twisted flat connection

$$D_\zeta = D - \zeta \mathrm{d}z.$$

and the associated elliptic complex $(L^2(V \otimes \Omega^\bullet), D_\zeta)$:

$$0 \to L^2(V) \xrightarrow{D_\zeta} L^2(V \otimes \Omega^1) \xrightarrow{D_\zeta} L^2(V \otimes \Omega^2) \to 0.$$

Easy to check that:

$$H^0(L^2(V \otimes \Omega^\bullet), D_\zeta) = 0 = H^2(L^2(V \otimes \Omega^\bullet), D_\zeta).$$

We define

$$\widehat{V}_\zeta = H^1(L^2(V \otimes \Omega^\bullet), D_\zeta).$$

Kodaira–Spencer: $\widehat{V}$ is a smooth vector bundle over $\widehat{\mathbf{C}} \setminus \widehat{P}$.

Hodge theory for twisted harmonic bundles

We introduce the Laplace operator

$$\Delta_\zeta = -D_\zeta D_\zeta^* - D_\zeta^* D_\zeta : \Omega^1 \otimes V \to \Omega^1 \otimes V.$$

Proposition

We have $\widehat{V}_\zeta = \ker(\Delta_\zeta)$ on its L^2-domain.

Paralelly, set

$$\theta_\zeta = \theta - \frac{\zeta}{2}\mathrm{d}z \quad \text{and} \quad D''_\zeta = \bar{\partial}^{\mathcal{E}} + \theta_\zeta.$$

Hodge relation:

$$\ker(\Delta_\zeta) \cong \ker(-D''_\zeta (D''_\zeta)^* - (D''_\zeta)^* D''_\zeta) = H^1(L^2(V \otimes \Omega^\bullet), D''_\zeta).$$

The algebraic Dolbeault complex

Proposition

There exists explicit subsheaves $\mathcal{F}, \mathcal{G}$ of $\mathcal{E}$ so that we have

$$H^1(L^2(V \otimes \Omega^\bullet), D''_\zeta) = \mathbf{H}^1\left(\mathcal{F} \xrightarrow{\theta_\zeta} \mathcal{G} \otimes K_{\mathbf{P}^1}(2 \cdot z_0 + z_1 + \cdots + z_n)\right).$$

(Analog for the Poincaré metric: C. Sabbah, T. Mochizuki.)

This endows $\widehat{V}$ with a holomorphic structure, denoted $\widehat{\mathcal{E}}$.

Local frame for $\mathcal{G}$

At logarithmic points z_i ($i > 0$) a local frame $\{\mathbf{g}_i^s\}_{s=1}^r$ for $\mathcal{G}$ may be defined from a compatible trivialization of $\mathcal{E}$ as follows:

- if $\alpha_i^{j(s)} = 0$ and $k(s) \geq -1$: set $\mathbf{g}_i^s = (z - z_i)\mathbf{e}_i^s$;
- otherwise set $\mathbf{g}_i^s = \mathbf{e}_i^s$.

At z_0, a local frame is defined by

- if $\alpha_0^{j(s)} = 0$ and $k(s) \geq -1$: set $\mathbf{g}_0^s = z^{-2}\mathbf{e}_0^s$;
- otherwise set $\mathbf{g}_0^s = z^{-1}\mathbf{e}_0^s$.

Local frame for $\mathcal{F}$

Let λ_i^s stand for the eigenvalue of $\mathrm{Gr}^{j(s)} \operatorname{res}_{z_i}(\theta)$ corresponding to the vector $\mathbf{e}_i^s$.

At z_i with $i > 0$ we define a local frame $\{\mathbf{f}_i^s\}_{s=1}^r$ for $\mathcal{F}$ as:

- if $\alpha_i^{j(s)} = 0 \neq \lambda_i^s$ and $k(s) \geq -1$: set $\mathbf{f}_i^s = (z - z_i)\mathbf{e}_i^s$;
- otherwise set $\mathbf{f}_i^s = \mathbf{e}_i^s$.

At z_0, we let $\mathbf{f}_0^s = \mathbf{g}_0^s$.

Local frame for $\mathcal{G}$

At logarithmic points z_i $(i > 0)$ a local frame $\{\mathbf{g}_i^s\}_{s=1}^r$ for $\mathcal{G}$ may be defined from a compatible trivialization of $\mathcal{E}$ as follows:

- if $\alpha_i^{j(s)} = 0$ and $k(s) \geq -1$: set $\mathbf{g}_i^s = (z - z_i)\mathbf{e}_i^s$;
- otherwise set $\mathbf{g}_i^s = \mathbf{e}_i^s$.

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- otherwise set $\mathbf{g}_0^s = z^{-1}\mathbf{e}_0^s$.

Local frame for $\mathcal{G}$

At logarithmic points z_i $(i > 0)$ a local frame $\{\mathbf{g}_i^s\}_{s=1}^r$ for $\mathcal{G}$ may be defined from a compatible trivialization of $\mathcal{E}$ as follows:

- if $\alpha_i^{j(s)} = 0$ and $k(s) \geq -1$: set $\mathbf{g}_i^s = (z - z_i)\mathbf{e}_i^s$;
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At z_0, a local frame is defined by

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- otherwise set $\mathbf{g}_0^s = z^{-1}\mathbf{e}_0^s$.

The algebraic Dolbeault complex

Proposition

There exists explicit subsheaves $\mathcal{F}, \mathcal{G}$ of $\mathcal{E}$ so that we have

$$H^1(L^2(V \otimes \Omega^\bullet), D''_\zeta) = \mathbf{H}^1 \left(\mathcal{F} \xrightarrow{\theta_\zeta} \mathcal{G} \otimes K_{\mathbf{P}^1}(2 \cdot z_0 + z_1 + \cdots + z_n) \right).$$

(Analog for the Poincaré metric: C. Sabbah, T. Mochizuki.)
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Extension over the singularities

Let $s_0, s_\infty \in H^0(\widehat{\mathbf{P}}^1, \mathcal{O}_{\widehat{\mathbf{P}}^1}(1))$ be sections such that over $\widehat{\mathbf{C}}$ we have

$$s_0(\zeta) = \zeta, \quad s_\infty(\zeta) = 1.$$

In the product $\mathbf{P}^1 \times \widehat{\mathbf{P}}^1$ let π_m stand for projection on the m^{th} factor.
We consider the following extension of the Higgs field over $\widehat{\mathbf{P}}^1$

$$\theta_\zeta = \theta \otimes s_\infty - \frac{1}{2}\mathrm{Id}_{\mathcal{E}} dz \otimes s_0 : \pi_1^* \mathcal{F} \to \pi_1^* \mathcal{G} \otimes K_{\mathbf{P}^1}(2 \cdot z_0 + z_1 + \cdots + z_n) \otimes \pi_2^* \mathcal{O}_{\widehat{\mathbf{P}}^1}(1)$$

Proposition

The hypercohomology groups of degree 0 and 2 of this complex vanish for all $\zeta \in \widehat{\mathbf{P}}^1$.

We get an extension of $\widehat{\mathcal{E}}$ as a locally free sheaf on $\widehat{\mathbf{P}}^1$ given by

$$\widehat{\mathcal{E}} = \mathbf{R}^1(\pi_2)_*(\theta_\zeta).$$

Topology of $\widehat{\mathcal{E}}$

A computation using Grothendieck–Hirzebruch formula shows that

$$\mathrm{rank}(\widehat{\mathcal{E}}) = \hat{r} = \sum_{i=1}^{n} \left(r - \delta_{0,\alpha_i^0} \dim \ker \mathrm{Gr}_i^0 \right),$$

$$\deg(\widehat{\mathcal{E}}) = \deg(\mathcal{F}) + r + \hat{r}.$$

Filtered Dolbeault complex

Equivalent definition of parabolic structure for $\mathcal{E}$ along a reduced effective divisor D: a decreasing family $\mathcal{E}_\bullet$ of coherent sheaves on $\mathbf{P}^1$ indexed by $\mathbf{R}$ so that for all $\alpha \in \mathbf{R}$

- left-continuity: there exists some $\varepsilon > 0$ with $\mathcal{E}_{\alpha-\varepsilon} = \mathcal{E}_\alpha$;
- quasi-periodicity: we have $\mathcal{E}_{\alpha+1} = \mathcal{E}_\alpha \otimes \mathcal{O}_X(-D)$.

For all $\alpha \in [0,1)$ we consider a filtered version of the Dolbeault complex

$$\theta_{\alpha,\zeta} : \mathcal{F}_\alpha \to \mathcal{G}_\alpha \otimes K_{\mathbf{P}^1}(z_1 + \cdots + z_n + 2 \cdot z_0),$$

and set

$$\widehat{\mathcal{E}}_\alpha = \mathbf{R}^1(\pi_2)_*(\theta_{\alpha,\zeta}).$$

Filtered sheaves

Definition of $\mathcal{F}_\alpha, \mathcal{G}_\alpha$ locally near z_i for $i > 0$:

- if $\alpha_i^{j(s)} = 0 = \lambda_i^s$ we set

$$\mathbf{f}_i^s(\alpha) = \mathbf{f}_i^s, \quad \mathbf{g}_i^s(\alpha) = \mathbf{g}_i^s;$$

- otherwise, if $\alpha \leq \alpha_i^{j(s)}$ we set

$$\mathbf{f}_i^s(\alpha) = \mathbf{f}_i^s, \quad \mathbf{g}_i^s(\alpha) = \mathbf{g}_i^s;$$

- otherwise, if $\alpha_i^{j(s)} < \alpha$ we set

$$\mathbf{f}_i^s(\alpha) = (z - z_i)\mathbf{f}_i^s, \quad \mathbf{g}_i^s(\alpha) = (z - z_i)\mathbf{g}_i^s.$$

For $i = 0$, same definitions up to replacing $(z - z_i)$ by z^{-1}.

Transformation of the parabolic structure

Extend $\widehat{\mathcal{E}}_\alpha$ to all $\alpha \in \mathbf{R}$ by quasi-periodicity and set

$$\widehat{D} = \mathrm{div}(\widehat{P}) + \infty.$$

Proposition

$\widehat{\mathcal{E}}_\alpha$ is a parabolic bundle with parabolic divisor $\widehat{D}$.

Transformed Higgs field

Multiplication by $-z/2d\zeta$ induces a morphism

$$\widehat{\theta} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}} \otimes K_{\widehat{\mathbf{P}}^1}(*\widehat{D}).$$

Theorem (Sz '16)

1. *The transformed Higgs field $\widehat{\theta}$ is meromorphic, has a first-order pole at the points of $\widehat{P}$ and a second-order pole at $\infty \in \widehat{\mathbf{P}}^1$ with semi-simple leading-order term, and no other poles.*
2. *The residues of $\widehat{\theta}$ at the parabolic points are compatible with the parabolic structure corresponding to the $\mathbf{R}$-parabolic structure $\widehat{\mathcal{E}}_\bullet$, and the leading-order term of $\widehat{\theta}$ at $\infty \in \widehat{\mathbf{P}}^1$ preserves the parabolic filtration.*
3. *The transformation is holomorphic for the Dolbeault complex structures of the moduli spaces.*

Stationary phase formula

For any $\zeta_\iota \in \widehat{P}$ denote by $(\mathcal{E}|_\infty)_\iota$ the ζ_ι-eigenspace of A and set

$$B_\iota = \mathrm{res}_{z=\infty}\, \theta|_{(\mathcal{E}|_\infty)_\iota}.$$

By compatibility with the parabolic structure, B_ι induces graded morphisms

$$\mathrm{Gr}^j B_\iota \in \mathrm{End}(\mathrm{Gr}^j (\mathcal{E}|_\infty)_\iota).$$

Assumption

For all ι, j the endomorphism $\mathrm{Gr}^j B_\iota$ is regular and for all $\iota, j \neq j'$ the eigenvalues of $\mathrm{Gr}^j B_\iota$ and $\mathrm{Gr}^{j'} B_\iota$ are distinct.

Theorem (Sz '17)

For all $\alpha_0^j > 0$ we have $\mathrm{Gr}^j \mathrm{res}_{\zeta=\zeta_\iota} \widehat{\theta} = -\mathrm{Gr}^j B_\iota$.
If $\alpha_0^0 = 0$ then we have $\mathrm{Gr}^0 \mathrm{res}_{\zeta=\zeta_\iota} \widehat{\theta} = -\mathrm{Gr}^0 B_\iota \oplus 0$, where 0 stands for the identically 0 endomorphism of appropriate dimension.

Transformed harmonic metric

For two elements $\hat{f}_1, \hat{f}_2 \in \widehat{V}_\zeta = \ker \Delta_\zeta$ let us set

$$\widehat{h}(\hat{f}_1, \hat{f}_2) = \int_{\mathbf{C}} h(\hat{f}_1(z), \hat{f}_2(z)). \quad (1)$$

This formula defines a Hermitian metric on $\widehat{V}$.

Theorem (Sz '07)

The triple $(\widehat{V}, \widehat{\theta}, \widehat{h})$ satisfies Hitchin's equations over $\widehat{\mathbf{P}}^1 \setminus \widehat{P}$.

Compatibility with the parabolic structure

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Theorem (Sz '17)

At any parabolic point ζ_ι (including $\zeta_0 = \infty$) and for any $\alpha \in (0,1)$ the parabolic weight induced by $\widehat{h}$ on $\mathrm{Gr}_\alpha \widehat{\mathcal{E}}$ is $\alpha - 1$.
The parabolic weight induced by $\widehat{h}$ on $\mathrm{Gr}_0 \widehat{\mathcal{E}}$ is

- 0 *on* $\ker \mathrm{Gr}_0 \mathrm{res}_{\zeta_\iota} \widehat{\theta}$
- *otherwise* 0 *on the weight* $k \geq -1$ *part, and*
- -1 *on the weight* $k < -1$ *part.*

Idea of Proof

Following an idea of Biquard–Jardim, it is sufficient to show that the parabolic weights induced by $\widehat{h}$ are bounded from below by the above stated quantities. One proves these estimates by providing explicit representatives of cohomology classes.
Namely, let $\varepsilon > 0$ be chosen very small and fix a cut-off function

$$\chi : \mathbf{C} \to [0,1]$$

satisfying

- the support of $d\chi$ is contained in the annulus $1/3 < |w| < 2/3$
- χ is identically 1 on the disc $|w| \leq 1/3$
- χ is identically 0 on the complement of the disc $|w| < 2/3$.


ζ → ∞
z_j(ζ) → 0
z_j(ζ)
0

Local sections

For simplicity assume $z_i = 0$. Let ς be any local section of $\mathcal{E}$ near $z = 0$ such that $\varsigma(0) \in \mathrm{Gr}_\alpha \mathcal{E}|_0$. Assume that $\varsigma(0)$ is a generalized eigenvector of $\mathrm{Gr}_\alpha \operatorname{res}_{z=0} \theta$ for some eigenvalue λ. It induces a local holomorphic section $\hat{\varsigma}$ of $\widehat{\mathcal{E}}_\alpha$ near $\zeta = \infty$ as follows. We let

$$v(z,\zeta)\mathrm{d}z = \chi\left(\varepsilon^{-1}|\zeta|(z - \lambda\zeta^{-1})\right)\varsigma(z)\zeta\frac{\mathrm{d}z}{z}.$$

and find $t(z,\zeta)\mathrm{d}\bar{z} \in C^\infty(\mathbf{P}^1, V \otimes \Omega^{0,1})$ such that

$$\bar{\partial}^{\mathcal{E}} v(z)\mathrm{d}z + \theta_\zeta t(z)\mathrm{d}\bar{z} = 0.$$

Then as ζ varies the section

$$v(z,\zeta)\mathrm{d}z + t(z,\zeta)\mathrm{d}\bar{z}$$

represents a local section $\hat{\varsigma}$ of $\widehat{\mathcal{E}}_\alpha$ near $\zeta = \infty$.


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represents a local section $\hat{\varsigma}$ of $\widehat{\mathcal{E}}_\alpha$ near $\zeta = \infty$.

Estimates

Proposition

For $\alpha \in (0,1)$ there exists a polynomial R such that we have

$$\int_{\mathbf{C}} |v(z,\zeta)|_h^2 \leq |\zeta|^{2-2\alpha} \cdot R(\log|\zeta|)$$

and

$$\int_{\mathbf{C}} |t(z,\zeta)|_h^2 \leq |\zeta|^{2-2\alpha} \cdot R(\log|\zeta|).$$

Local sections

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and

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Isometry

In what follows, we write

$$\mathcal{M} = \mathcal{M}(\mathbf{P}^1, r, \{z_i\}, A, \{\dim \mathrm{Gr}^j_{F_i}\}, \{\alpha^j_i\}, \{\mathcal{O}^j_i\}).$$

Theorem (Sz '14)

If the orbits $\mathcal{O}^j_i$ are regular semi-simple then Nahm transform is a hyper-Kähler isometry between moduli spaces

$$\mathcal{M} \to \widehat{\mathcal{M}}$$

corresponding to singularity behaviours as specified above.

Strategy of proof: show

$$I \mapsto \widehat{I}$$

$$J \mapsto \widehat{J}$$

$$\Omega_I \mapsto \Omega_{\widehat{I}}$$

Complex structures

The transformation of the Dolbeault complex structure I follows by the algebraic definition of $(\widehat{\mathcal{E}}, \widehat{\theta})$.
The transformation of the complex structure J follows from identification with minimal extension followed by Fourier–Laplace transform of the underlying holonomic $\mathcal{D}$-module (Sz '12).
It remains to show the transformation of Ω_I.

Hilbert scheme of curves

Consider the ruled surface

$$Z = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus L) \xrightarrow{\pi} \mathbf{P}^1$$

and for fixed r consider the Hilbert scheme

$$\mathrm{Hilb}(r)$$

of curves $S \subset Z$ having the same Hilbert polynomial as a generic r to 1 cover of $\mathbf{P}^1$ in Z, and let

$$B \subset \mathrm{Hilb}^0(r)$$

be the Zariski open subset parameterising smooth irreducible curves S not contained in D_∞.

Moduli spaces of sheaves on Z

Consider moreover the relative Picard bundle

$$\mathrm{Pic}^d(Z) \to B$$

whose fiber over $b \in B$ is the set of isomorphism classes of degree d line bundles M over S_b.
$\mathrm{Pic}^d(Z)$ has a canonical Poisson structure. The orbits $\mathcal{O}_i^j$ naturally single out points on Z, whose ideal sheaf is denoted by $\mathcal{I}$. There is a symplectic leaf

$$\mathcal{L}(\{z_i\}, A, \{\mathcal{O}_i^j\}) \subset \mathrm{Pic}^d(Z)$$

consisting of sheaves on curves passing through the points defining $\mathcal{I}$.
The deformation space of $\mathcal{L}$ at a point is given by

$$\mathrm{Ext}^1_{\mathcal{I}}(M, M).$$

BNR-correspondence

The induced Mukai symplectic structure Ω_{Muk} on $\mathcal{L}$ is defined by the Yoneda product

$$\Omega_{\mathsf{Muk}} : \mathrm{Ext}^1_{\mathcal{J}}(M, M) \times \mathrm{Ext}^1_{\mathcal{J}}(M, M) \to \mathrm{Ext}^2_{\mathcal{J}}(M, M) \cong \mathbf{C}.$$

There exists a Zariski open subset $\mathcal{M}^0 \subset \mathcal{M}$ such that we have a biholomorphism

$$\Pi : \mathcal{L}(\{z_i\}, A, \{\mathcal{O}^j_i\}) \to \mathcal{M}^0.$$

It is given as follows: given $M \in \mathrm{Pic}^d(Z)$ associate to it

$$\Pi(M) = (\pi_* M, \pi_* \zeta).$$

An intermediate symplectomorphism

Proposition

We have $\Pi^*\Omega_I = \Omega_{Muk}$.

A similar result: J. Harnad, J Hurtubise (2008).
Idea of proof: Π induces an isomorphism on deformation spaces

$$\mathrm{Ext}^1_{\mathfrak{J}}(M, M) \to \mathbf{H}^1(\mathbf{P}^1, \mathrm{ad}(\theta)).$$

One may compute the Yoneda product using any projective resolution of the sheaves M. The definition of the spectral sheaf

$$0 \to \pi^*\mathcal{E} \otimes L^\vee \xrightarrow{\theta_\zeta} \pi^*\mathcal{E} \to M \to 0$$

provides one such resolution. Using this resolution we get the claim.

The symplectomorphism between Dolbeault structures

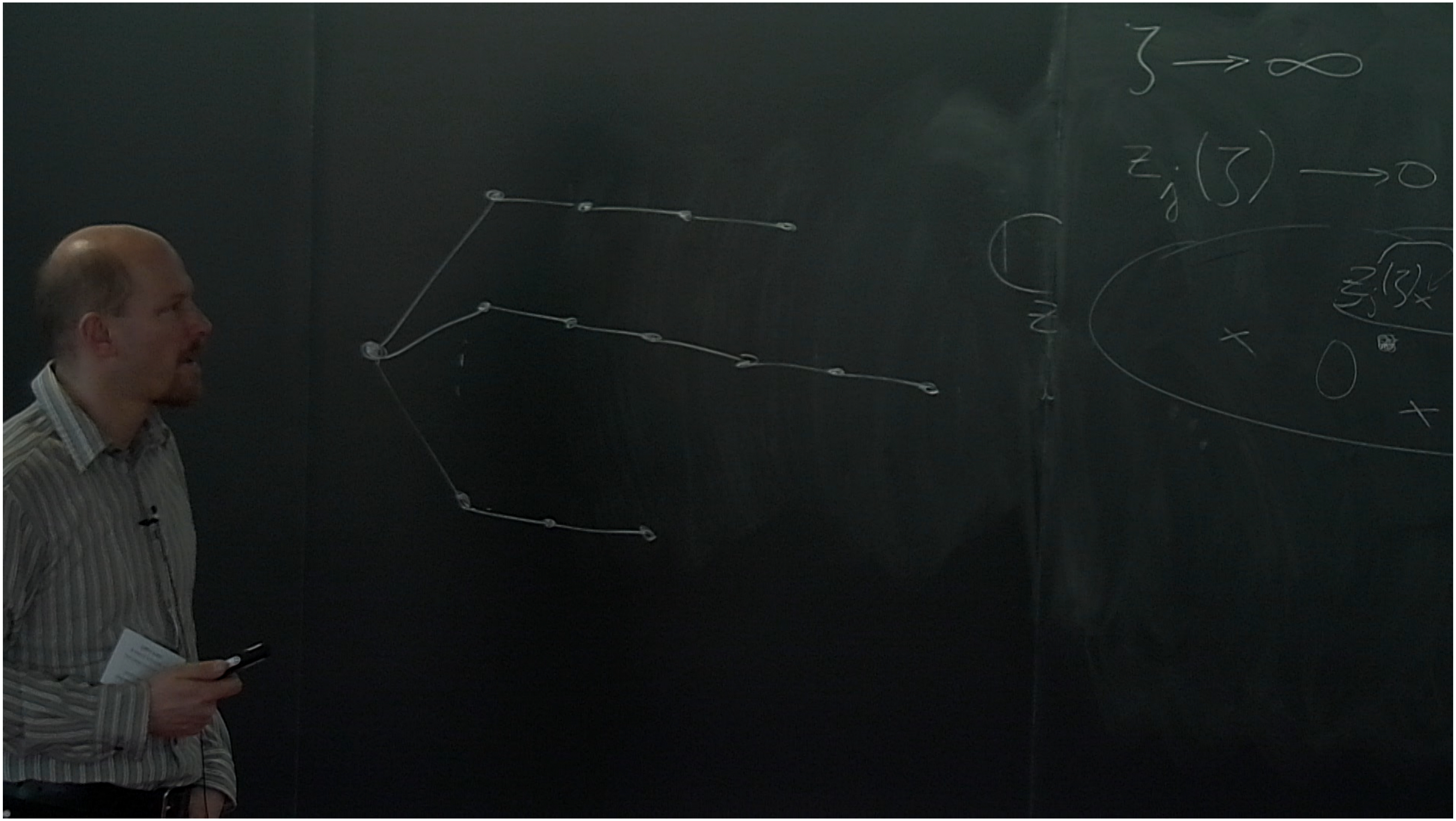
Proposition

The Higgs bundles $(\mathcal{E}, \theta)$ and $(\widehat{\mathcal{E}}, \widehat{\theta})$ have isomorphic spectral sheaves:

$$M_{(\mathcal{E},\theta)} \cong M_{(\widehat{\mathcal{E}},\widehat{\theta})}.$$

Applying the previous Proposition to both of these resolutions, we get

$$\Pi^* \Omega_I = \Omega_{\mathsf{Muk}} = \widehat{\Pi}^* \Omega_{\widehat{I}}.$$


ζ → ∞
Z_j(ζ) → 0

Topology of $\widehat{\mathcal{E}}$

A computation using Grothendieck–Hirzebruch formula shows that

$$\operatorname{rank}(\widehat{\mathcal{E}}) = \hat{r} = \sum_{i=1}^{n} \left(r - \delta_{0,\alpha_i^0} \dim \ker \mathrm{Gr}_i^0 \right),$$

$$\deg(\widehat{\mathcal{E}}) = \deg(\mathcal{F}) + r + \hat{r}.$$