Title: Holomorphic symplectic Morita equivalence and the generalized Kahler potential

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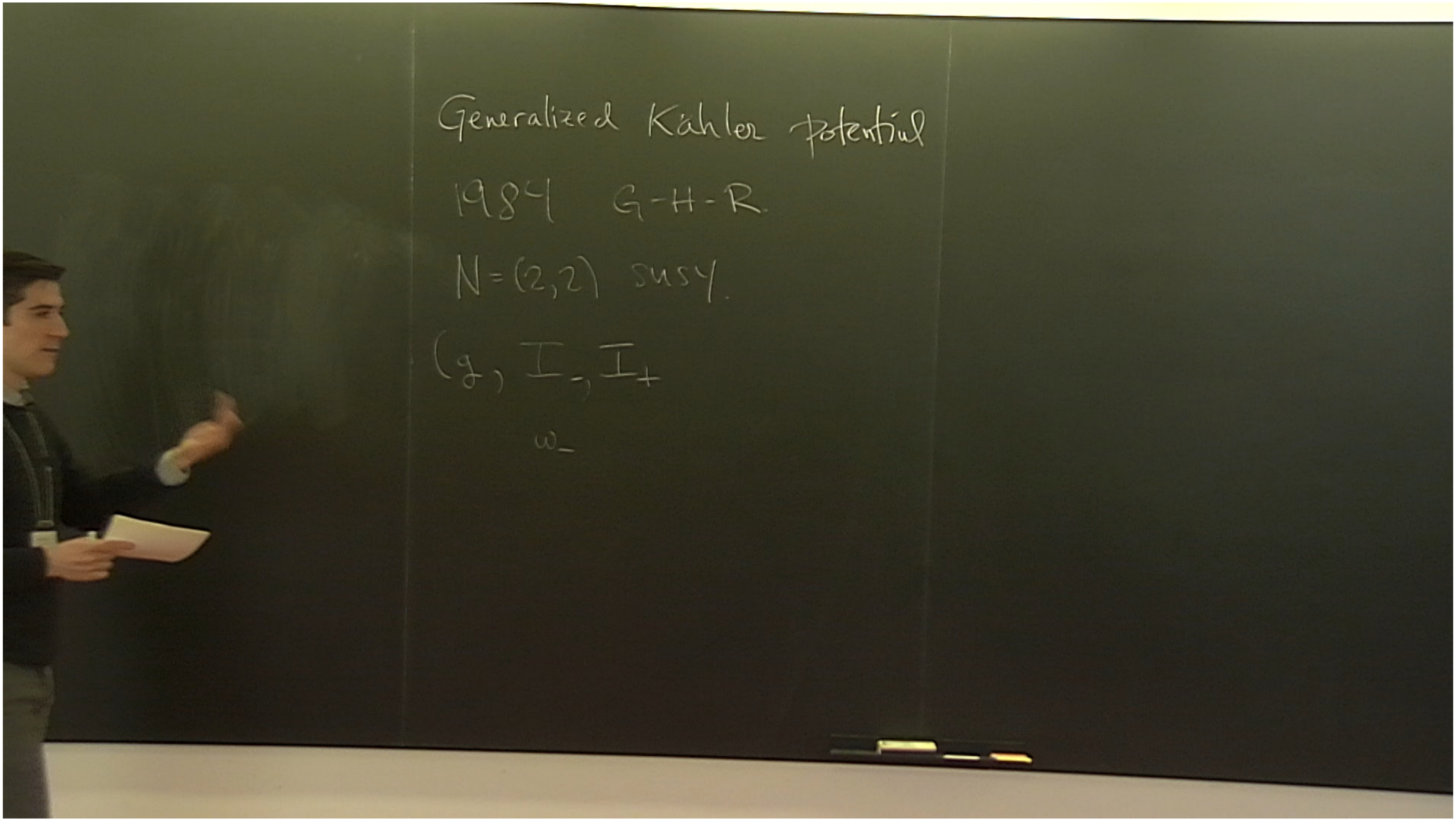
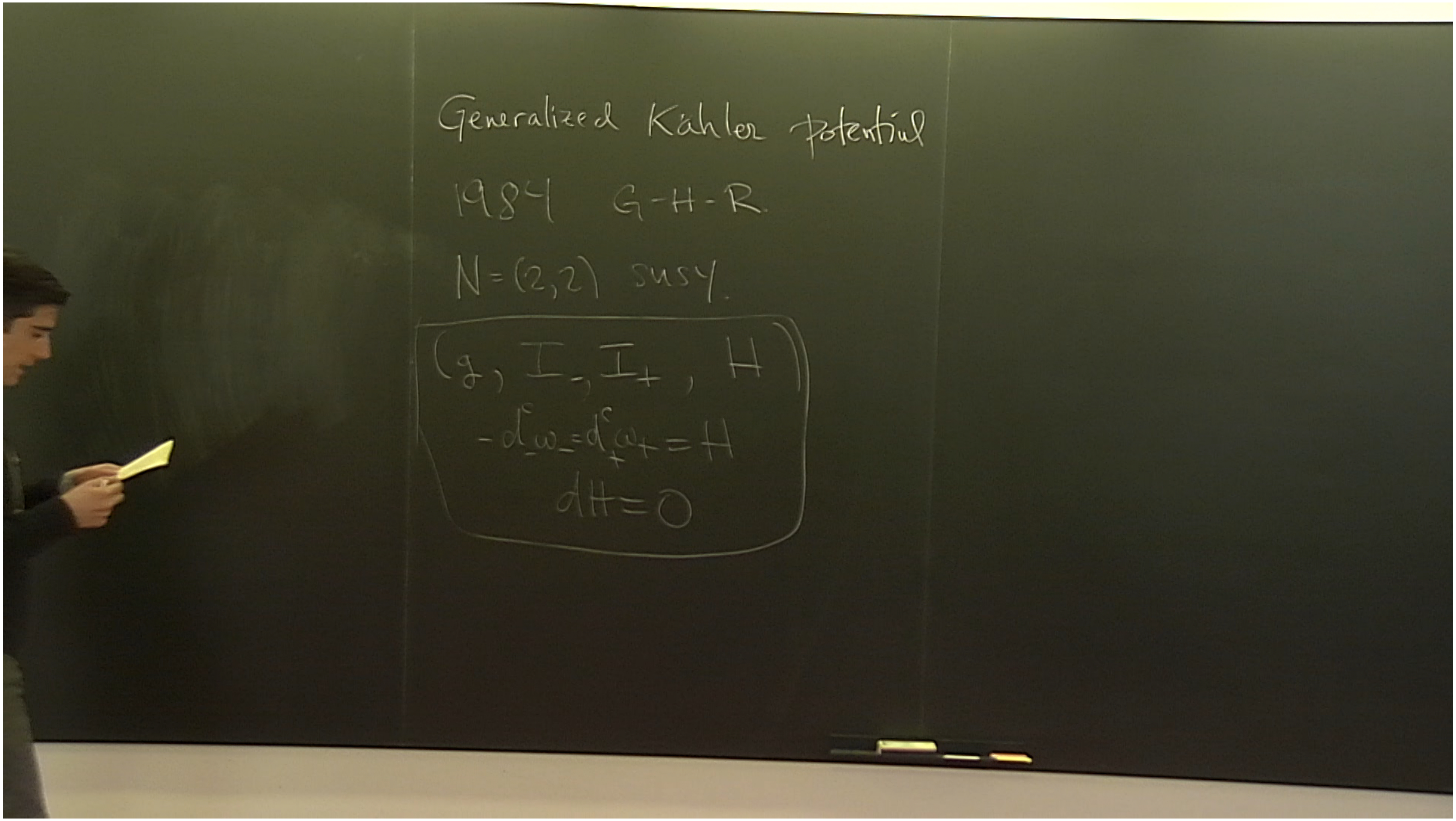
URL: http://pirsa.org/17020019

Abstract: Since the introduction of generalized Kahler geometry in 1984 by Gates, Hull, and Rocek in the context of two-dimensional supersymmetric sigma models, we have lacked a compelling picture of the degrees of freedom inherent in the geometry. In particular, the description of a usual Kahler structure in terms of a complex manifold together with a Kahler potential function is not available for generalized Kahler structures, despite many positive indications in the literature over the last decade. I will explain recent work showing that a generalized Kahler structure may be viewed in terms of a Morita equivalence between holomorphic Poisson manifolds; this allows us to solve the problem of existence of a generalized Kahler potential.


Generalized Kähler Potential
1984 G-H-R.
N=(2,2) susy.
(g, I_-, I_+
ω_-


Generalized Kähler potential
1984 G-H-R
N=(2,2) susy
(g, I_-, I_+, H)
$-d^c_-\omega_- = d^c_+\omega_+ = H$
$dH=0$

$N=(2,2)$ susy.

$$\boxed{\begin{gathered}(g,\ I_-,\ I_+,\ H)\\ -d^c_-\omega_- = d^c_+\omega_+ = H\\ dH=0\end{gathered}}$$

$d^c_\pm = i(\partial_\pm - \bar{\partial}_\pm)$

d

...lized Kähler potential

...H G-H-R.

...(2,2) susy.

$$\boxed{I_-,\ I_+,\ H \qquad \ldots\omega_- = d^c_+\omega_+ = H \qquad dH = 0}$$

$d^c_\pm = i(\partial_\pm - \bar\partial_\pm)$

d

Underlying geom. strs

① repackage:

$$\begin{matrix}(+)\ \mathbb{J}_A \\ (-)\ \mathbb{J}_B\end{matrix} = \frac{1}{2}\begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ (\omega_+ \mp \omega_-) & -(I_+^* \pm I_-^*) \end{pmatrix} : \begin{matrix} T \\ \oplus \\ T^* \end{matrix} \circlearrowleft$$

$$\mathbb{J}_A^2 = \mathbb{J}_B^2 = -1.$$

...lized Kähler potential

...l G-H-R.

(2,2) susy.

$$\boxed{\begin{aligned} &I_-,\ I_+,\ H \\ &\omega_- = d^c_+ \omega_+ = H \\ &dH = 0 \end{aligned}}$$

$d^c_\pm = i(\partial_\pm - \bar\partial_\pm)$

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$$\mathbb{J}_A^2 = \mathbb{J}_B^2 = -1.$$

$$\mathbb{J}_A \mathbb{J}_B = \mathbb{J}_B \mathbb{J}_A \qquad \text{GKähler.}$$

$$\left.\begin{matrix} \ldots^{-1}_{+} \mp \omega_-^{-1}) \\ \ldots^{*}_{+} \pm I_-^{*}) \end{matrix}\right) : \begin{matrix} T \\ \oplus \\ T^* \end{matrix} \circlearrowleft$$

ähler.

(2) Hitchin

$$Q = -\frac{1}{2}[I_+, I_-]g^{-1}$$

Im part of
hol. Poisson strs.

$$\sigma_+ = I_+ Q + iQ$$

$$\sigma_- = I_- Q + iQ.$$

$$\omega_+^{-1} \mp \omega_-^{-1}$$

$$I_+^* \pm I_-^* \Big) : T \oplus T^* \circlearrowleft$$

Kähler.

(2) Hitchin

$$Q = -\frac{1}{2}[I_+, I_-]g^{-1}$$

Im part of hol. Poisson strs.

$$\sigma_+ = I_+ Q + i Q$$

$$\sigma_- = I_- Q + i Q.$$

(3) hol Courant algs, Dirac strs.

otential

Underlying geom. strs

① repackage:

$$(+)\ \mathbb{J}_A = \frac{1}{2}\begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ (\omega_+ \mp \omega_-) & -(I_+^* \pm I_-^*) \end{pmatrix} : T \oplus T^* \circlearrowleft$$

$(-)\ \mathbb{J}_B$ //

$d_\pm^c = i(\partial_\pm - \bar{\partial}_\pm)$

d

$$\mathbb{J}_A^2 = \mathbb{J}_B^2 = -1.$$

$$\mathbb{J}_A \mathbb{J}_B = \mathbb{J}_B \mathbb{J}_A$$

G Kähler.

② Hitchin

$Q = -\frac{1}{2}[I_+, I$

Im part of

hol. Poisson str

$\sigma_+ = I_+ Q + i Q$

$\sigma_- = I_- Q + i Q$

③ hol Courant algs,

Dirac strs.

$= -\frac{1}{2}[I_+, I_-]g^{-1}$

art of

Poisson strs.

$= I_+Q + iQ$

$= I_-Q + iQ$.

Courant algs,

rac strs.

Special Case:

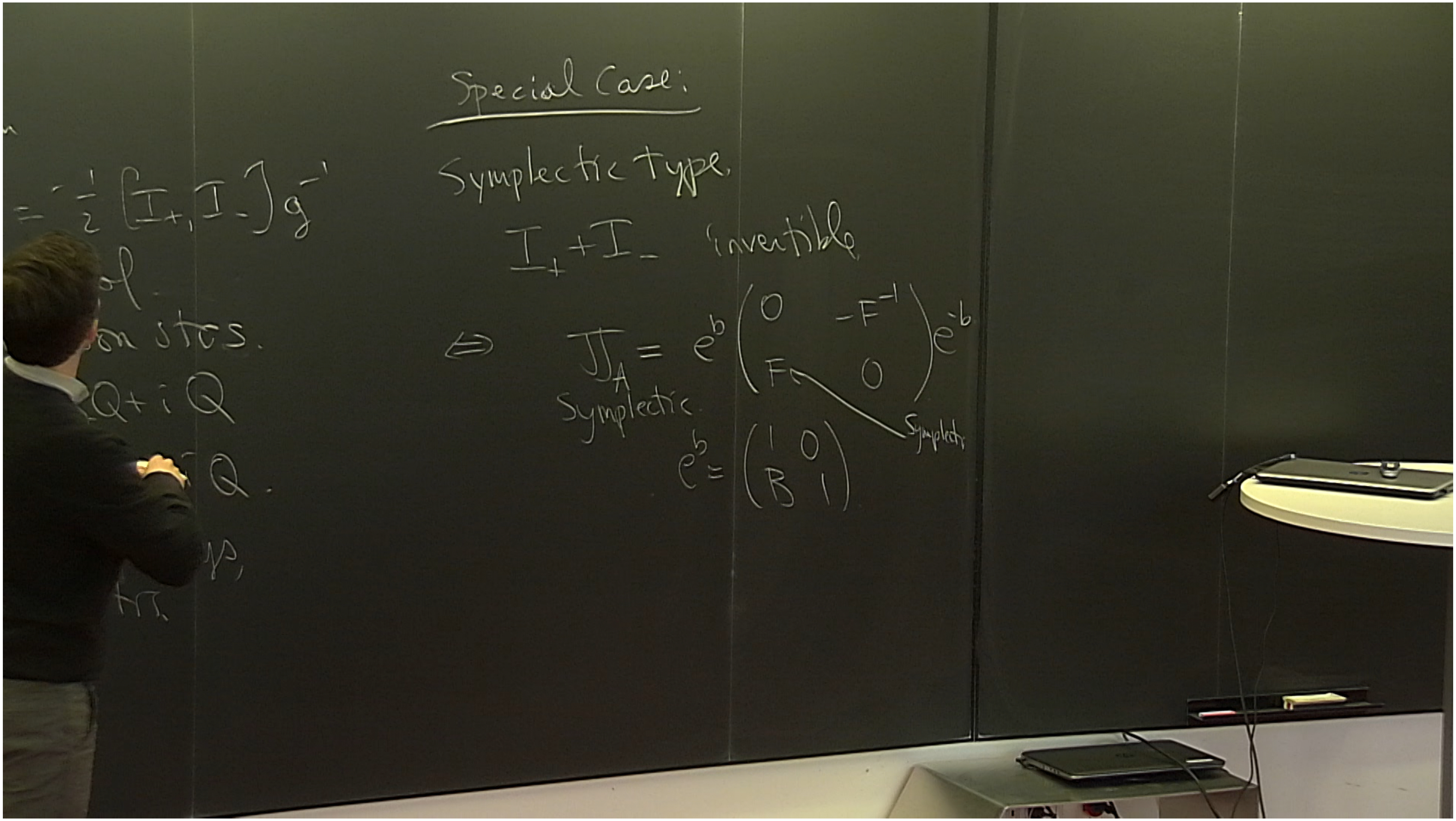
Symplectic type.

$I_+ + I_-$ invertible

$$\Leftrightarrow \quad \mathbb{J}_A = e^b \begin{pmatrix} 0 & -F^{-1} \\ F & 0 \end{pmatrix} e^{-b}$$

Symplectic

$$e^b = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$


Special Case:
Symplectic type.
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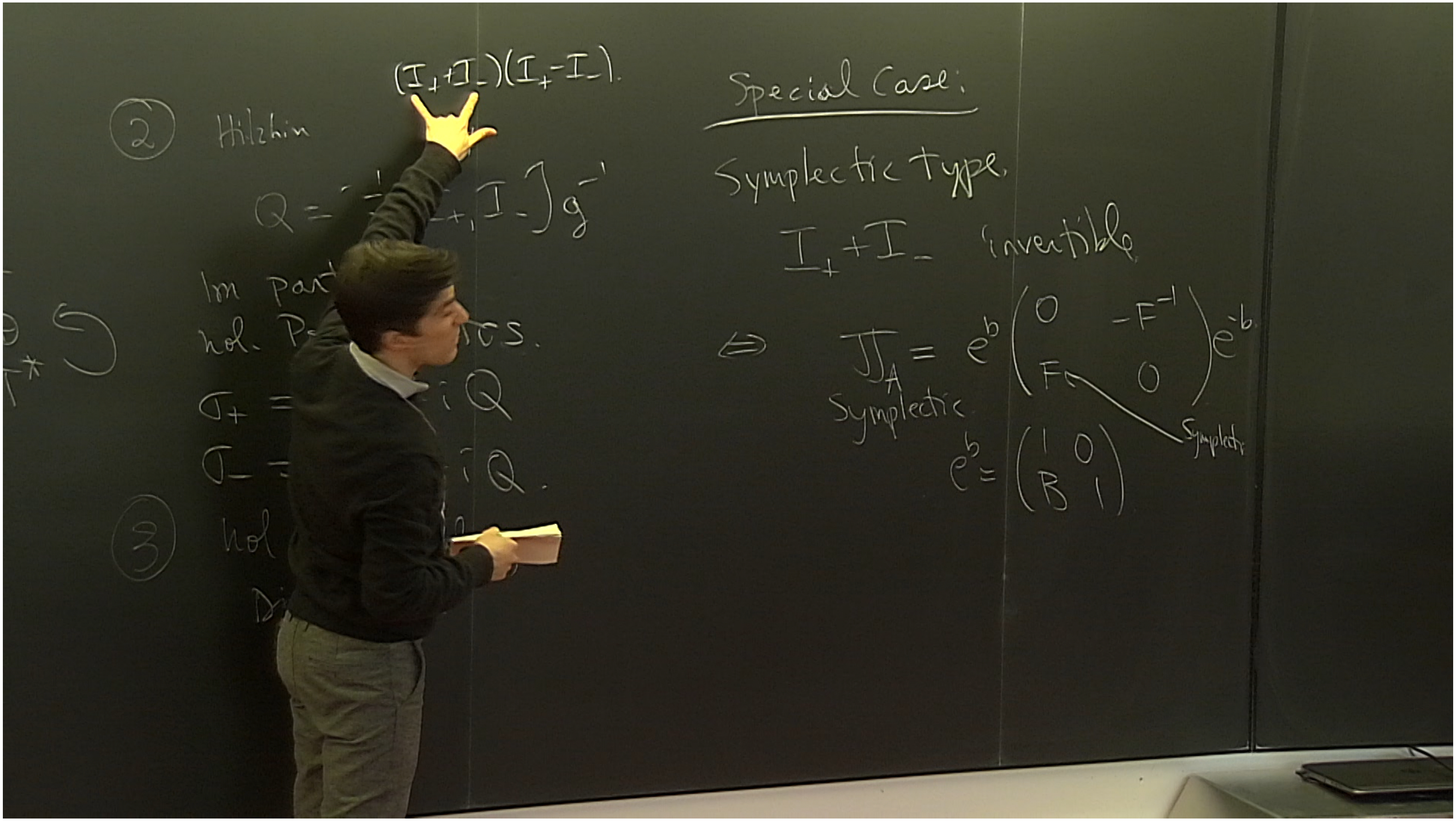
Symplectic
Symplectic
$e^b = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$


(I+ + I-)(I+ - I-).
Special Case:
Symplectic type.
I+ + I- invertible
Hitchin
Symplectic
Symplectic

Generalized Kähler potential

Thm (MG) In this case: the pair of

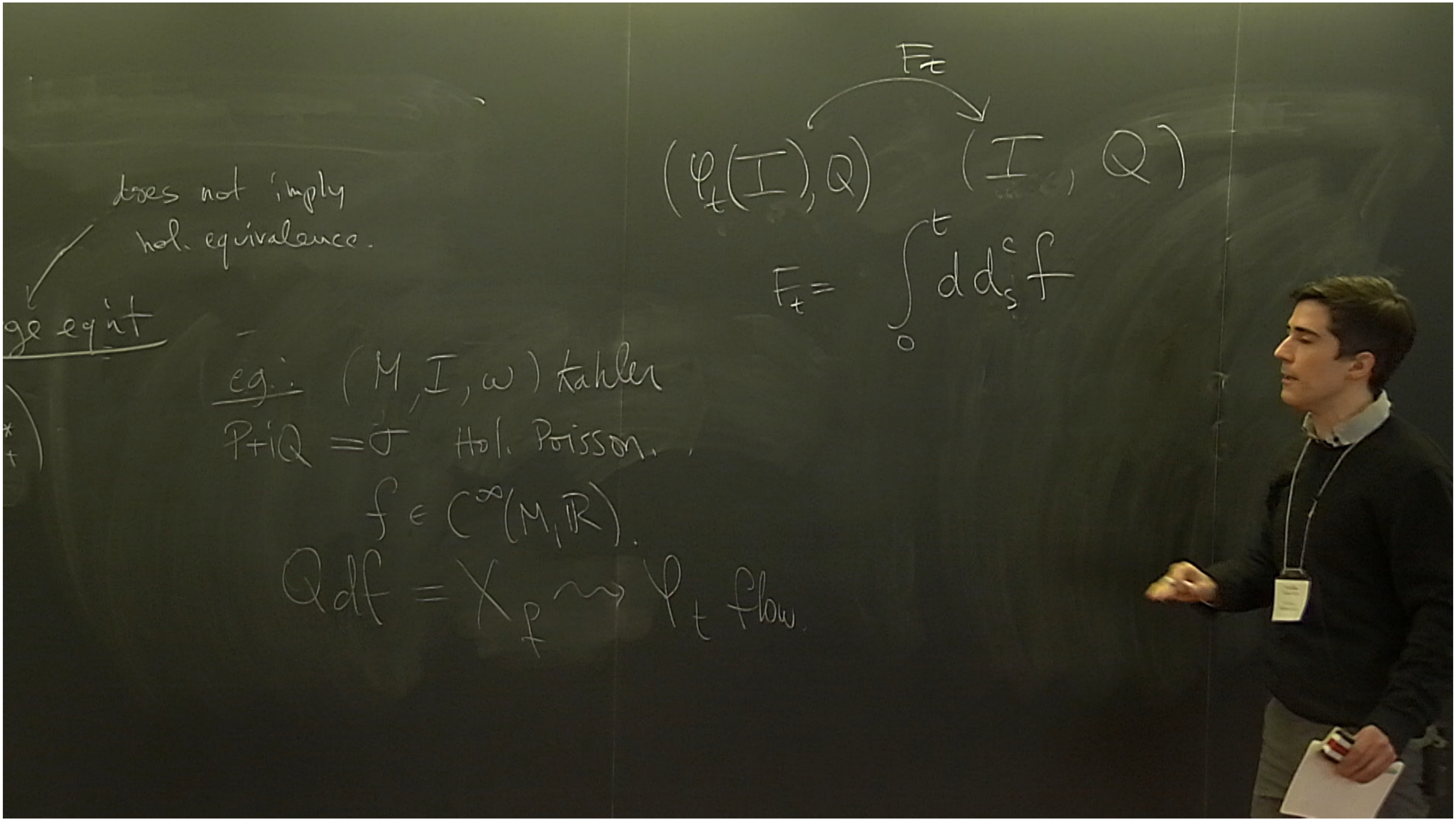
hol. Poisson strs are Gauge eq'nt

does not imply hol. equivalence.

$$\begin{pmatrix} I_- & Q \\ 0 & -I_-^* \end{pmatrix} \qquad e^F = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \qquad \begin{pmatrix} I_+ & Q \\ 0 & -I_+^* \end{pmatrix}$$

$$(I_-, Q) \longrightarrow (I_+, Q)$$

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathbb{P}^1 \to \mathbb{F}^2 \to \mathbb{P}^1$$


does not imply
hol. equivalence.
ge eq'nt
eg.: (M, I, ω) kahler
P+iQ = σ Hol. Poisson.
f ∈ C^∞(M, ℝ).
Q df = X_f ⇝ φ_t flow.
F_t
(φ_t(I), Q)   (I, Q)
F_t = ∫_0^t d d_s^c f

does not imply
hol. equivalence.

ge eq'nt

$$(\varphi_t(I), Q) \xleftarrow{F_t} (I, Q)$$

$$F_t = \int_0^t d\, \underset{\varphi_s(I)}{d^c_s} f$$

eg: (M, I, ω) Kahler

$P + iQ = \sigma$ Hol. Poisson.

$f \in C^\infty(M, \mathbb{R})$.

$Q\,df = X_f \rightsquigarrow \varphi_t$ flow.

Generalized Kähler Potential

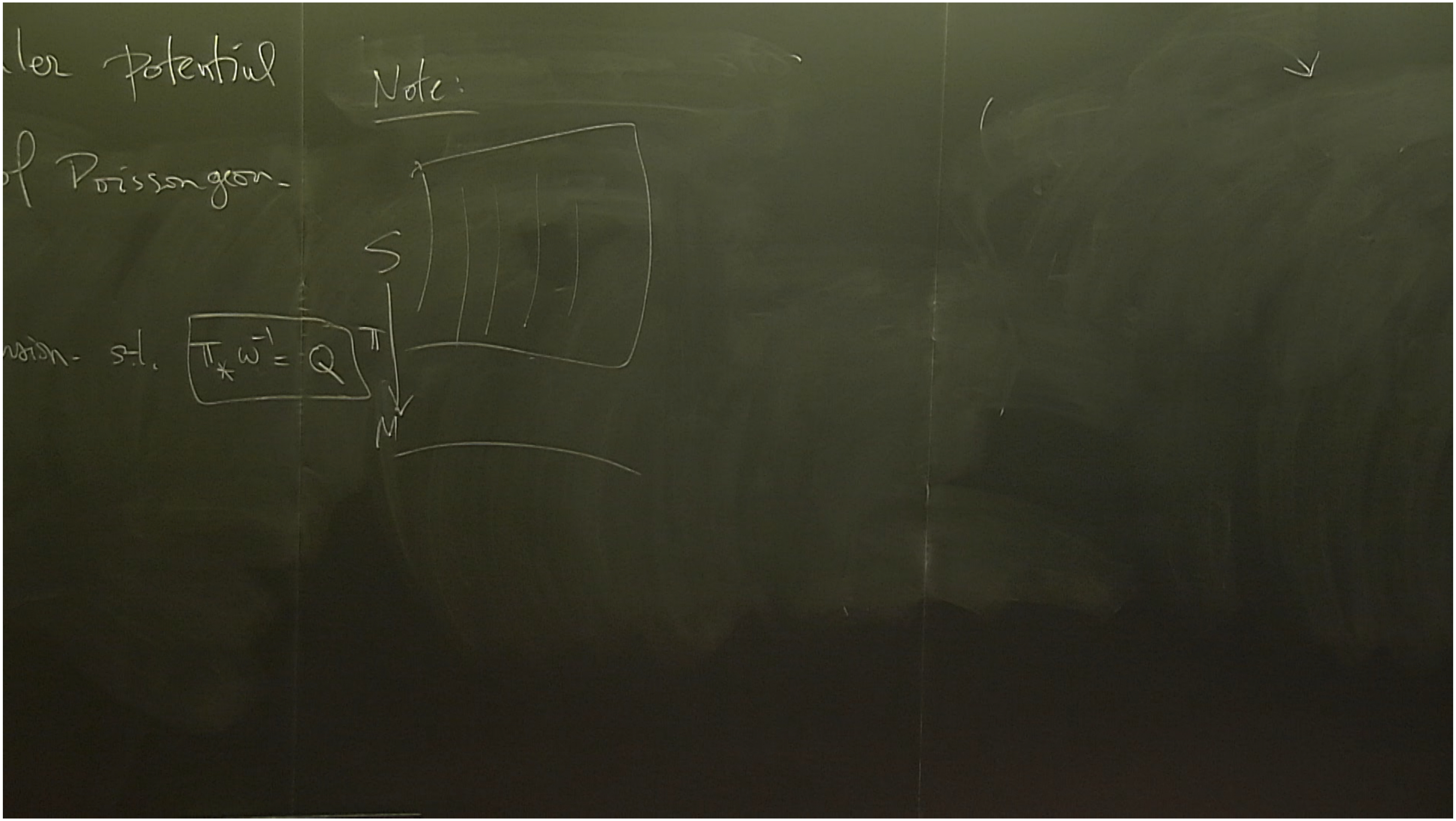
Weinstein school of Poisson geom.

Symp. Realization

$$\begin{array}{c}(S, \omega) \\ \downarrow \pi \\ (M, Q)\end{array}$$

surj submersion. s.t. $\boxed{\pi_* \omega^{-1} = Q}$

Poisson mfld.

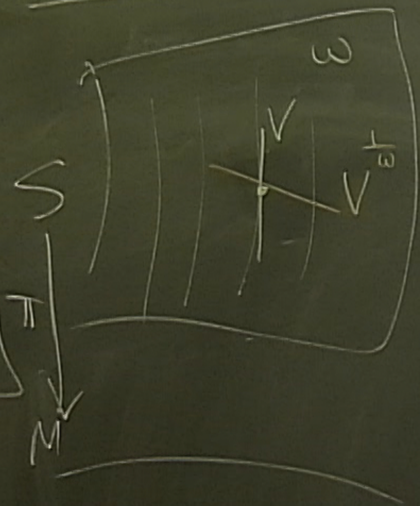

ler Potential
of Poissongeom.
sion. s.t. $\pi_* \omega^{-1} = Q$
Note:
S
π
M

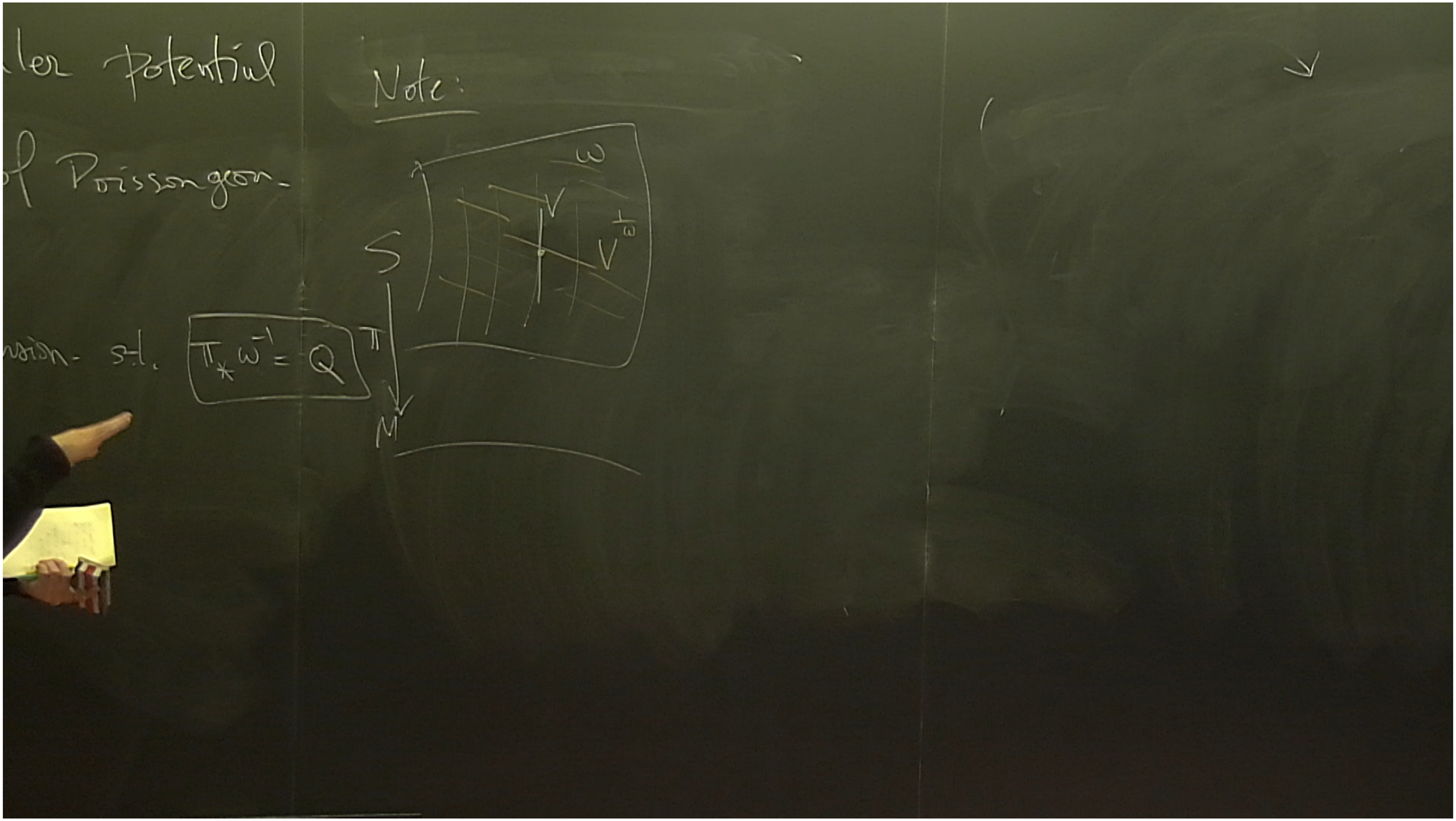
...ler Potential

...f Poissongeom.

...nsion. s.t. $\boxed{\pi_* \omega^{-1} = Q}$

Note:




ler Potential
of Poissongeom.
sion. s.t. $\boxed{\pi_* \omega^{-1} = Q}$
Note:
S
π
M
ω
V
$V^{\perp_\omega}$

...ler Potential

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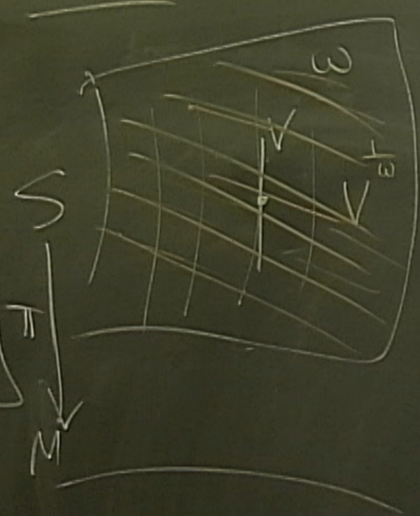
Note:



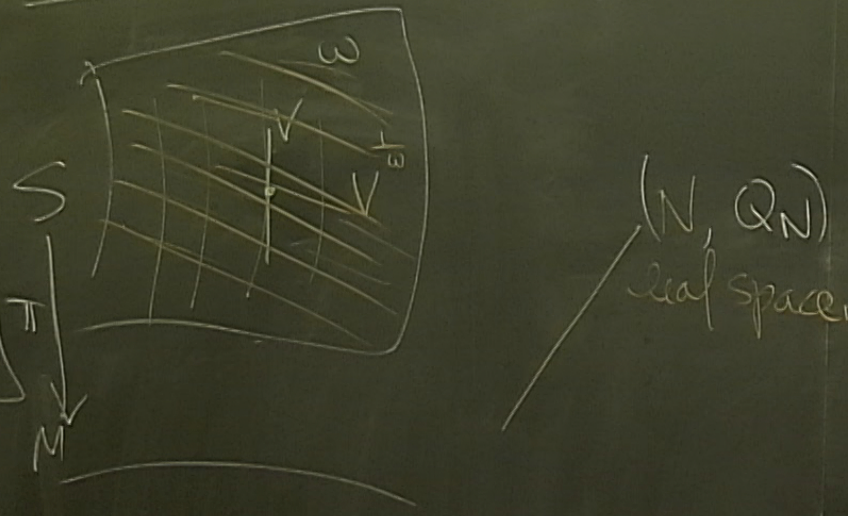
(N, Q_N)
leaf space

$V^{\perp\omega}$ is involutive

quot. is Poisson.

$\boxed{\omega^{-1} = Q}$

Note:



$V^{\perp\omega}$ is involutive

quot: is Poisson.

This is instance of

Symplectic M. Equivalence

$$\begin{array}{ccc} & (S,\omega) & \\ \swarrow^{p_M} & & \searrow^{p_N} \\ (M, Q_M) & & (N, Q_N) \end{array}$$

p_M, p_N surj subm. Poisson

s.t. $\omega(\ker p_{M*}, \ker p_{N*}) = 0$

This is instance of

Symplectic M. Equivalence

$$\begin{tikzcd} & (S,\omega) \arrow[dl, "p_M"'] \arrow[dr, "p_N"] & \\ (M, Q_M) & & (N, Q_N) \end{tikzcd}$$

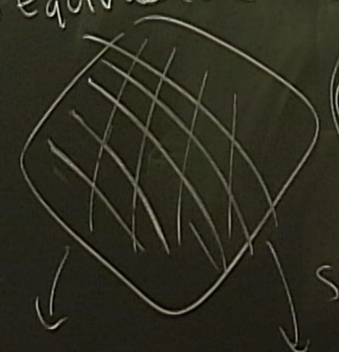
p_M, p_N surj subm. Poisson

s.t. $\omega(\ker p_{M*}, \ker p_{N*}) = 0$.

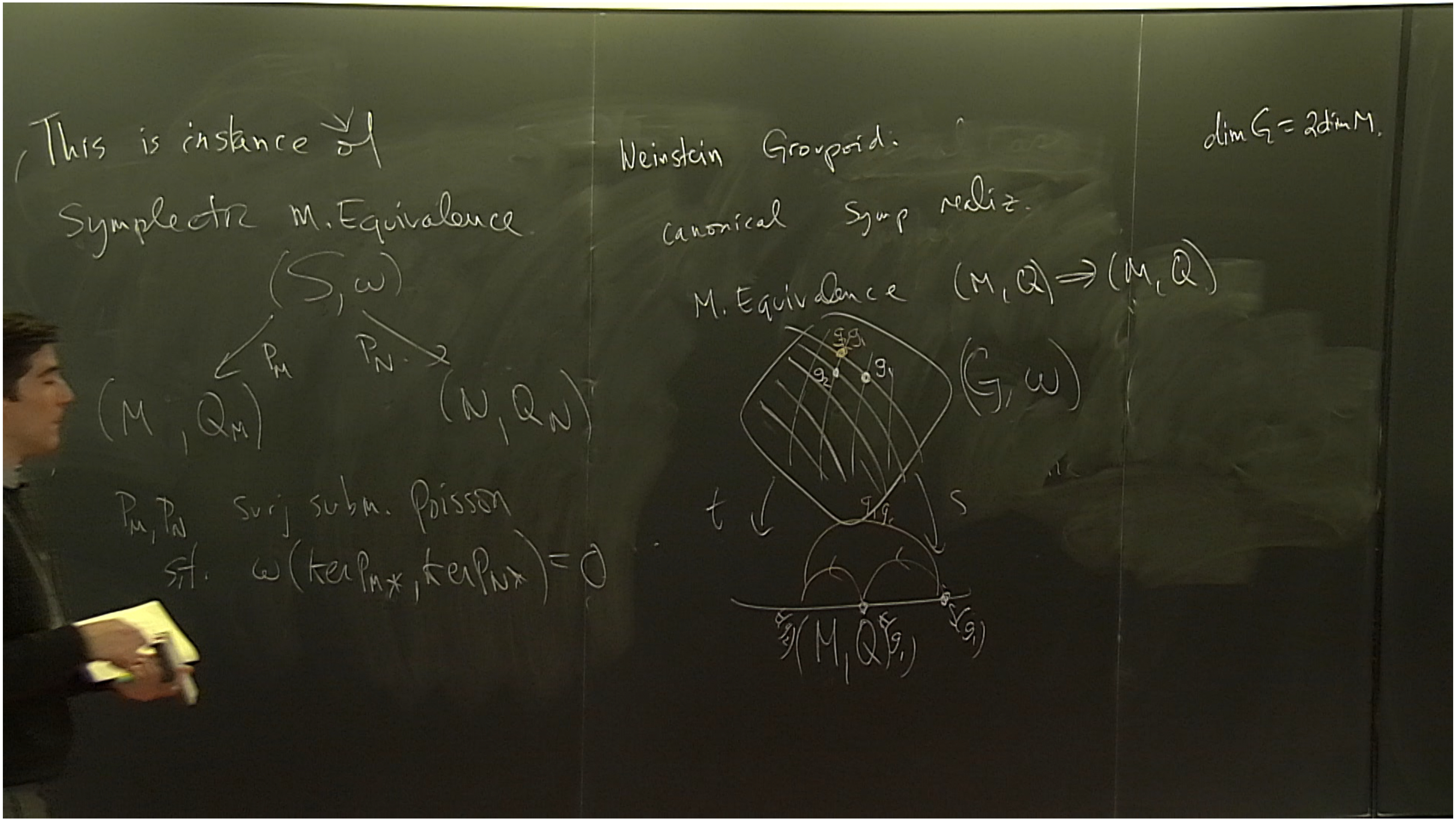
Weinstein Groupoid:

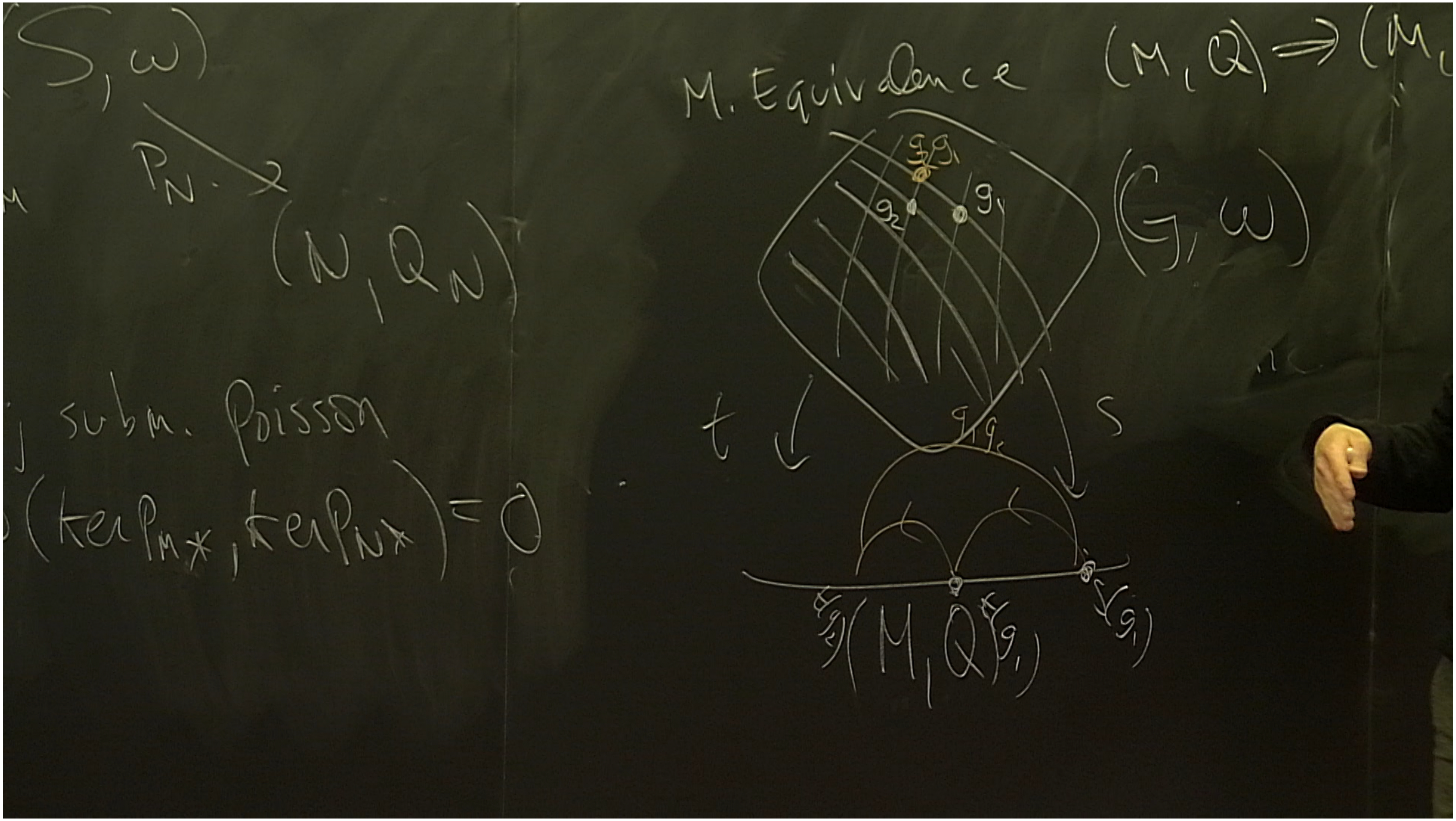
canonical Symp realiz.

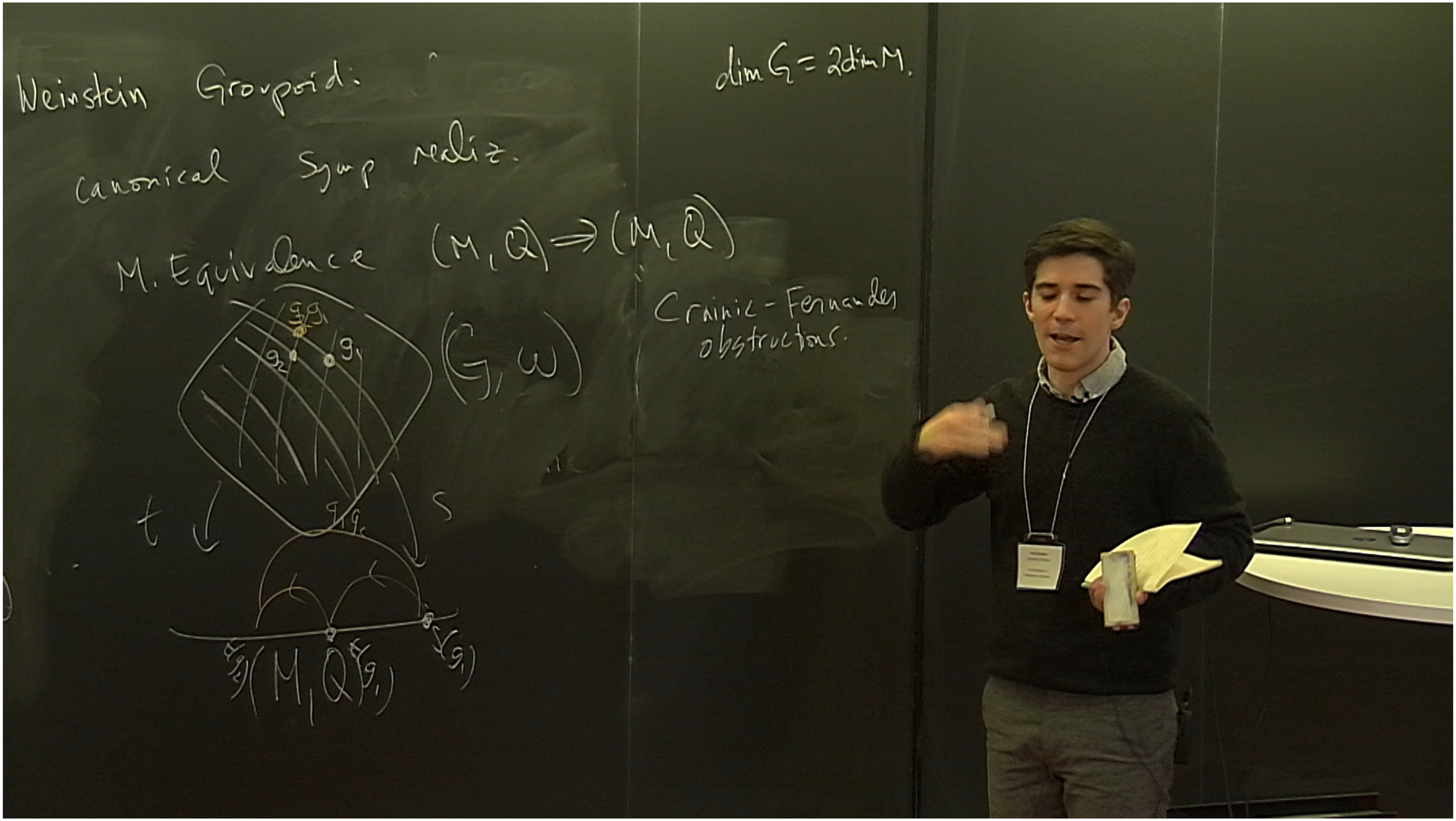
M. Equivalence $(M,Q) \Rightarrow (M,Q)$



$$\begin{tikzcd} & (G,\omega) \arrow[dl, "t"'] \arrow[dr, "s"] & \\ & (M,Q) & \end{tikzcd}$$


This is instance of
Symplectic M. Equivalence
(S, ω)
P_M
P_N
(M, Q_M)
(N, Q_N)
P_M, P_N surj subm. Poisson
s.t. ω(ker P_M*, ker P_N*) = 0
Weinstein Groupoid:
canonical Symp realiz.
M. Equivalence (M, Q) ⇒ (M, Q)
dim G = 2 dim M.
(G, ω)
g_2
g_1
t
s
(M, Q)


(S, ω)
P_N
(N, Q_N)
subm. Poisson
(ker P_M*, ker P_N*) = 0
M. Equivalence
(M, Q) ⇒ (M
(G, ω)
t
s
(M, Q)



Weinstein Groupoid:
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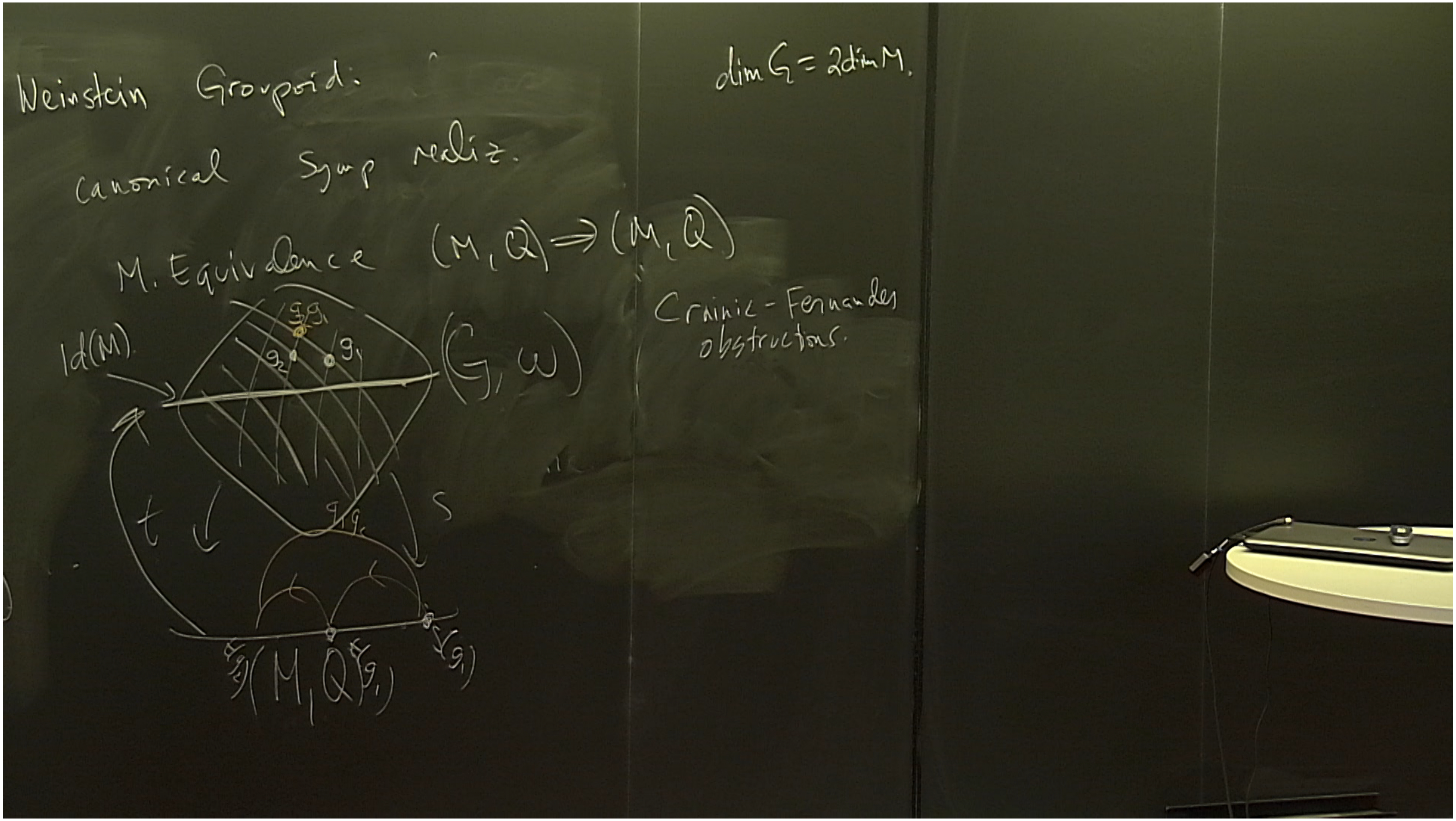
Crainic - Fernandes obstructions.
(G, ω)
t
s
(M, Q)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Id(M)
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t
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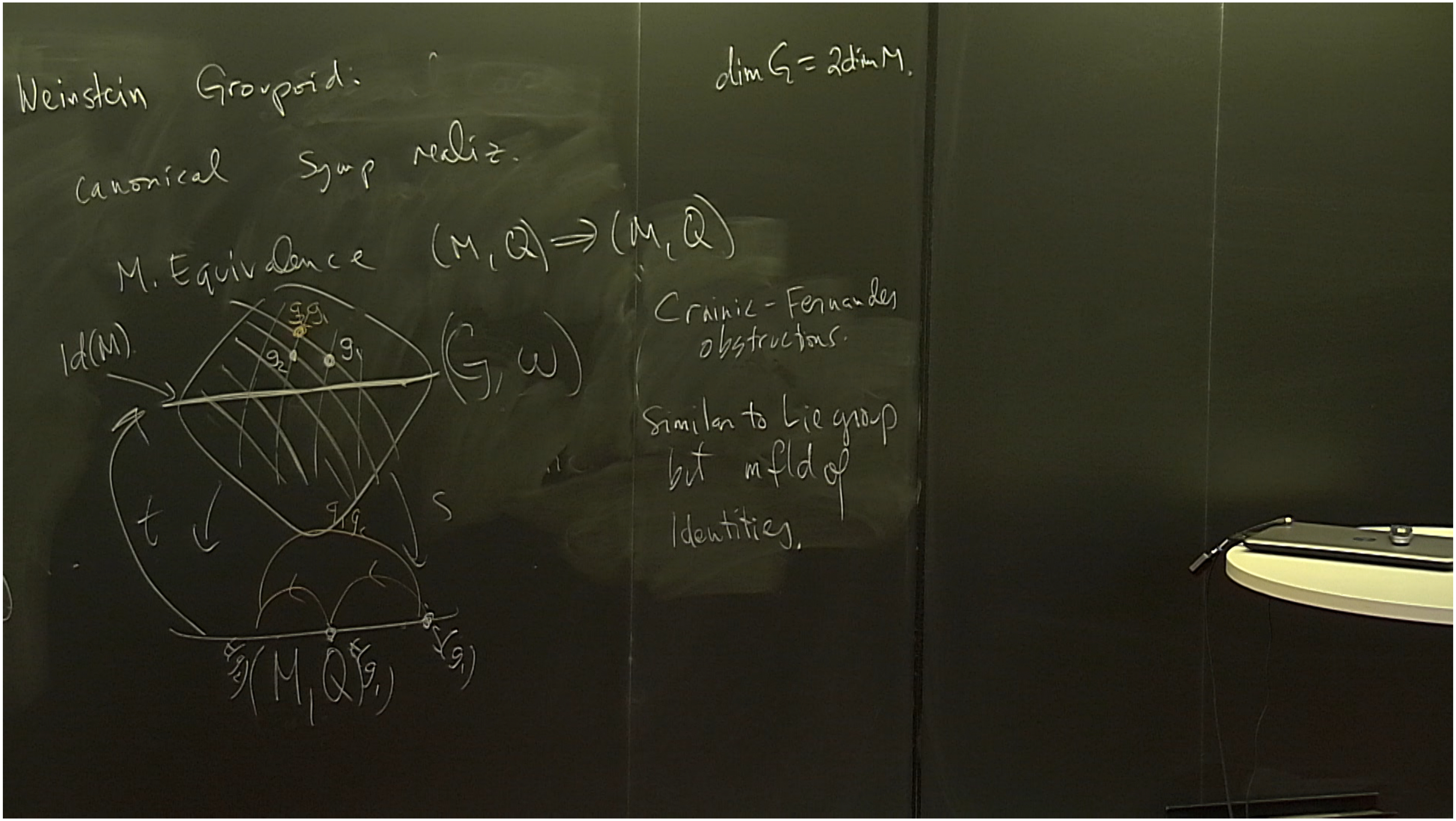
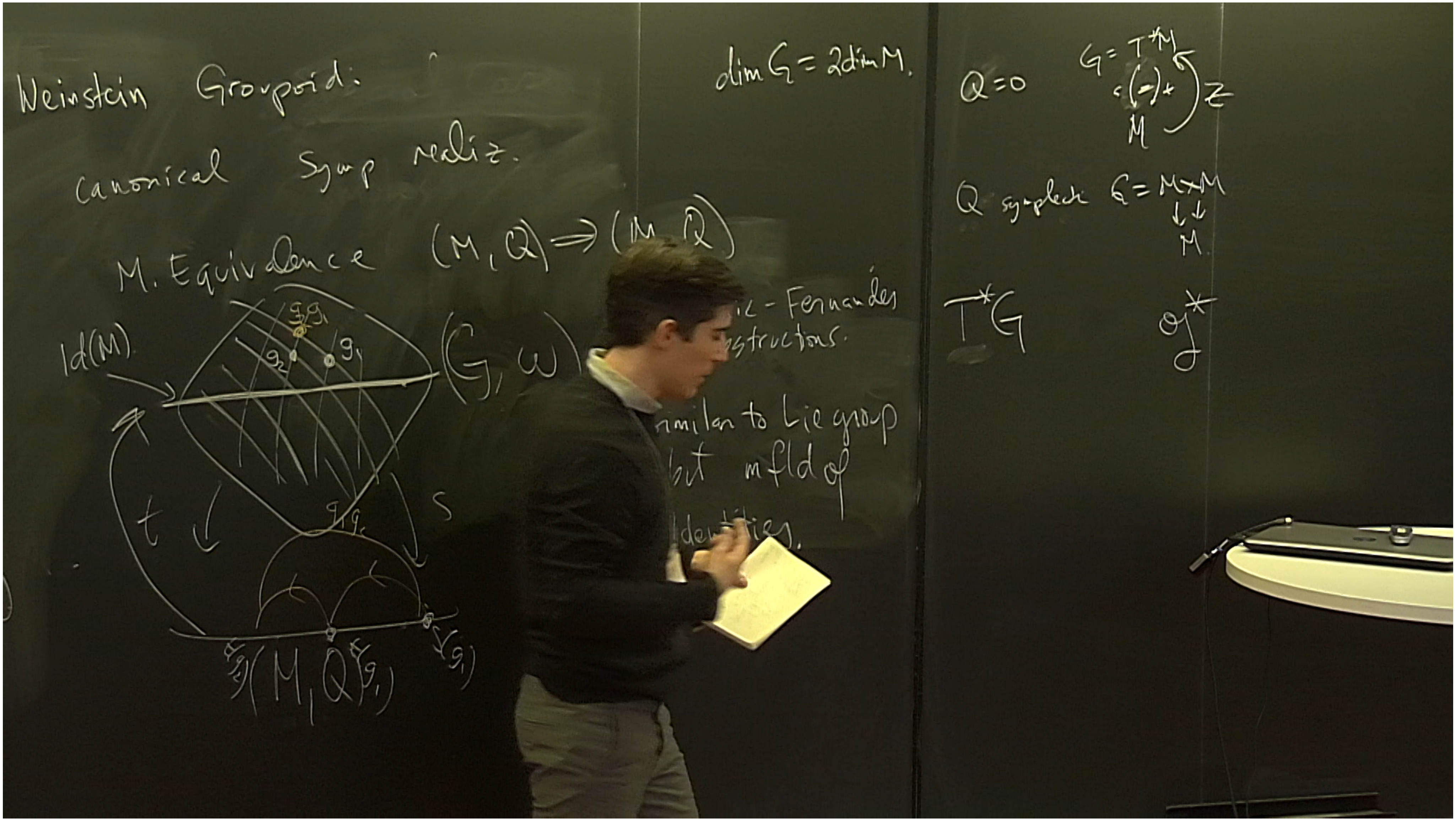
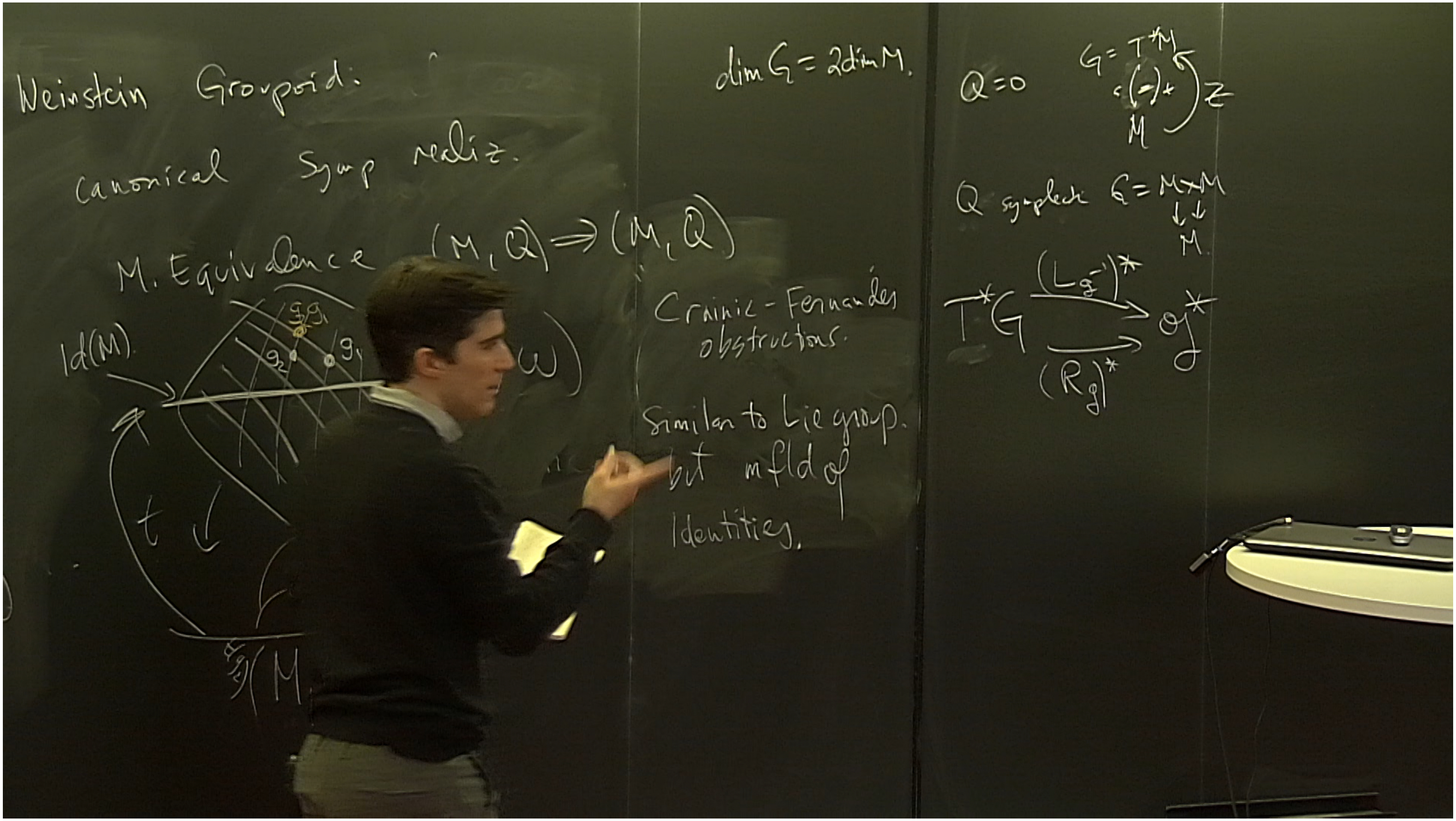

Weinstein Groupoid:
canonical Symp realiz.
M. Equivalence (M, Q) ⇒ (M, Q)
dim G = 2dim M.
Crainic - Fernandes
obstructions.
Similar to Lie group
but mfld of
Identities.
Id(M)
(G, ω)
t
s
(M, Q)

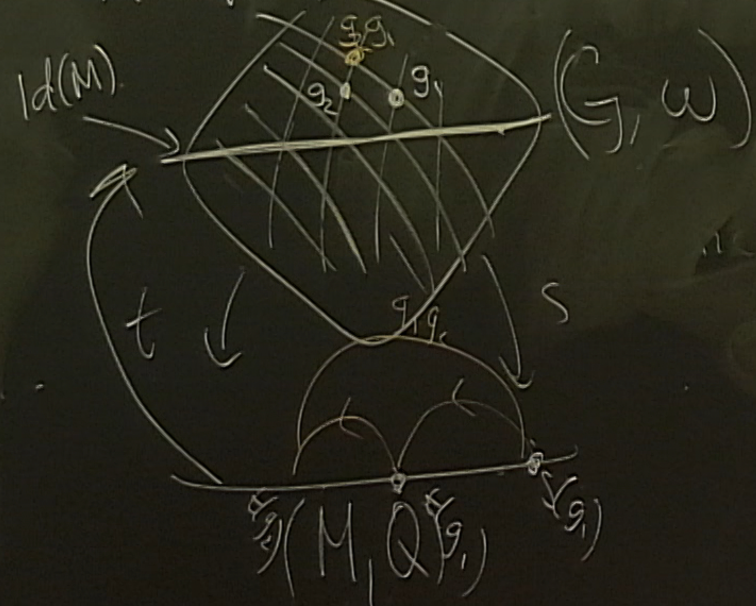

Weinstein Groupoid:
canonical symp realiz.
M. Equivalence (M, Q) ⇒ (M, Q)
Id(M)
(G, ω)
t
s
(M, Q)
dim G = 2dim M.
- Fernandes
structions.
similar to Lie group
but mfld of
identities.
Q=0
G = T*M
Z
M
Q symplectic G = M×M
M
T*G
g*


Weinstein Groupoid:
canonical symp realiz.
M. Equivalence $(M,Q) \rightrightarrows (M,Q)$
$\dim G = 2\dim M$.
Crainic - Fernandes obstructions.
Similar to Lie group. but mfld of Identities.
Id(M)
$Q=0$ $G=T^*M$
Q symplectic $G=M\times M$
$T^*G \xrightarrow{(L_{g^{-1}})^*} \mathfrak{g}^*$
$(R_g)^*$

Weinstein Groupoid:

canonical symp realiz.

M. Equivalence $(M, Q) \rightrightarrows (M, Q)$



$\dim G = 2 \dim M.$

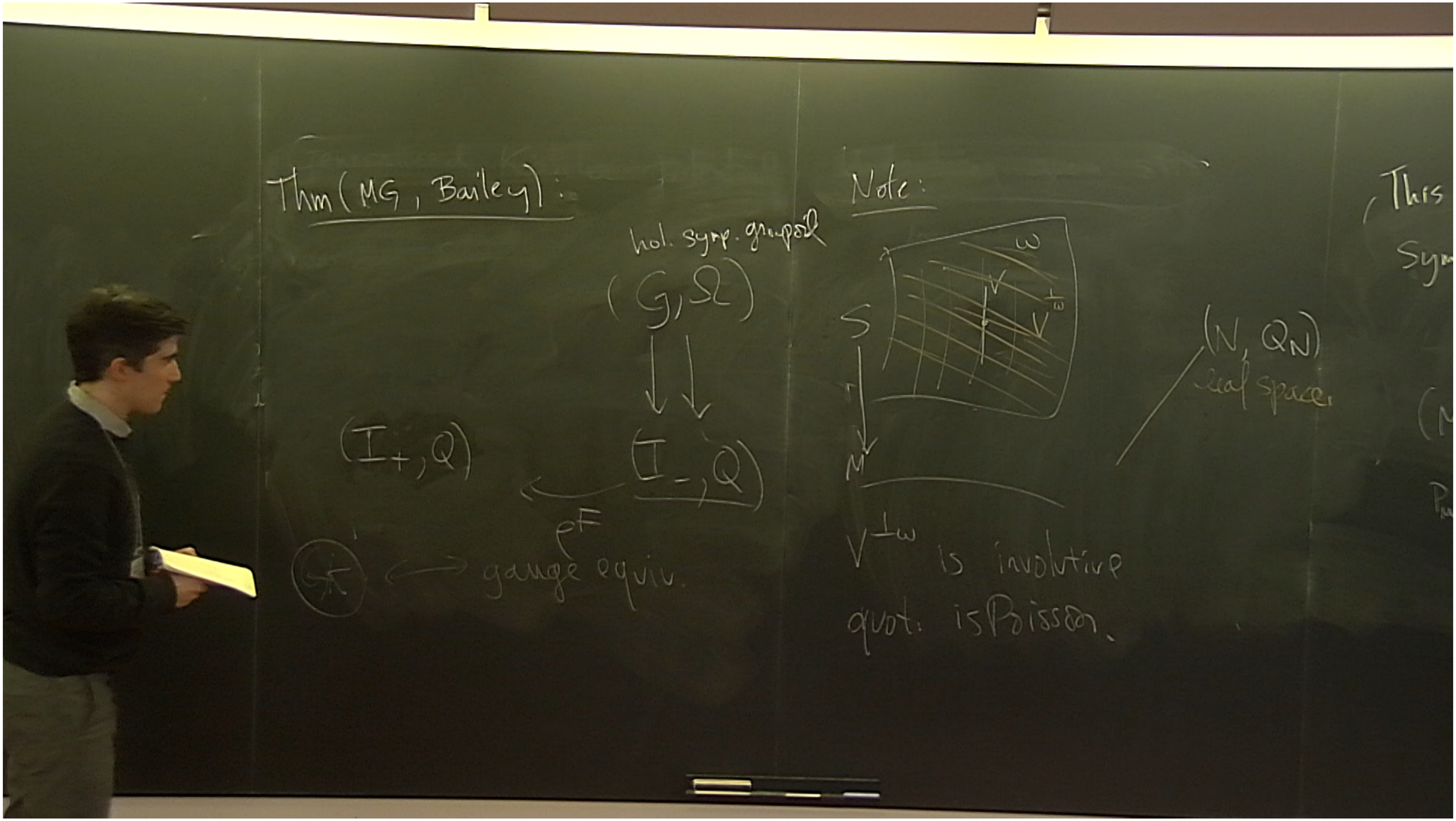
Crainic - Fernandes obstructions.

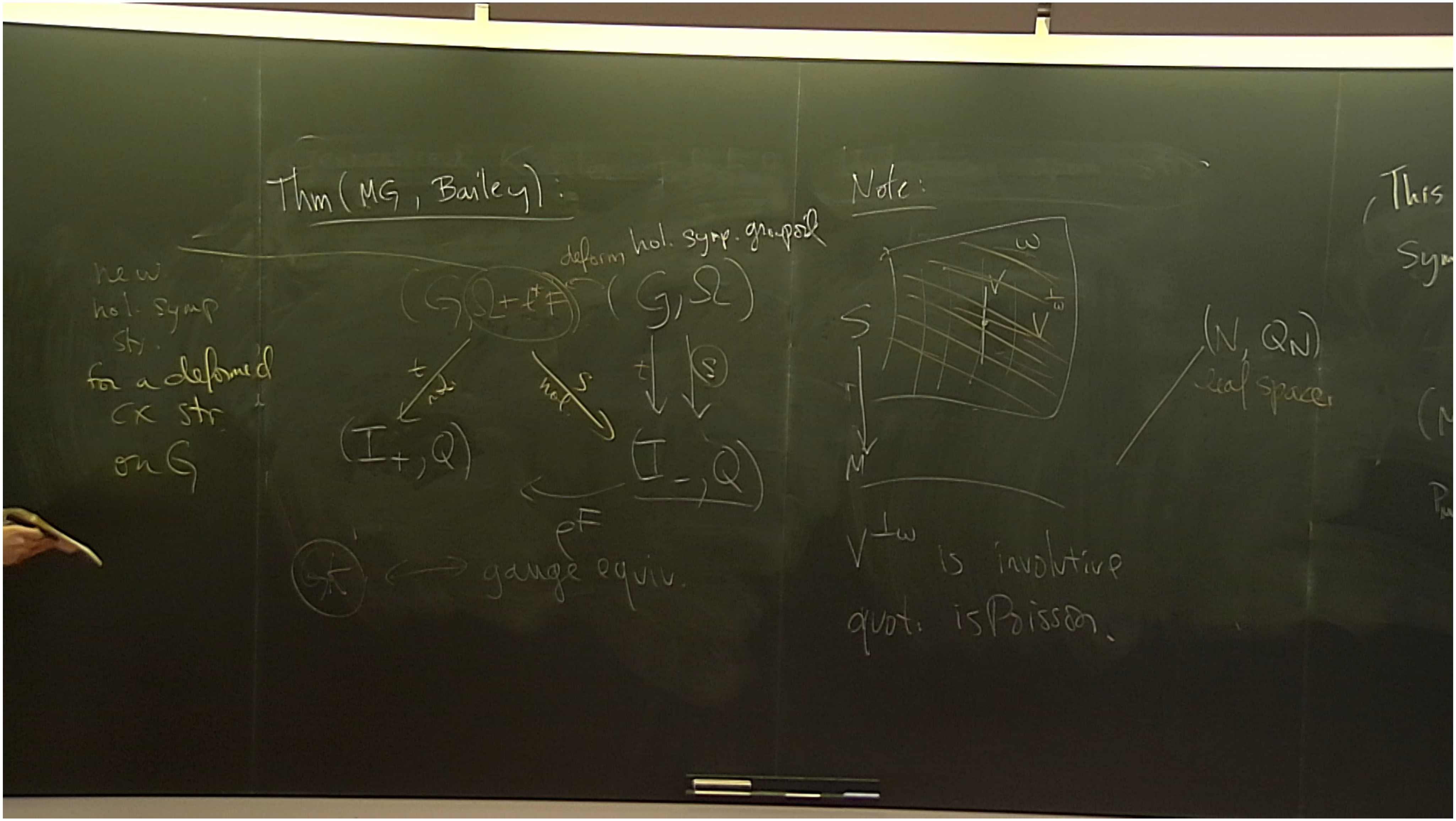
Similar to Lie group. but mfld of Identities.

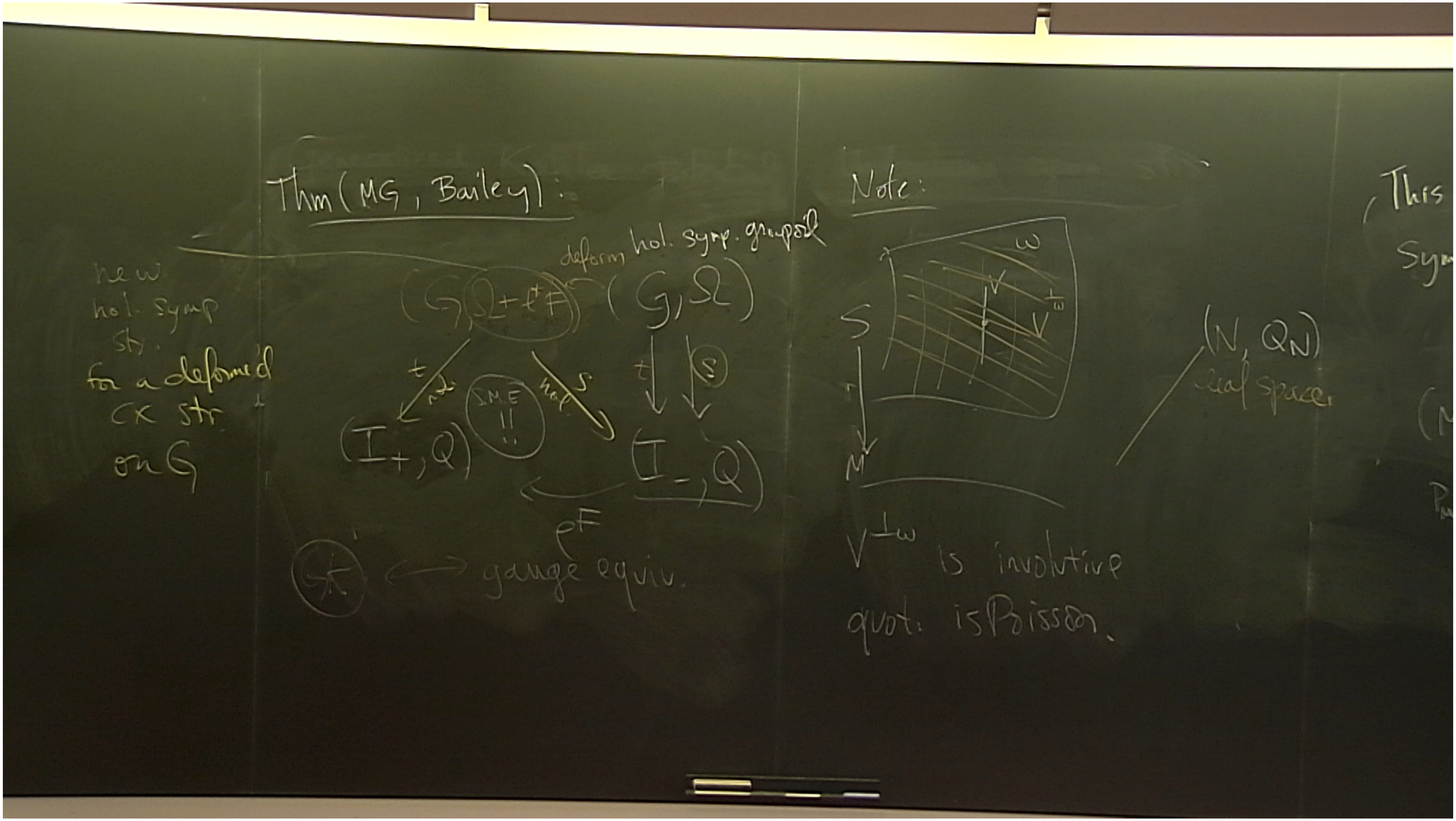
$Q = 0$ $\quad G = T^*M \rightrightarrows M$ $\quad Z$

Q symplectic $\quad G = M \times M \rightrightarrows M$

$$T^*G \xrightarrow{(L_{g^{-1}})^*} \mathfrak{g}^*$$
$$T^*G \xrightarrow{(R_g)^*} \mathfrak{g}^*$$
$$T^*G = \mathfrak{g}$$


Thm (MG, Bailey):
hol. symp. groupoid
(G, Ω)
(I_+, Q)
(I_-, Q)
e^F
gauge equiv.
Note:
ω
V
$V^{\perp_\omega}$
S
M
$V^{\perp_\omega}$ is involutive
quot: is Poisson.
(N, Q_N)
leaf space.
This
Sym


Thm (MG, Bailey):
new hol. symp str. for a deformed cx str on G
deform hol. symp. groupoid
$(G, \Omega + \rho^* F)$
(G, Ω)
(I_+, Q)
(I_-, Q)
e^F
gauge equiv.
Note:
ω
V
$V^{\perp_\omega}$
S
M
(N, Q_N)
real space
$V^{\perp_\omega}$ is involutive
quot. is Poisson.
This
Sym


Thm (MG, Bailey):
new hol. symp str.
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(G, Ω + ρ*F)
(G, Ω)
t
s
hol.
S.M.E !!
(I₊, Q)
(I₋, Q)
ρF
gauge equiv.
Note:
ω
V
S
M
V^⊥ω is involutive
quot: is Poisson.
(N, Q_N)
real space.
This
Sym

, Bailey):

deform hol. symp. groupoid

$(\mathcal{G}, \Omega + t^*F) \longleftarrow (\mathcal{G}, \Omega)$

$t \downarrow \ \downarrow s$

S.M.E !!

$(\ , Q) \qquad (I_-, Q)$

e^F

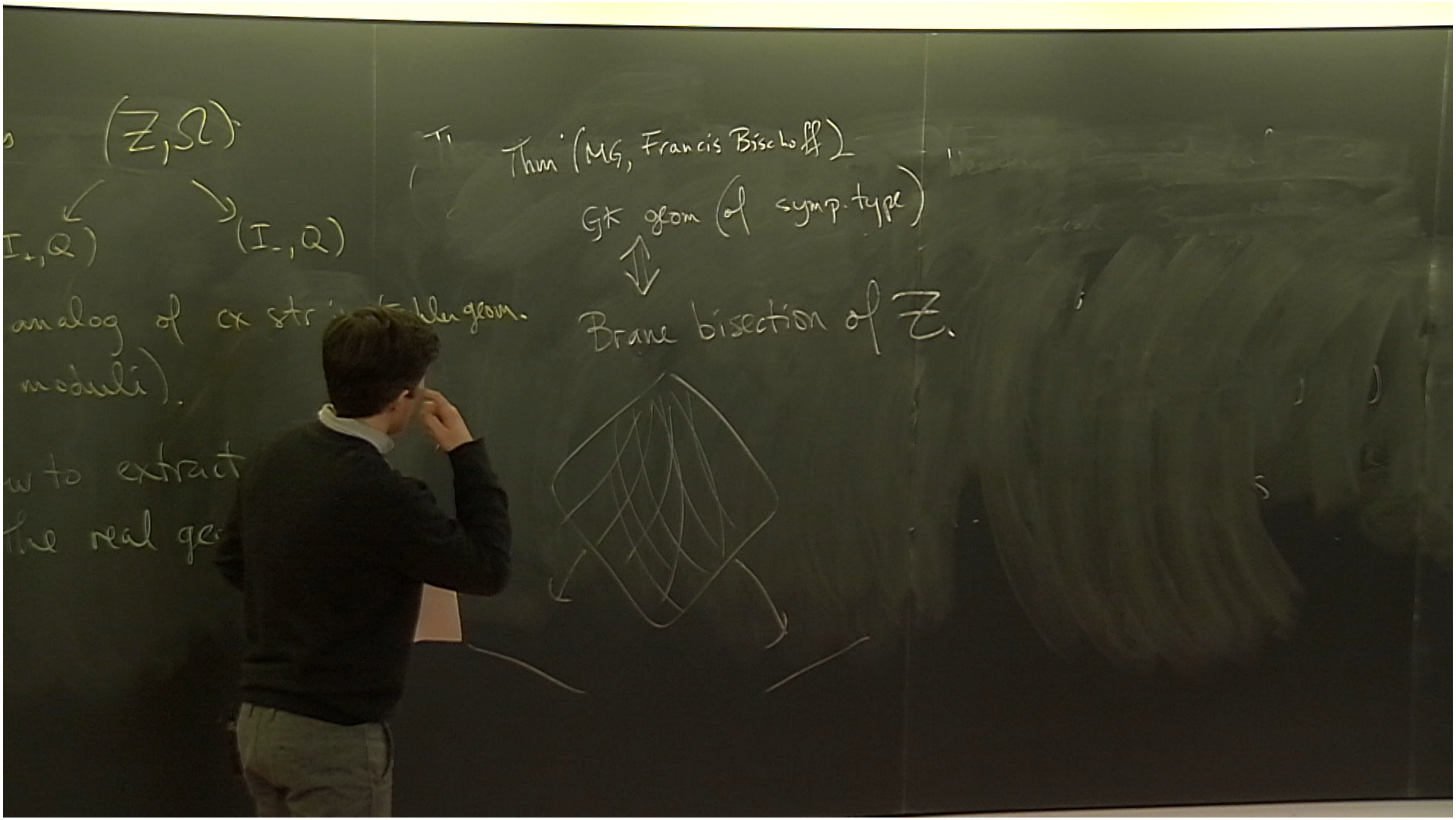
$\longrightarrow$ gauge equiv.

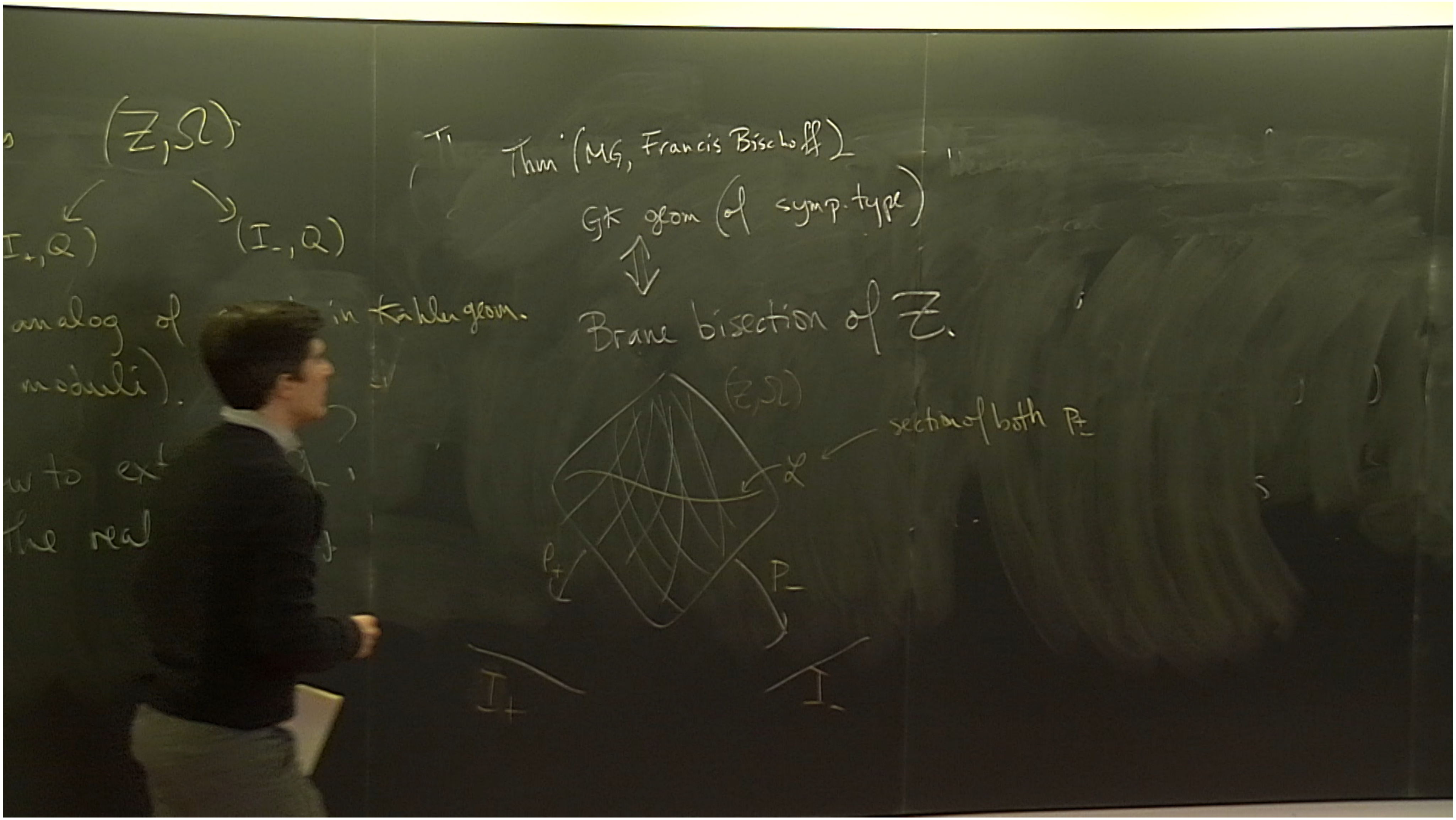
<u>Claim</u>: this

$$\begin{tikzcd} & (Z,\Omega) \arrow[dl] \arrow[dr] & \\ (I_+,Q) & & (I_-,Q) \end{tikzcd}$$

is the analog of cx str in Kähler geom.
(holom. moduli).

<u>Q</u>: how to extract g?, the real geometry


(Z, Ω)
(I+, Q)
(I-, Q)
analog of cx str
hler geom.
moduli).
w to extract
The real ge
Thm (MG, Francis Bischoff)
GK geom (of symp. type)
Brane bisection of Z.


(Z, Ω)
(I_+, Q)
(I_-, Q)
analog of
in Kähler geom.
moduli).
w to ex
The real
Thm (MG, Francis Bischoff)
GK geom (of symp. type)
$\Updownarrow$
Brane bisection of Z.
(Z, Ω)
section of both $P_\pm$
$\mathcal{L}$
P_+
P_-
I_+
I_-

(Z,Ω)

$(I_+, Q) \qquad (I_-, Q)$

analog of cx str in Kähler geom.

moduli).

w to extract g?

the real geometry

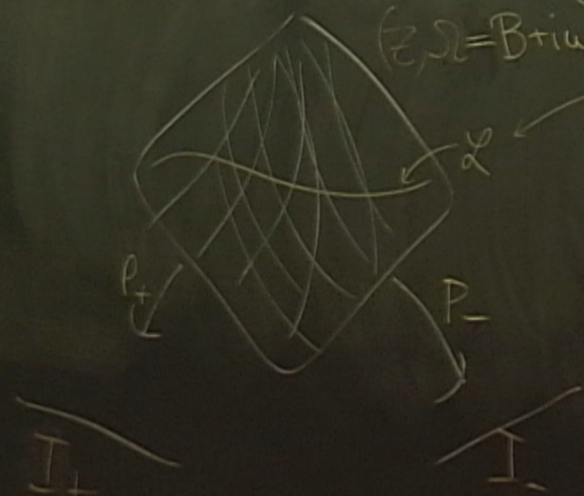
Thm (MG, Francis Bischoff)

GK geom (of symp. type)

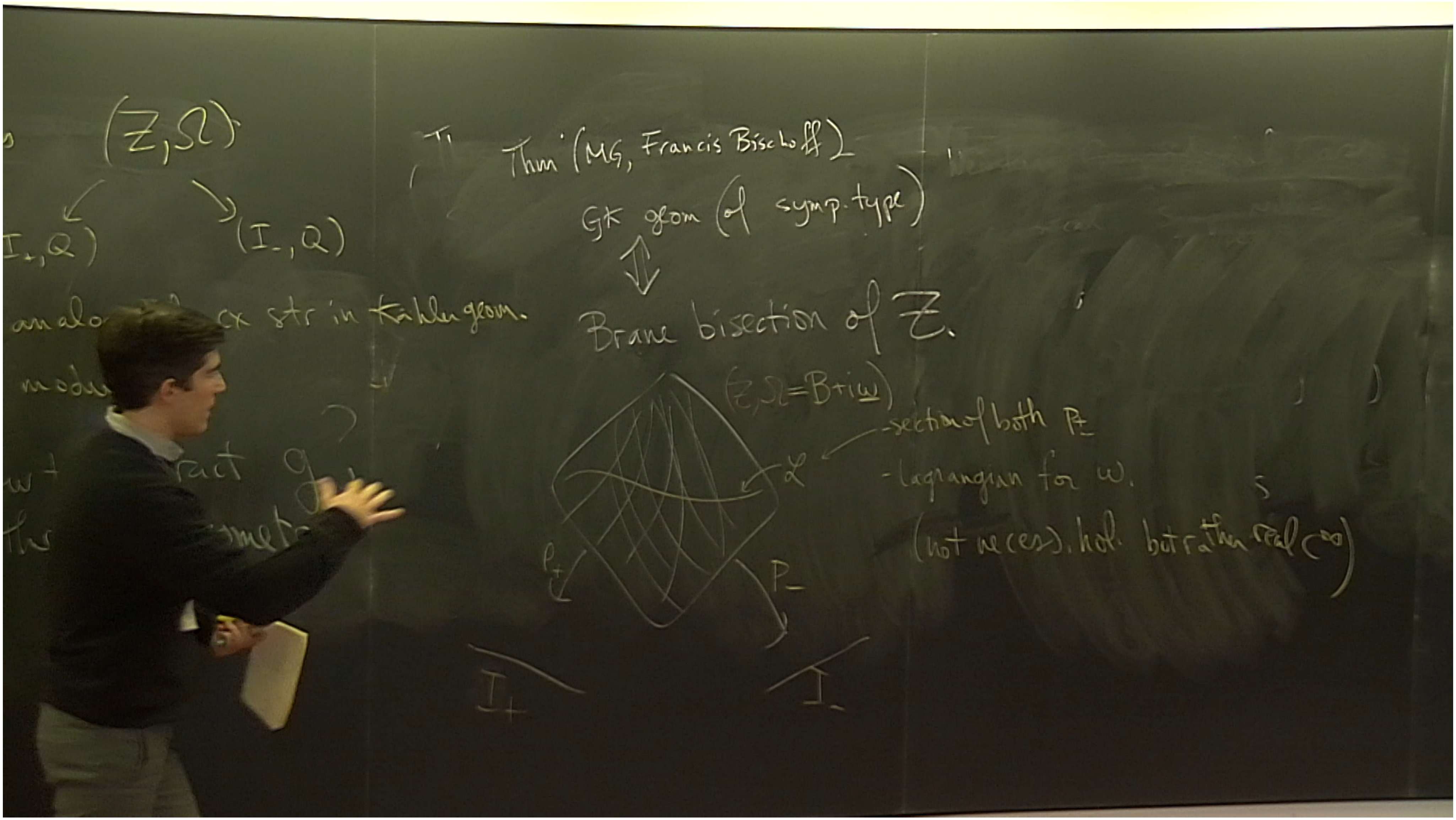
$\Updownarrow$

Brane bisection of Z.

$(Z, \Omega = B + i\omega)$



- section of both $P_\pm$
- Lagrangian

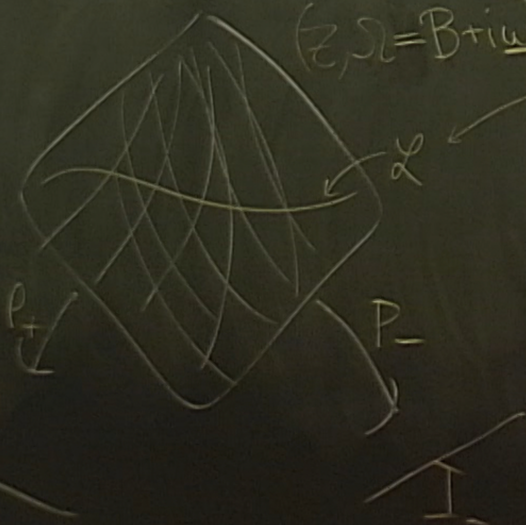

(Z, Ω)
(I+, Q)
(I-, Q)
cx str in Kähler geom.
Thm (MG, Francis Bischoff)
GK geom (of symp. type)
Brane bisection of Z.
(Z, Ω = B + iω)
- section of both P±
- Lagrangian for ω.
(not necess. hol. but rather real C∞)
P+
P-
I+
I-

Thm (MG, Francis Bischoff)

G^* geom (of symp. type)

$\Updownarrow$

Brane bisection of Z.



$(Z, \Omega = B + i\underline{\omega})$

- section of both $P_\pm$
- Lagrangian for ω.

(not necess. hol. but rather real C^∞)

notice: $\Omega|_{\mathcal{L}} = (B + i\omega)|_{\mathcal{L}} = B|_{\mathcal{L}} = F$ $\dim G = 2\dim M$.

recovers the 2-form F

$Q = 0$ $G = T^*(-)$ M

Q symplectic $G =$

$T^*G \xrightarrow{(L_{g^{-1}})^*}$ $\xrightarrow[(R_g)^*]{}$

$\|$

$\mathfrak{g}$

Thm (MG, Francis Bischoff)

GK geom (of symp. type)

$\Updownarrow$

Brane bisection of Z.

Kähler geom.

$(Z, \Omega = B + i\omega)$

- section of both $P_\pm$
- Lagrangian for ω.

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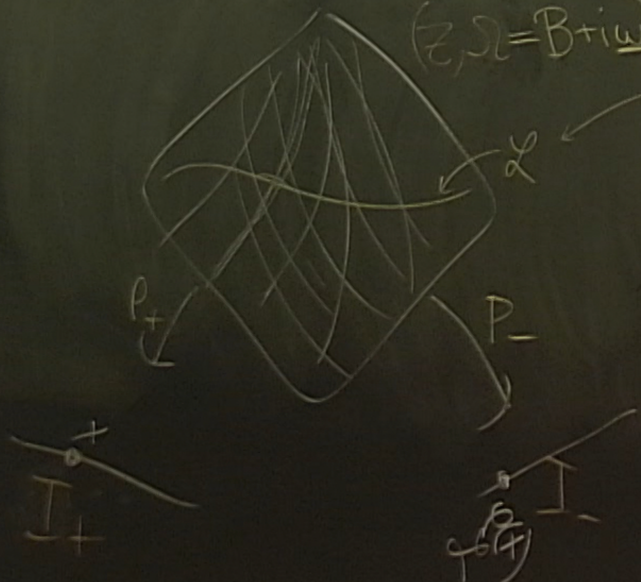
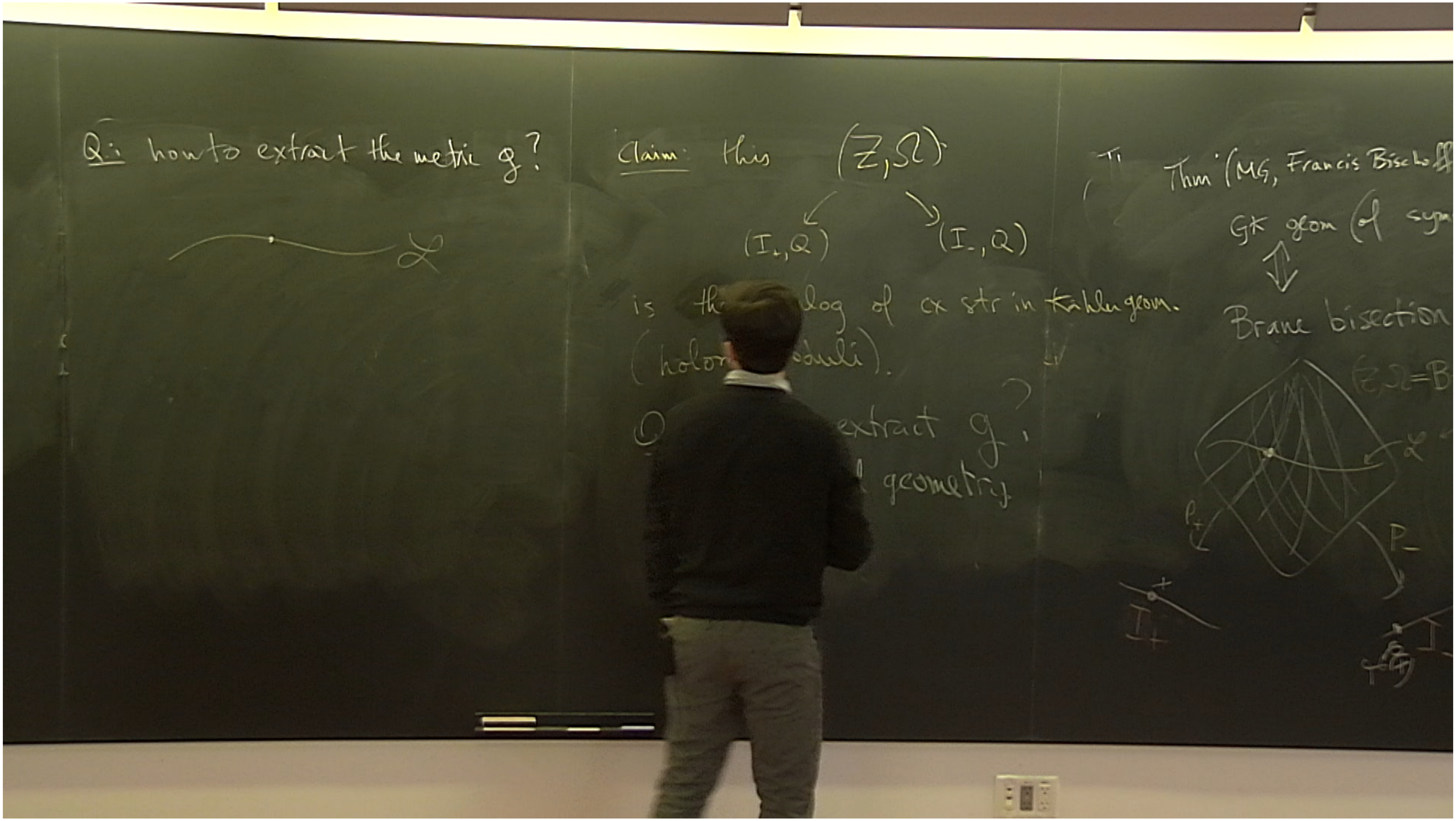
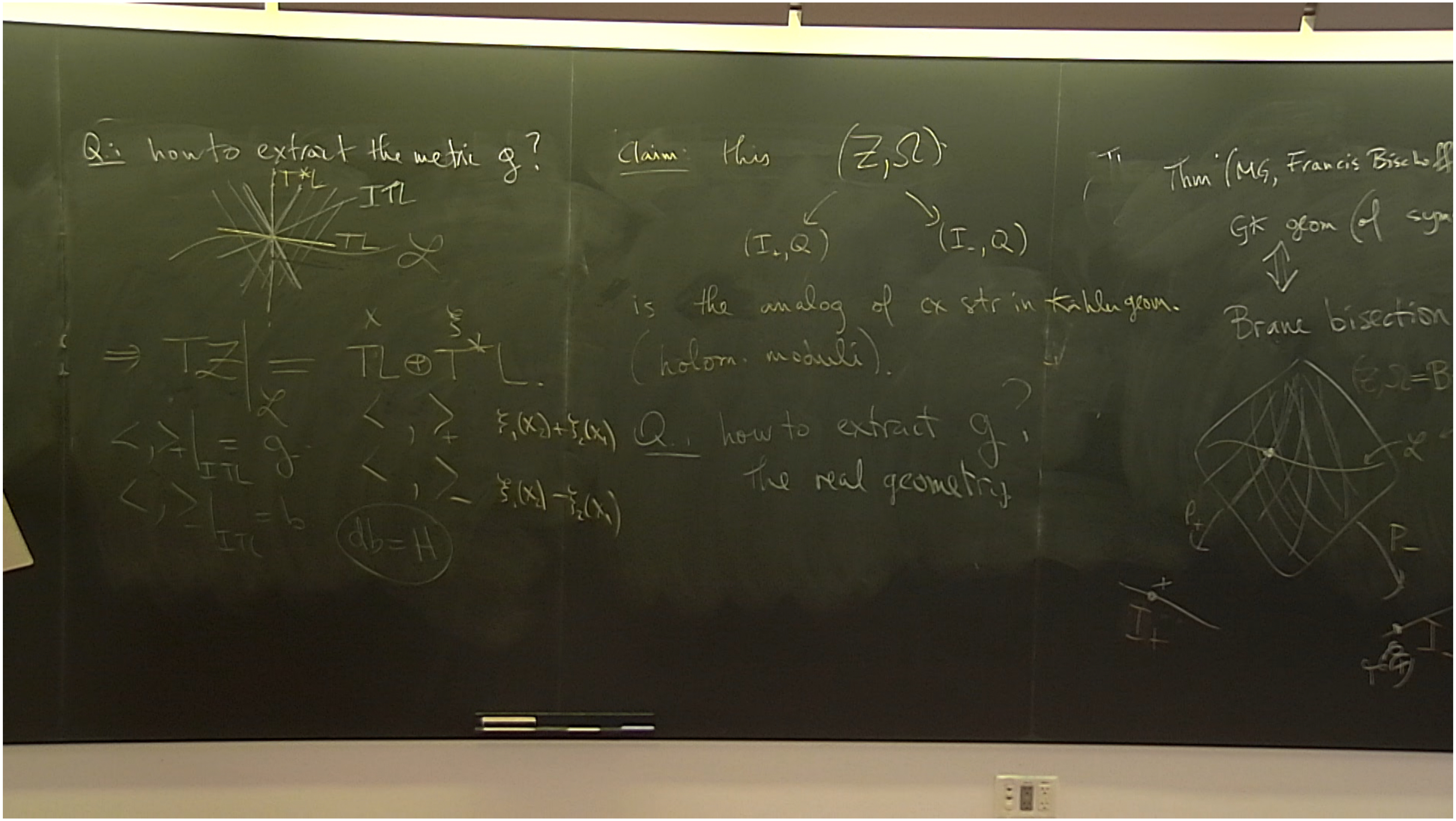


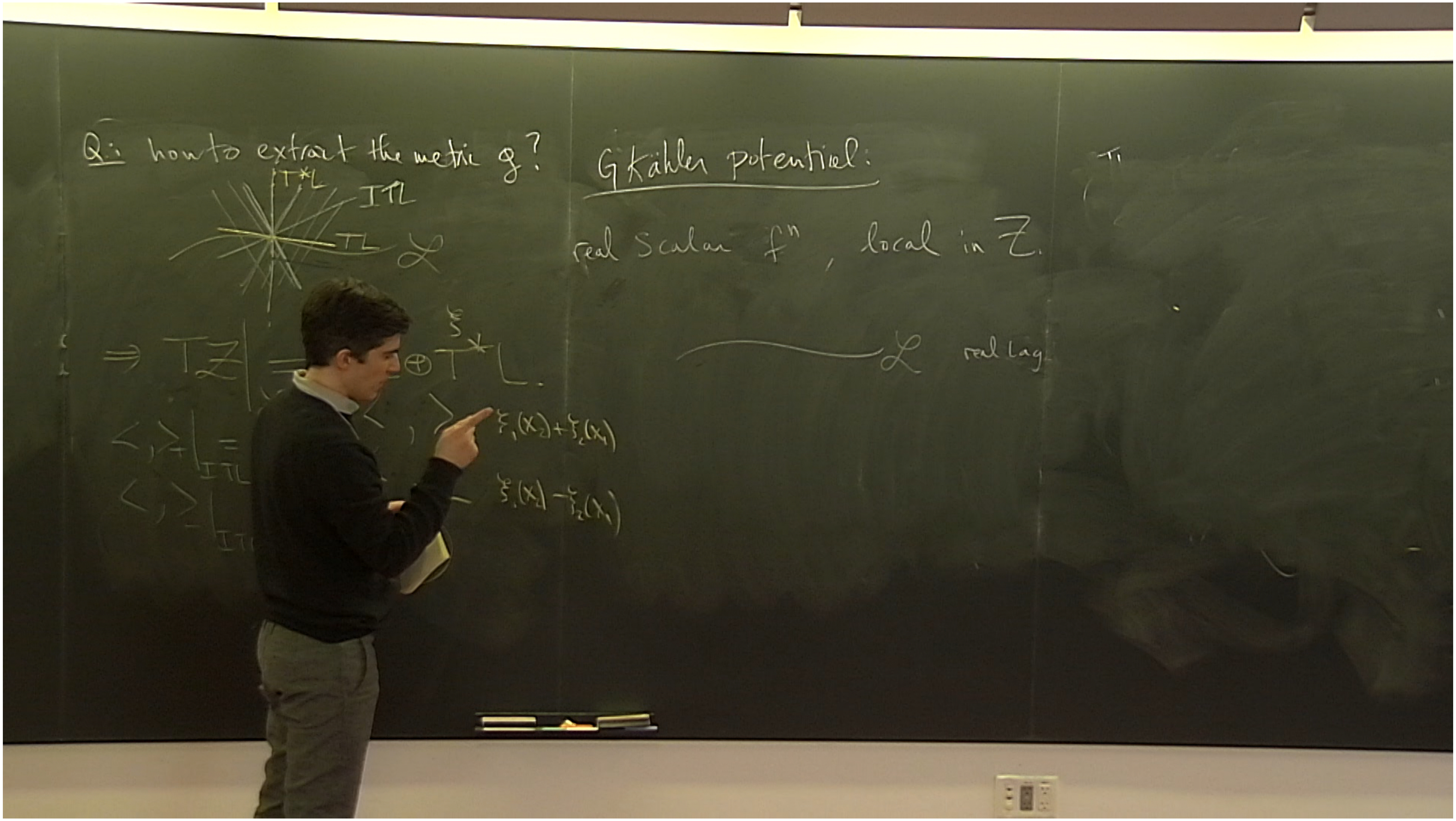
notice: ① $\Omega|_{\mathcal{L}} = (B + i\omega)|_{\mathcal{L}} = B|_{\mathcal{L}} = F$ $\dim G = 2\dim M$.

recovers the 2-form $F \in \Omega^2(\mathcal{L}, \mathbb{R})$

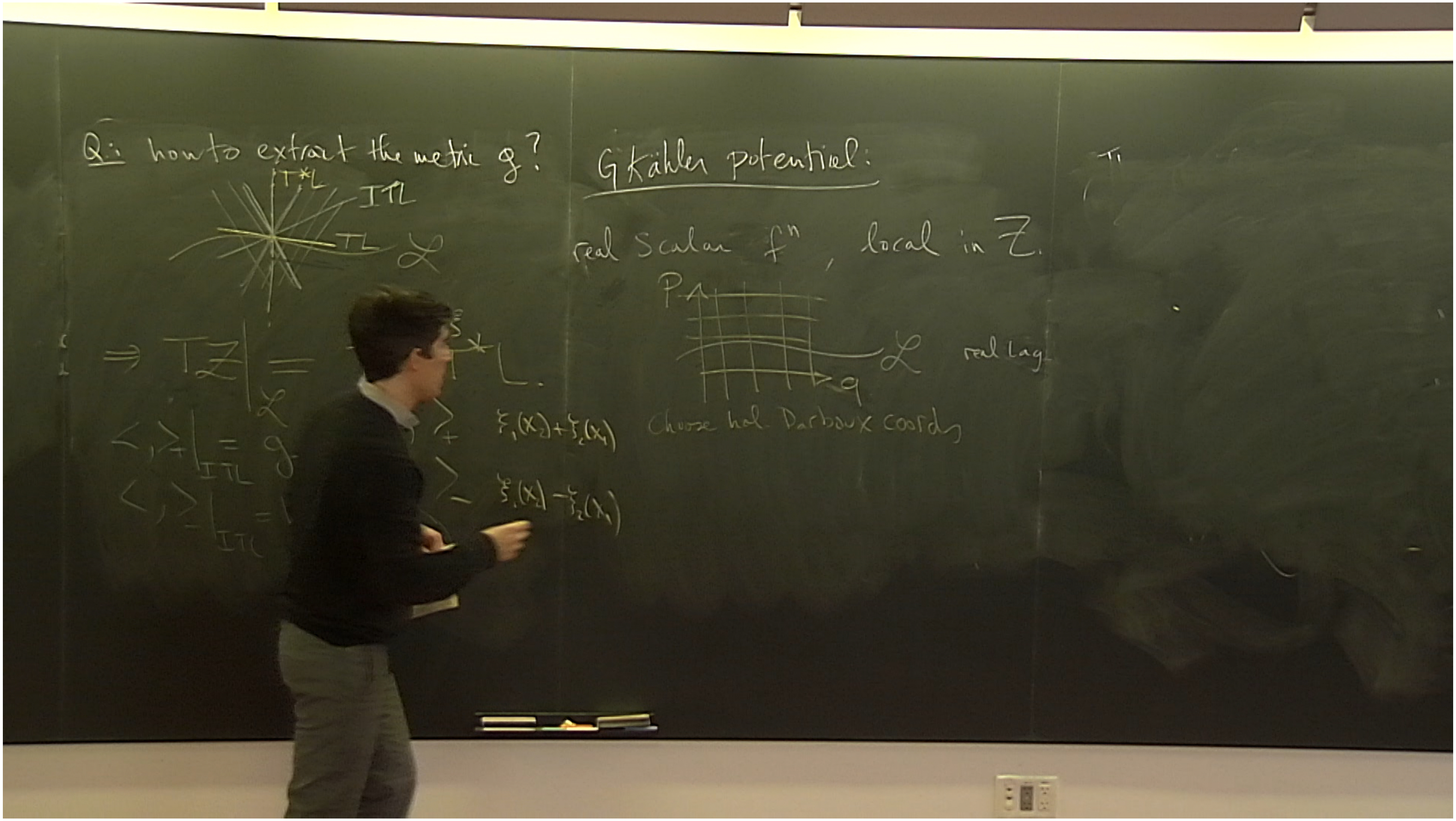
② def. a diffeo. $\varphi_{\mathcal{L}} : (M, I_+) \to (N, I_-)$

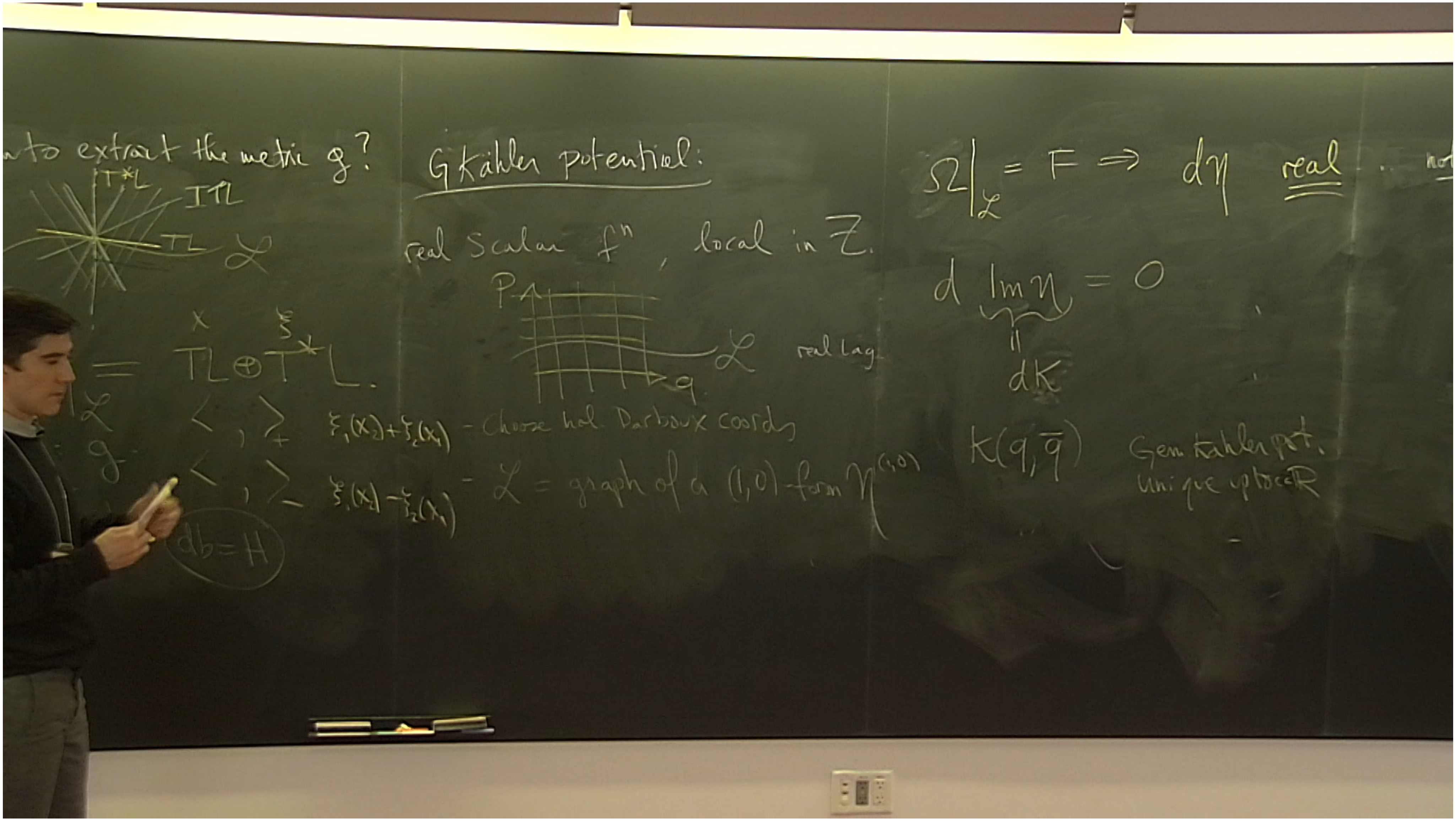

Q: how to extract the metric g?
Claim: this (Z, Ω)
(I+, Q)
(I-, Q)
of cx str in Kähler geom.
extract g?
geometry
Thm (MG, Francis Bischoff
GK geom
Brane bisection
P+
P-
I+

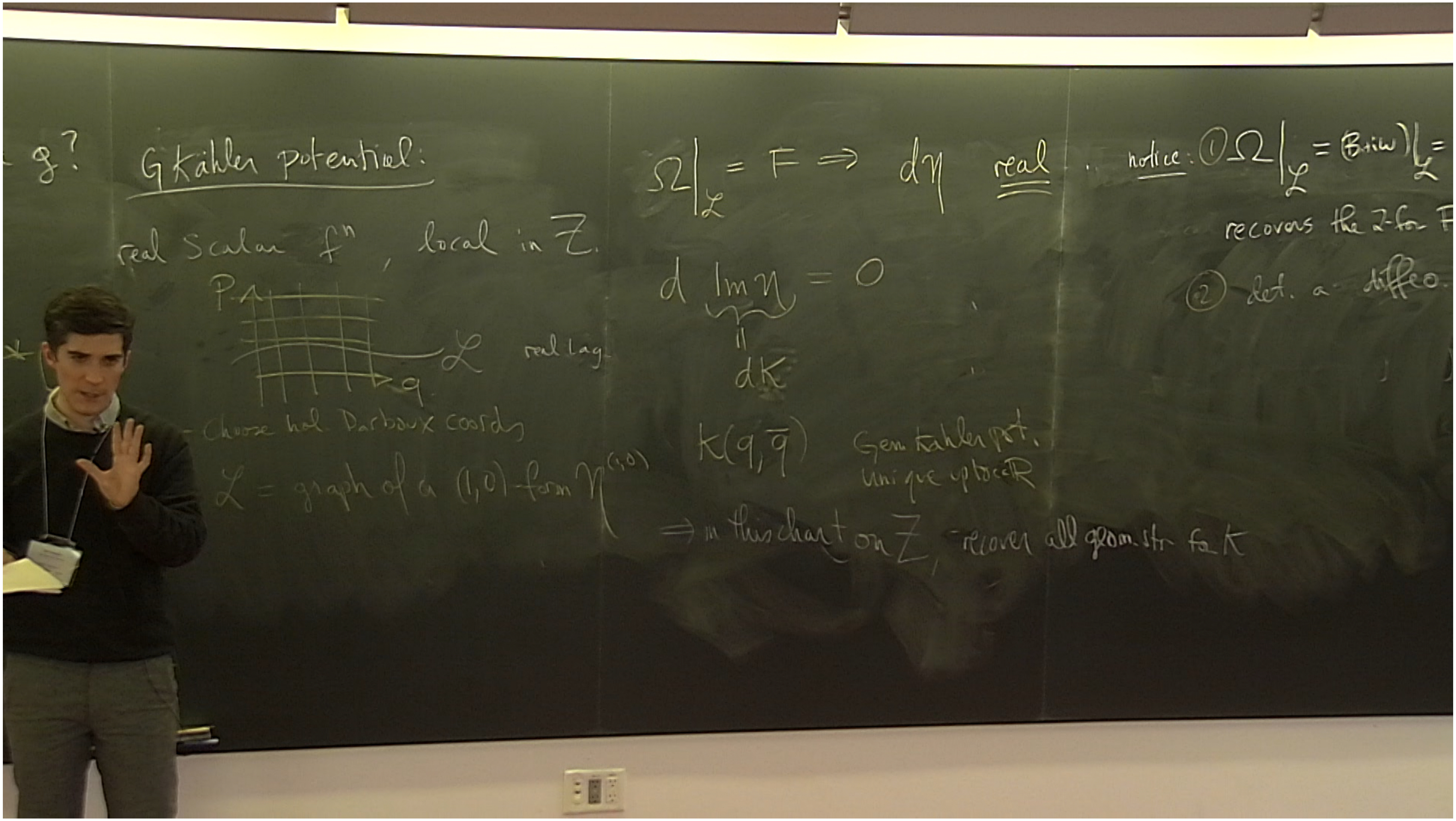

Q: how to extract the metric g?
T*L
ITL
TL
⇒ TZ|_L = TL ⊕ T*L.
⟨ , ⟩_+ ξ_1(X_2)+ξ_2(X_1)
⟨ , ⟩_- ξ_1(X_2)−ξ_2(X_1)
⟨,⟩_+|_ITL = g
⟨,⟩_-|_ITL = b
db=H
Claim: this (Z,Ω)
(I_+,Q)
(I_-,Q)
is the analog of cx str in Kähler geom.
(holom. moduli).
Q: how to extract g?
the real geometry
Thm (MG, Francis Bischoff
G* geom
Brane bisection
P_+
P_-
I_+



Q: how to extract the metric g?
Kähler potential:
real scalar f^n, local in Z.
real Lag.


Q: how to extract the metric g?
Kähler potential:
real scalar f^n, local in Z.
real Lag.
Choose hol. Darboux coords


to extract the metric g?
T*L
TL
$\mathcal{L}$
$= TL \oplus T^*L.$
$\langle\, , \rangle_+$ $\xi_1(X_2)+\xi_2(X_1)$ - Choose hol. Darboux coords
g $\langle\, , \rangle_-$ $\xi_1(X_2)-\xi_2(X_1)$ - $\mathcal{L}$ = graph of a (1,0)-form $\eta^{(1,0)}$
db=H
G Kähler potential:
real Scalar f^n, local in Z.
P
q
$\mathcal{L}$ real Lag.
$\Omega|_{\mathcal{L}} = F \Rightarrow d\eta$ real
$d\,\mathrm{Im}\,\eta = 0$
dK
$K(q,\bar{q})$ Gen. kähler pot.
Unique up to c∈R


g?
G Kähler potential:
real Scalar f^n, local in Z.
real Lag.
Choose hol. Darboux coords
L = graph of a (1,0)-form η^(1,0)
Ω|_L = F ⇒ dη real
d Im η = 0
dK
K(q, q̄) Gen kähler pot.
unique up to c∈R
⇒ in this chart on Z, recover all geom. str. for K
notice: ① Ω|_L = (B+iω)|_L =
recovers the 2-form F
② def. a diffeo.