Title: Generalizing Quivers: Bows, Slings, Monowalls

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Abstract: Quivers emerge naturally in the study of instantons on flat four-space (ADHM), its orbifolds and their deformations, called ALE space (Kronheimer-Nakajima). Pursuing this direction, we study instantons on other hyperkaehler spaces, such as ALF, ALG, and ALH spaces. Each of these cases produces instanton data that organize, respectively, into a bow (involving the Nahm equations), a sling (involving the Hitchin equations), and a monopole wall (Bogomolny equation).

Generalizing Quivers: Bows, Slings, Monowalls

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Hitchin Systems in Mathematics and Physics
Perimeter Institute
Feb. 13, 2017

An Egyptian sling from 1900 B.C.
from: warfarehistorynetwork.com

Outline

- Hitchin System and its generalization and degenerate limits: Quivers, Bows, Slings, Monopoles, and Instantons.
- String Theory motivation.
- Analysis of Instantons on ALF space: Dirac Index.
- Algebraic geometry of Instantons on ALF space: the Monad.
- Monowalls and tropical geometry.

Relations

Equations:

ADHM — Nahm Eqs. — Hitchin System — Bogomolny Eq. — Anti-self-duality

ADHM	Nahm Eqs.	Hitchin System	Bogomolny Eq.	Anti-self-duality
$[B_{01}, B_{10}] = IJ$ $[B_{01}, B_{01}^\dagger] + [B_{10}, B_{10}^\dagger] = II^\dagger - J^\dagger J$	$\frac{d}{ds}T_i = [T_j, T_k]$	$\bar{D}\Phi = 0$ $[D, \bar{D}] = [\Phi, \Phi^\dagger]$	$F_A = - * D_A \Phi$	$F_A = - * F_A$

Generalizations:

	Quiver	Bow	Sling	Monowall
	Inst/ALE	Inst/ALF	Inst/ALG	Inst/ALH
Infinity of base space:	$\mathbb{R}^4/\Gamma$	Circle fibration	Elliptic fibration over cone	Elliptic fibration over cylinder

Main Question:

Bows and Monowalls suggest two very different vantage points.

How do these views carry over to the Hitchin System?

String theory Motivation

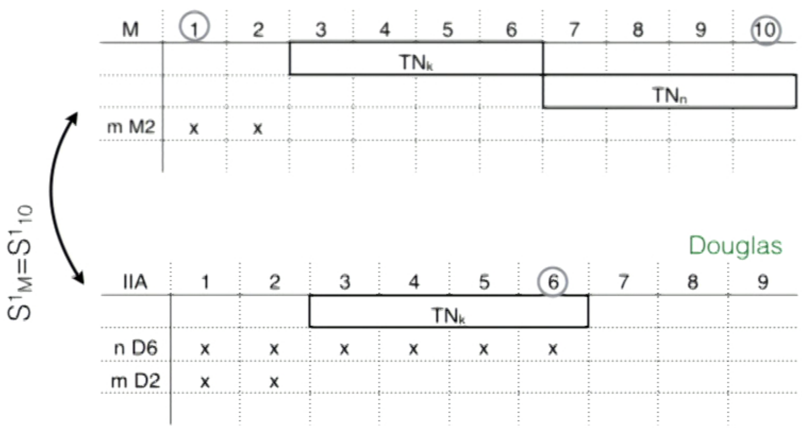
Douglas

IIA	1	2	3	4	5	(6)	7	8	9
			TN_k						
n D6	x	x	x	x	x	x			
m D2	x	x							

m U(n) Instantons
in the world-volume of D6.

Bow solution
in the world-volume of D3.

String theory Motivation



m U(n) Instantons
in the world-volume of D6.

Bow solution
in the world-volume of D3.

String theory Motivation

M	(1)	2	3	4	5	6	7	8	9	(10)
			TN_k							
							TN_n			
m M2	x	x								

$S^1_M = S^1_{10}$

Douglas

IIA	1	2	3	4	5	(6)	7	8	9
			TN_k						
n D6	x	x	x	x	x	x			
m D2	x	x							

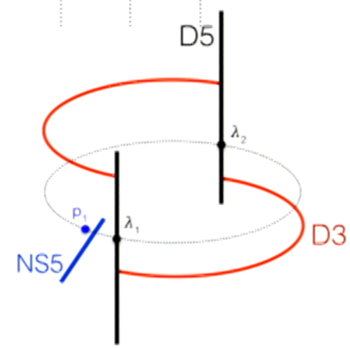
m U(n) Instantons
in the world-volume of D6.

T_6

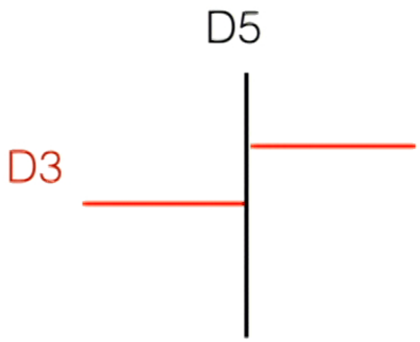
Chalmers-Hanany-Witten

IIB	1	2	3	4	5	(6)	7	8	9
k NS5	x	x	ν^1_σ	ν^2_σ	ν^3_σ	p_σ	x	x	x
n D5	x	x	x	x	x	λ_j			
m D3	x	x				x			

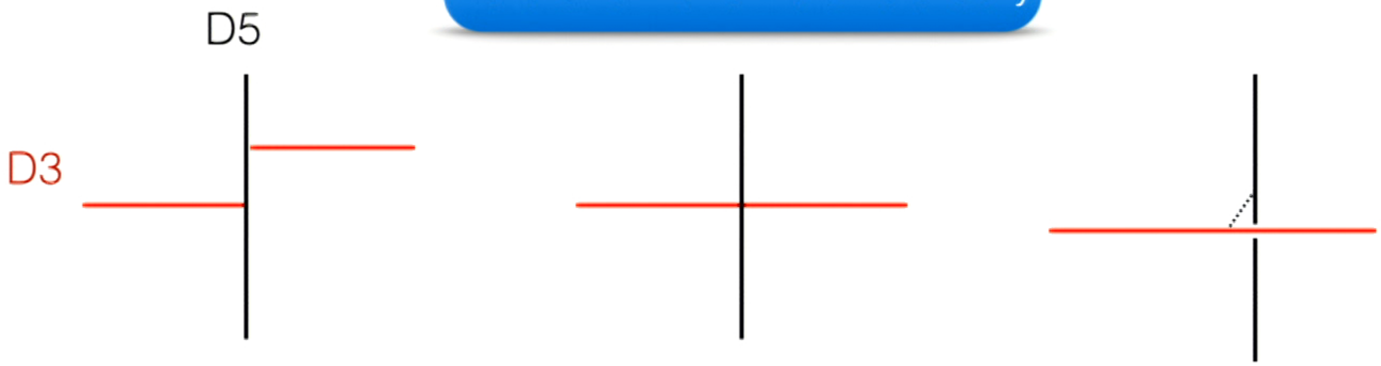
Bow solution
in the world-volume of D3.



D3-brane world-volume theory

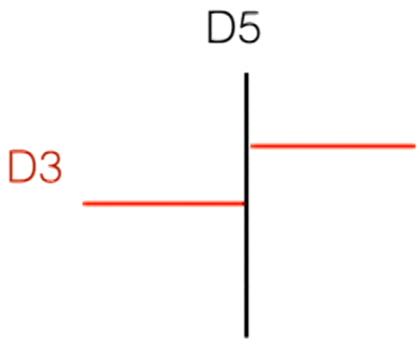


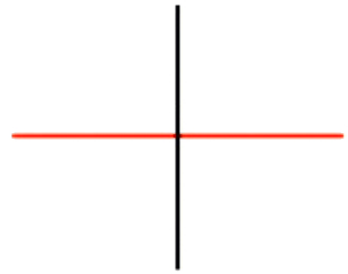
D3-brane world-volume theory

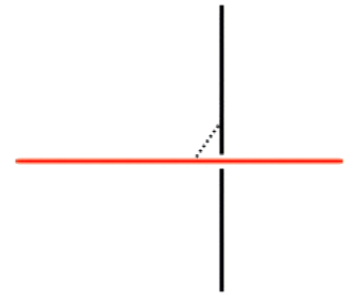


Massive fundamental hypermultiplet
from D3-D5 open string mode

D3-brane world-volume theory

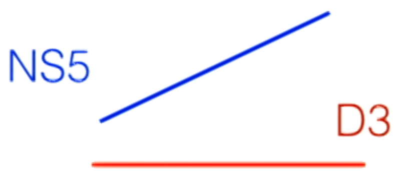


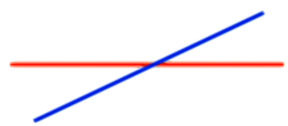


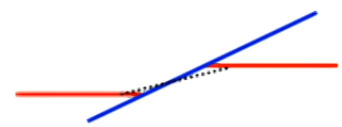


Massless fundamental
hypermultiplet: Q

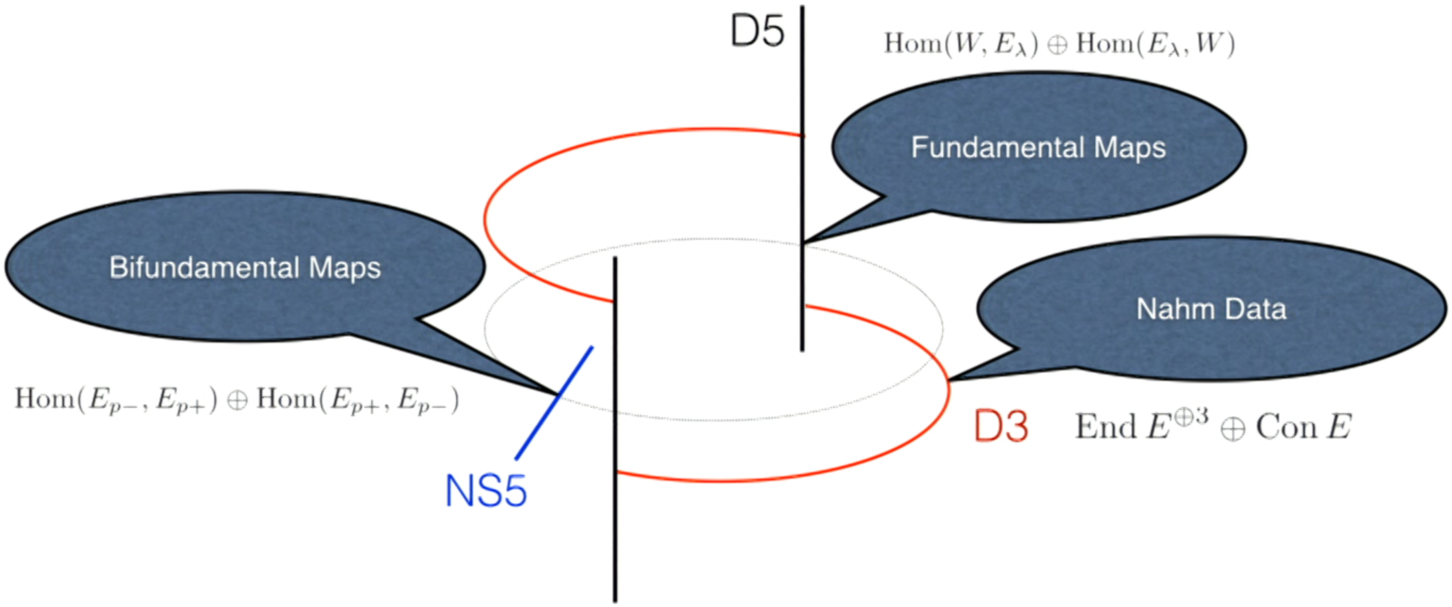
Massive fundamental hypermultiplet
from D3-D5 open string mode

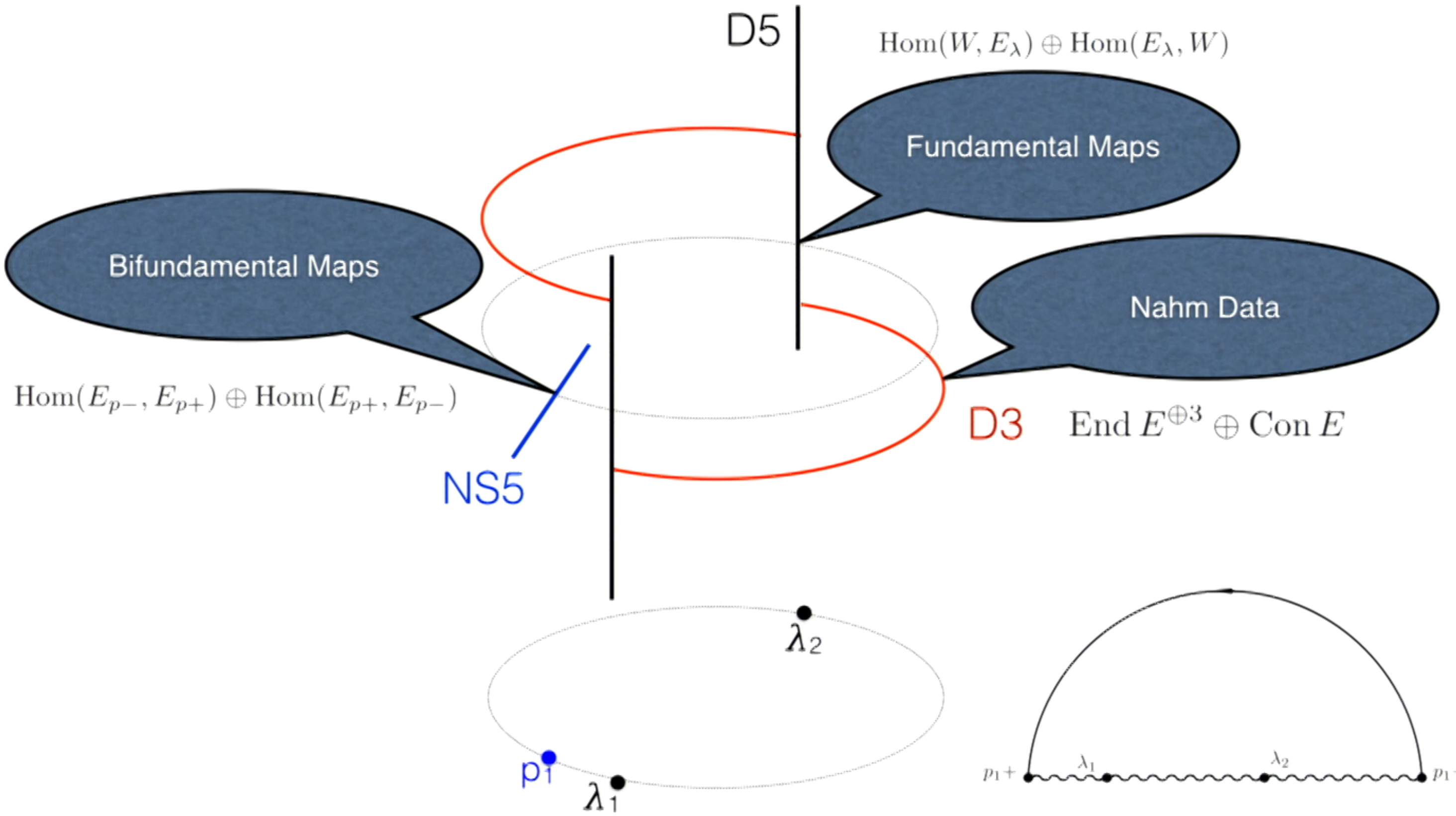






Massive bifundamental hypemultiplet
from D3-D3 open string mode

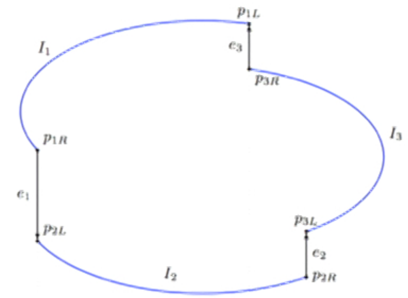

D5
$\mathrm{Hom}(W, E_\lambda) \oplus \mathrm{Hom}(E_\lambda, W)$
Fundamental Maps
Bifundamental Maps
Nahm Data
$\mathrm{Hom}(E_{p-}, E_{p+}) \oplus \mathrm{Hom}(E_{p+}, E_{p-})$
D3
$\mathrm{End}\, E^{\oplus 3} \oplus \mathrm{Con}\, E$
NS5


D5
$\mathrm{Hom}(W, E_\lambda) \oplus \mathrm{Hom}(E_\lambda, W)$
Fundamental Maps
Bifundamental Maps
Nahm Data
$\mathrm{Hom}(E_{p-}, E_{p+}) \oplus \mathrm{Hom}(E_{p+}, E_{p-})$
NS5
D3
$\mathrm{End}\, E^{\oplus 3} \oplus \mathrm{Con}\, E$
λ_2
p_1
λ_1
p_1+
λ_1
λ_2
p_1-

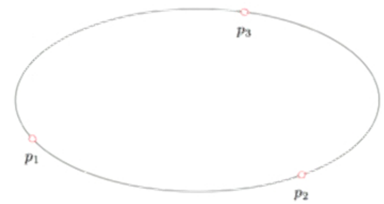
Bows

Bow (A_k):

Bow:



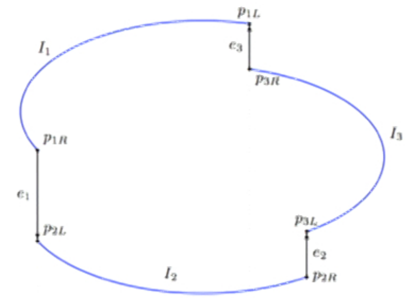
Circle diagram:



Bows

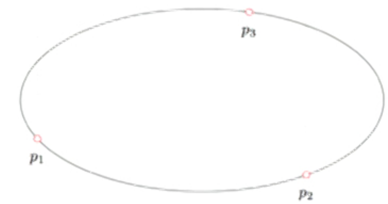
Bow (A_k):

Bow:



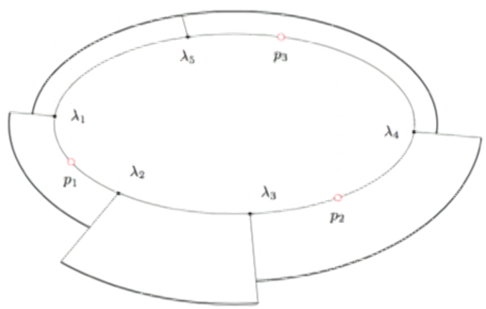
Representation *R* of the bow:

Circle diagram:



$$\{\lambda_j\}, R(s)$$

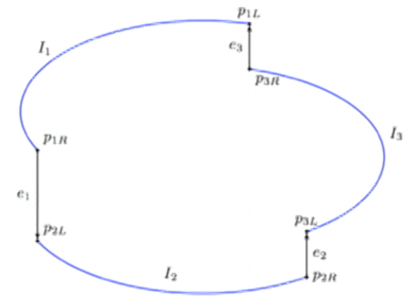
$$\{W_\lambda = \mathbb{C}\}_{R(\lambda-)=R(\lambda+)}$$



Bows

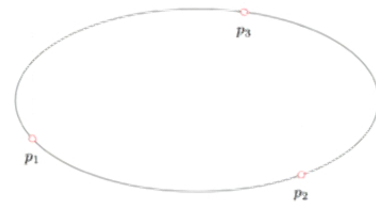
Bow (A_k):

Bow:



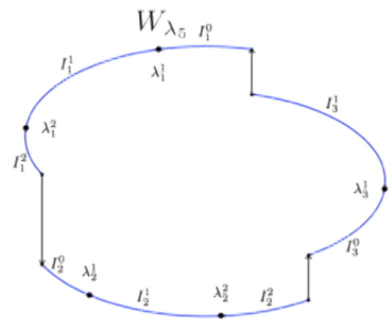
Representation *R* of the bow:

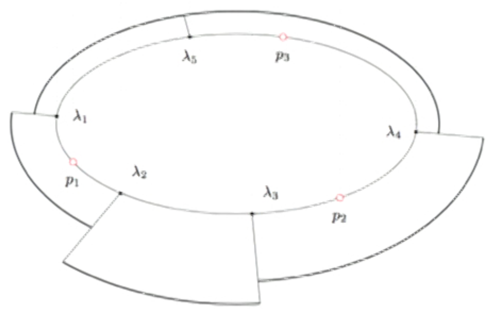
Circle diagram:



$\{\lambda_j\}, R(s)$

$\{W_\lambda = \mathbb{C}\}_{R(\lambda-)=R(\lambda+)}$





Let S be a 2d representation of quaternions,
and e_1, e_2, and e_3 be quaternionic units.

Affine space: Dat(R)=B⊕F⊕N is hyperkähler

B: $B_\sigma^+ = \begin{pmatrix} B_{\sigma,\sigma+1}^\dagger \\ B_{\sigma+1,\sigma} \end{pmatrix} \in \mathrm{Hom}(E_{p_\sigma -}, S \otimes E_{p_\sigma +})$

F: $Q_\lambda = \begin{pmatrix} J_\lambda^\dagger \\ I_\lambda \end{pmatrix} \in \mathrm{Hom}(W_\lambda, S \otimes E_\lambda)$

N: $D = \frac{d}{ds} + T_0 + e_j T_j \in \mathrm{Con}(S \otimes E)$

Gauge group $\mathcal{G}$ acts triholomorphically on Dat(R)!

$$B_\sigma^+ \mapsto g(p_\sigma -) B_\sigma^+ g(p_\sigma +),$$

$$Q_\lambda \mapsto g(\lambda) Q_\lambda,$$

$$T_0(s) \mapsto g^{-1}(s) T_0 g(s) + g^{-1}(s) \frac{d}{ds} g(s),$$

$$T_j(s) \mapsto g^{-1}(s) T_j g(s).$$

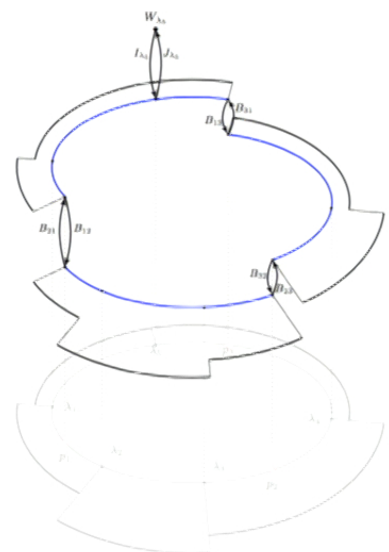
Higgs branch=Bow rep. moduli space:

$$\mathcal{M}^{\text{Bow}} = \mathrm{Dat}(R) /\!/\!/ \mathcal{G} = \mu^{-1}(\nu)/\mathcal{G}$$

* The moment map conditions

$$\mu(T, Q, B)$$

of the hyperkähler reduction.
* The level ν = NS5 positions.



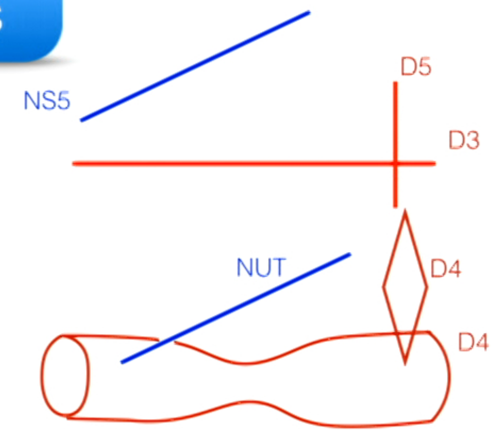
Hints of Slings

M	1	2	3	4	(5)	(6)	7	8	9	(10)
			per. TN_k=ALG							
							TN_n			
m M2	x	x								

IIA	1	2	3	4	(5)	(6)	7	8	9
			per. TN_k=ALG						
n D6	x	x	x	x	x	x			
m D2	x	x							

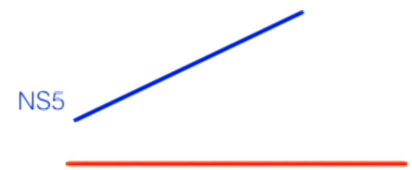
IIB	1	2	3	4	(5)	(6)	7	8	9
k NS5	x	x					x	x	x
n D5	x	x	x	x	x				
m D3	x	x				x			

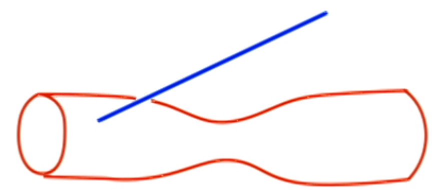
IIA	1	2	3	4	(5)	(6)	7	8	9
k NS5			$\widetilde{TN}_k$						
n D4	x	x	x	x					
m D4	x	x			x	x			



(twisted) Hitchin system on a curve with a pole.

What is the significance of NS5?





Hitchin systems on reduced curves with massless bifundamentals.

- One way to a sling is via the Down transform:

Instanton on ALF space —> Index Bundle —> Bow

Instantons on ALG space —> Index Bundle —> Sling

- It is a version of ADHM-Nahm transform.
- For Bows, it can be done:
  - Analytically: Dirac index (with Mark Stern and Andres Larrain-Hubach)

    and

  - Algebro-geometrically: Monad (with Jacques Hurtubise).

Analysis of Instantons

with Mark Stern and
Andres Larrain-Hubach
arXiv:1608.00018

Uhlenbeck '79: Instanton curvature on $\mathbb{R}^4$ decays quartically:

$$|F_A|<C/r^4.$$

We extend this result proving the following **theorems**:

- Curvature of any finite action solution of the Yang-Mills Eqs. on ALF space decays at infinity.
  $D_A {*F_A}=0$ & $\int \mathrm{tr}\, F\wedge *F<\infty$ => $|F_A|\to 0$ as $r\to\infty$.
- Instanton curvature on ALF space decays quadratically:
  $$|F_A|<C/r^2.$$
- Instanton on ALF space with generic asymptotic holonomy has the form
  $A=-i\,\mathrm{diag}(a_1,a_2,\ldots,a_n)+O(\ln r/r^2)$ with
  $$a_j = \left(\lambda_j + \frac{m_j}{2r}\right)\frac{d\phi+\eta}{V} - \frac{m_j}{k}\eta$$
- Harmonic spinors decay exponentially (if no λ_j=0)
  or quadratically.

Index of the Dirac Operator

with Mark Stern and
Andres Larrain-Hubach
arXiv:1608.00018

$$\mathrm{ind}_{L^2} D^+ = \frac{k}{2}\mathrm{tr}(\{\Lambda\}^2 - \{\Lambda\}) + \frac{1}{2}\mathrm{tr}\left(\int_{S^2_\infty} \frac{\{\Lambda\}}{\pi} iF^0_{23} - \int_{S^2_\infty} \frac{iF^0_{23}}{2\pi}\right) d\mathrm{Vol}_{S^2} + \frac{1}{8\pi^2}\int \mathrm{tr} F \wedge F.$$

Observation:

As λ_j decreases below 0, the index changes by m_j, the corresponding monopole charge!

Down Transform

- TN_k is equipped with abelian instantons:

  one associated to each NUT: $a^{(\sigma)} = \frac{1}{2|t-\nu_\sigma|}\frac{d\phi+\eta}{V} - \eta_\sigma \qquad d\eta_\sigma = *d\frac{1}{2|t-\nu_\sigma|}$

  and one more: $a^{(0)} = \frac{d\phi+\eta}{V}$

  These organize into a family parameterized by a bow:

  $a_s := s a^{(0)} + \sum_{p_\sigma < s} a^{(\sigma)}$ instanton connection on a line bundle $e_s = L^s \otimes \bigotimes_{p_\sigma < s} R^{(\sigma)}$

- Given an instanton A on a Hermitian bundle $\mathcal{E}$ over TN_k,

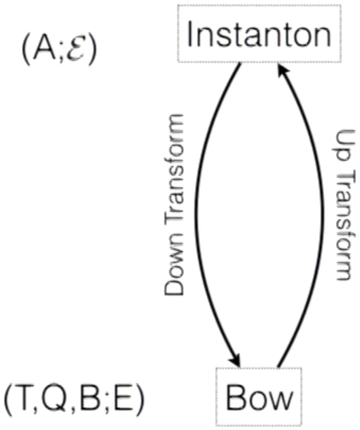
  consider a family $A \otimes 1_{e_s} + 1_{\mathcal{E}} \otimes a_s$ on $\mathcal{E} \otimes e_s$.

  It has a family of associated Dirac operators D_s

- Eigenvalues of holonomy of A at infinity = $\exp(2\pi i\, \lambda_j)$.

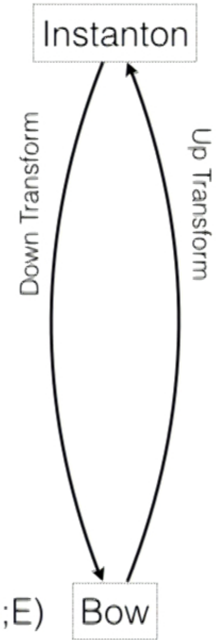
  $\mathrm{Ind}\, D^\dagger_s = R(s)$

  Bow fiber $E_s = \mathrm{Ker}\, D^\dagger_s = \{\Psi \mid D^\dagger_s \Psi = 0\}$



Instanton => Bow Representation.

$(A;\mathcal{E})$

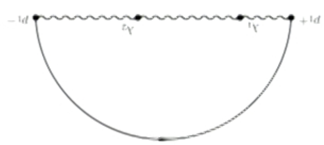


$(T,Q,B;E)$

Family of Dirac operators:

$$D_s = D_{A\otimes 1_{e_s} + 1\otimes a_s}$$

Bow Representation=Index Bundle:



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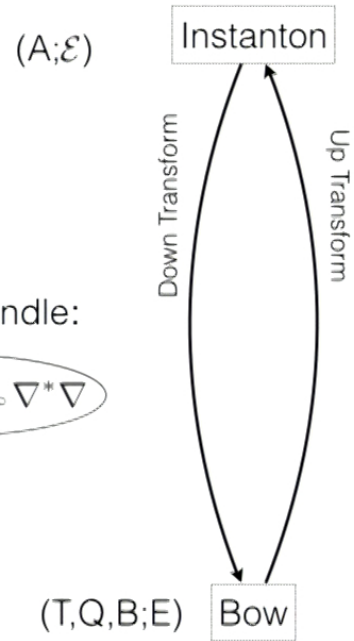
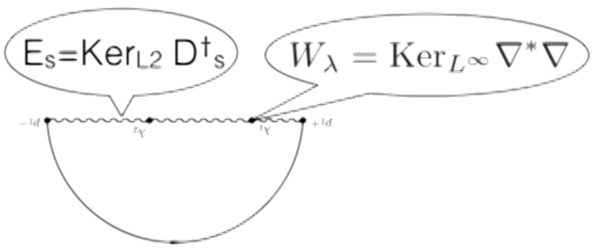


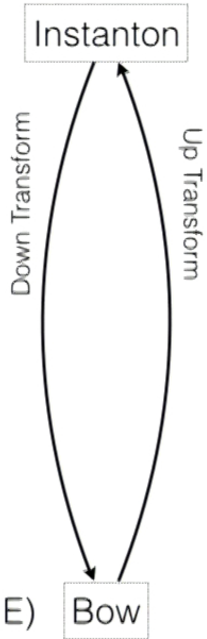
Bow Representation=Index Bundle:



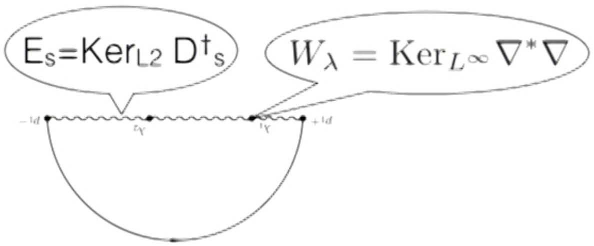
$(\mathrm{A};\mathcal{E})$

Family of Dirac operators:

$$D_s = D_{A\otimes 1_{e_s}+1\otimes a_s}$$



Bow Representation=Index Bundle:



Bow Solution: $(\mathrm{T,Q,B;E})$

$\{f_\sigma\}$ orthonormal basis of $\mathrm{Ker}_{L^\infty} \nabla^*\nabla$

$\{\Psi_a\}$ orthonormal basis of $\mathrm{Ker}_{L^2} D_s^\dagger$

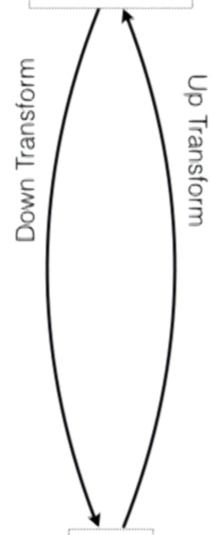
$$T^0_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger i \frac{d}{ds} \Psi_b d\mathrm{Vol},\ T^j_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger t^j \Psi_b d\mathrm{Vol}$$

$$Q_{a\sigma} = \int_{\mathrm{TN}} \Psi_a^\dagger D_\lambda f_\sigma d\mathrm{Vol},$$

$$B^p_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger b^p \Psi_b d\mathrm{Vol},$$

(A;$\mathcal{E}$)

Instanton



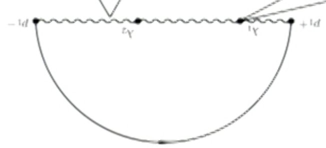
Family of Dirac operators:

$$D_s = D_{A\otimes 1_{e_s} + 1\otimes a_s}$$

Bow Representation=Index Bundle:

$E_s = \mathrm{Ker}_{L^2} D^\dagger_s$

$W_\lambda = \mathrm{Ker}_{L^\infty} \nabla^* \nabla$



Bow Solution: (T,Q,B;E) Bow

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$\{\Psi_a\}$ orthonormal basis of $\mathrm{Ker}_{L^2} D_s^\dagger$

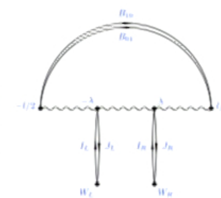
$$T^0_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger i \frac{d}{ds} \Psi_b d\mathrm{Vol},\ T^j_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger t^j \Psi_b d\mathrm{Vol}$$

$$Q_{a\sigma} = \int_{\mathrm{TN}} \Psi_a^\dagger D_\lambda f_\sigma d\mathrm{Vol},$$

$$B^p_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger b^p \Psi_b d\mathrm{Vol},$$

Bow Dirac operator:

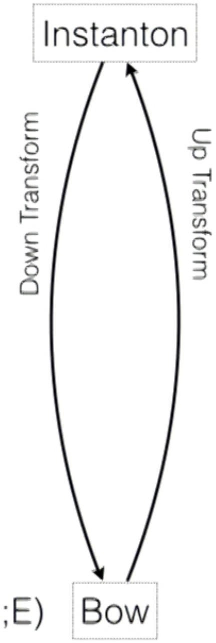
$$\mathcal{D} = \begin{pmatrix} i\frac{d}{ds} + T_0 + e_j T_j \\ B_p^\dagger \\ (B_p^c)^\dagger \\ Q^\dagger \end{pmatrix}$$



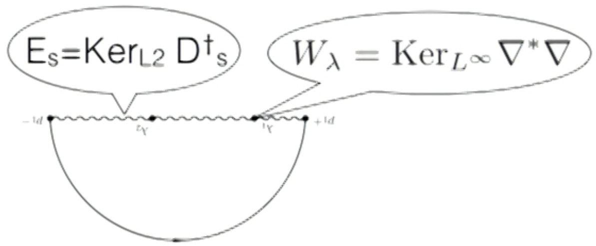
of a fixed solution (T,Q,B)
of a large rep. R.

Family of Dirac operators: (A;$\mathcal{E}$)

$$D_s = D_{A\otimes 1_{e_s} + 1\otimes a_s}$$



Bow Representation=Index Bundle:



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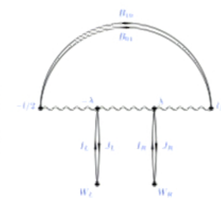
$$Q_{a\sigma} = \int_{\mathrm{TN}} \Psi^\dagger_a D_\lambda f_\sigma d\mathrm{Vol},$$

$$B^p_{ab} = \int_{\mathrm{TN}} \Psi^\dagger_a b^p \Psi_b d\mathrm{Vol},$$

Bow Dirac operator:

$$\mathcal{D} = \begin{pmatrix} i\frac{d}{ds} + T_0 + e_j T_j \\ B^\dagger_p \\ (B^c_p)^\dagger \\ Q^\dagger \end{pmatrix}$$



of a fixed solution (T,Q,B) of a large rep. R.

Point on a TN = (t,b), solution of a small rep. s.

Has corresponding bow Dirac operator $\eth$

Instanton

Down Transform

Up Transform

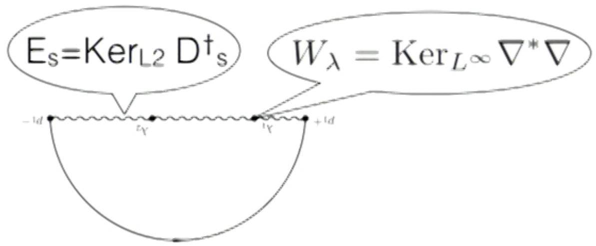
Bow

(A;$\mathcal{E}$)

Family of Dirac operators:

$$D_s = D_{A\otimes 1_{e_s} + 1\otimes a_s}$$

Bow Representation=Index Bundle:



Bow Solution: (T,Q,B;E)

$\{f_\sigma\}$ orthonormal basis of $\mathrm{Ker}_{L^\infty} \nabla^* \nabla$

$\{\Psi_a\}$ orthonormal basis of $\mathrm{Ker}_{L^2} D_s^\dagger$

$$T^0_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger i \frac{d}{ds} \Psi_b d\mathrm{Vol},\ T^j_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger t^j \Psi_b d\mathrm{Vol}$$

$$Q_{a\sigma} = \int_{\mathrm{TN}} \Psi_a^\dagger D_\lambda f_\sigma d\mathrm{Vol},$$

$$B^p_{ab} = \int_{\mathrm{TN}} \Psi_a^\dagger b^p \Psi_b d\mathrm{Vol},$$

Family of bow Dirac operators:

$\mathcal{D}_{(t,b)} = \mathcal{D} \otimes 1_e + 1_E \otimes \mathfrak{d}^c$ on $S \otimes E \otimes e^*$

Bow Dirac operator:

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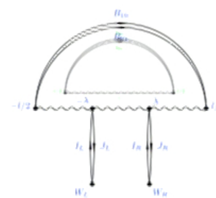


of a fixed solution (T,Q,B) of a large rep. R.

Point on a TN = (t,b), solution of a small rep. s.

Has corresponding bow Dirac operator $\mathfrak{d}$

- How are Up and Down transforms related?

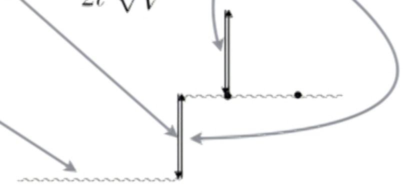
  Up$\circ$Down=?
  and
  Down$\circ$Up=?

- Why should bow and instanton moduli spaces be isometric?

- To relate Down and Up transforms express solution (χ,ν,f) of Bow Dirac equation through solutions ψ of TN Dirac equation:

Poisson type equations on TN_k:

$$\nabla^*\nabla\chi = \frac{c^0}{\sqrt{V}}\Psi \qquad \nabla^*\nabla\nu_+ = i\frac{b_+^\dagger}{2t}\frac{c^0}{\sqrt{V}}\Psi_t \qquad \nabla^*\nabla\nu_- = i\frac{b_+^\dagger}{2t}\frac{c^0}{\sqrt{V}}\Psi_h \qquad \nabla^*\nabla f_\lambda = 0$$



These form an orthonormal basis of solutions of the Bow Dirac equation!

$$(i\partial_s + I^j(t^j - T^j))\chi + (\delta(s-t)b_+ - \delta(s-h)B_+)\nu_+ + (\delta(s-t)B_- - \delta(s-h)b_-)\nu_- - \delta(s-\lambda)Q = 0$$

in short $\qquad \mathcal{D}^\dagger_{(t,b)}\begin{pmatrix}\chi\\ \nu_+\\ \nu_-\\ f\end{pmatrix} = 0$

Moreover, just as $(\chi,\nu_+,\nu_-,f_\lambda)$ satisfy the Poisson equation on TN_k, Ψ satisfy the Poisson equation on the bow:

$$\left((i\frac{d}{ds} - T^0)^2 + (t^j - T^j)^2\right)\Psi = \frac{c^0}{\sqrt{V}}\chi$$

Moral: The seeds of the Up transform are in the Down transform.

- These Dirac and Poisson relations

$$D_s \Psi = 0 \qquad \mathcal{D}^{\dagger}_{(t,b)} \begin{pmatrix} \chi \\ \nu_+ \\ \nu_- \\ f \end{pmatrix} = 0$$

$$\nabla^* \nabla \chi^c = \frac{c^0}{\sqrt{V}} \Psi \qquad \left((i \frac{d}{ds} - T^0)^2 + (t^j - T^j)^2 \right) \Psi = \frac{c^0}{\sqrt{V}} \chi$$

- together with the appropriate index theorem

$$\mathrm{ind}_{L^2} D^+ = \frac{k}{2} \mathrm{tr}(-\{A_\infty\}^2 + \{A_\infty\}) + \frac{1}{2} \mathrm{tr}(\int_{S^2_\infty} \frac{\{A_\infty\}}{\pi} i F_{23} - \int_{S^2_\infty} \frac{i F_{23}}{2\pi}) - \frac{1}{8\pi^2} \int \mathrm{tr} F \wedge F$$

- and our decay rate theorems

prove that

Up∘Down=1 and Down∘Up=1

Monowall

with Richard Ward

A monowall is a BPS monopole on $S^1_x \times S^1_y \times \mathbb{R}_z$,

i.e. an n-dimensional hermitian vector bundle $E \to S^1 \times S^1 \times \mathbb{R}$

with a connection one-form A (with curvature F=dA+A∧A)
and an endomorphism Φ satisfying the Bogomolny equation

$$*D_A\Phi = -F$$

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A Simple Example:

Abelian case (gauge group U(1)) $\Phi = \mathtt{i}\phi \quad A = \mathtt{i}a$

Bogomolny Equation is linear $*d\phi = -da$

thus the function ϕ is harmonic

- A typical abelian solution (Constant Energy Density solution)

$$\phi = 2\pi(Qz + M), \quad a = 2\pi(Qy\,dx - p\,dx - q\,dy)$$

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Charge (Q); Monodromy along x and y (p, q)

Our Asymptotic Conditions:

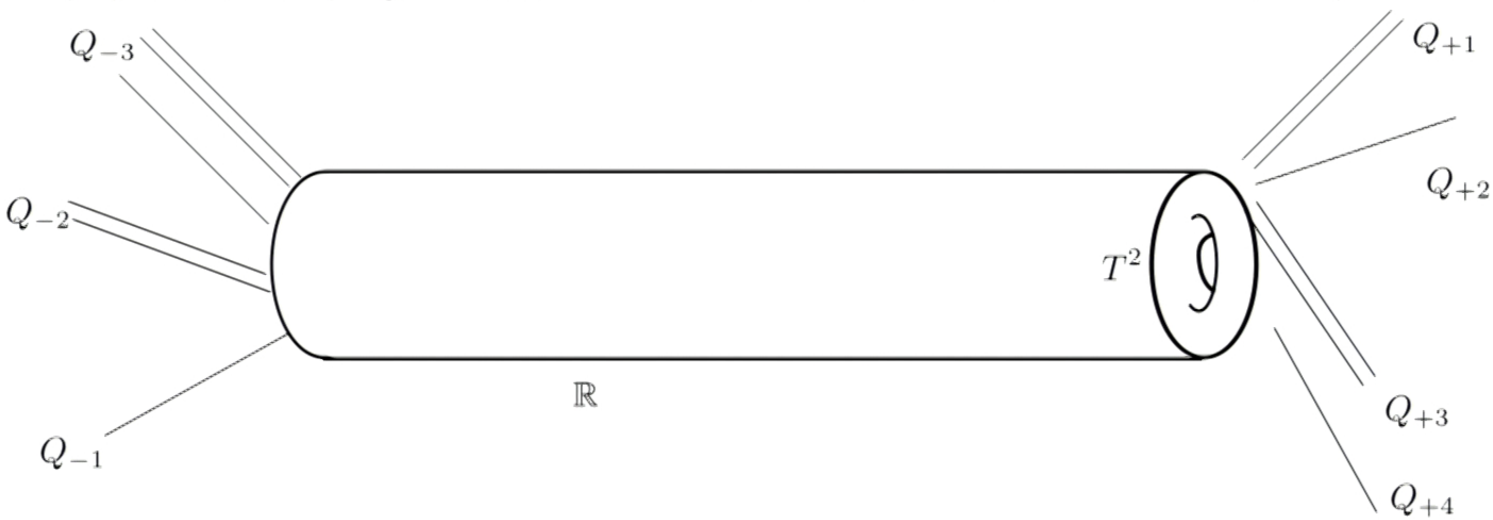
For a U(n) monopole wall, as $z \to \pm\infty$,

$$\mathrm{EigVal}\,\Phi = \{2\pi \mathrm{i}\,(Q_{\pm,l} z + M_{\pm,l}) + o(1/z)\,|\,l = 1,\ldots,n\}\,.$$

Simplest case is Maximal Symmetry Breaking: the pairs $(Q_{+,l}, M_{+,l})$ are all distinct, and so are $(Q_{-,l}, M_{-,l})$. It splits the bundle at infinity into subbundles.

$$E|_z = \oplus_{j=1}^{f_\pm} E_{\pm j};$$

Asymptotic holonomy eigenvalues are $e^{2\pi i p_{\pm,l}}$ around the x-direction and $e^{2\pi i q_{\pm,l}}$ around y.



The charges $Q_{\pm,l}$ are rational, with the denominator equal to the multiplicity of $(Q_{\pm,l}, M_{\pm,l})$

$$E|_z = \oplus_{j=1}^{f_\pm} E_{\pm j};$$

$$\int_{T_z} c_1(E_{\pm j}) = \frac{\mathrm{i}}{2\pi}\int_{T_z} \mathrm{tr}\, F_{\pm j} = -\frac{\mathrm{i}}{2\pi}\int_{T_z} \mathrm{tr} * D\Phi_{\pm j} = \mathrm{rk}(E_{\pm j})\, Q_{\pm j}.$$

$$Q_{\pm j} = \frac{\alpha_{\pm j}}{\beta_{\pm j}},$$

$$\mathrm{rk}(E_{\pm j}) = r_{\pm j}\beta_{\pm j}.$$

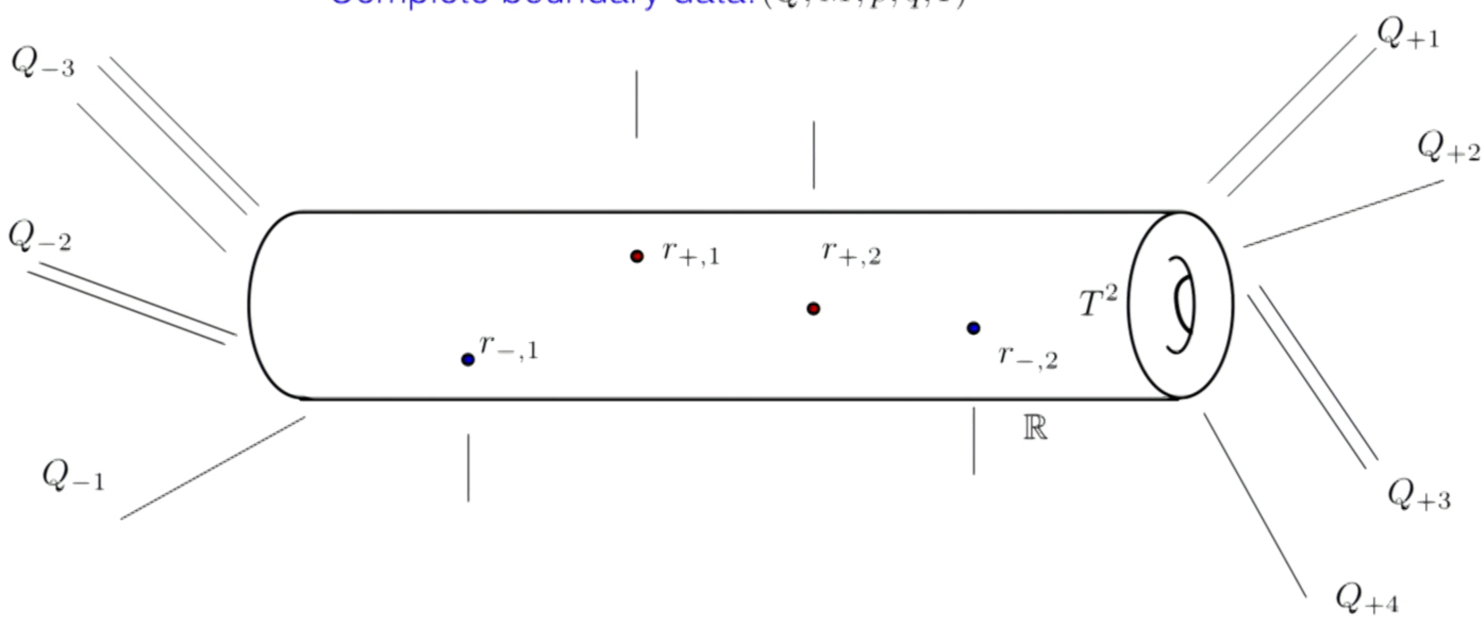
Singularities:

We also allow prescribed positive and negative Dirac singularities at some points $\vec{r}_\alpha \in T^2 \times \mathbb{R}$ with the Higgs field behavior

$$\Phi = \mathrm{i}\begin{pmatrix} \frac{+1}{2|\mathbf{r}-\mathbf{r}_{+,\nu}|} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix} + O(|\mathbf{r}-\mathbf{r}_{+,\nu}|),$$

$$\Phi = \mathrm{i}\begin{pmatrix} \frac{-1}{2|\mathbf{r}-\mathbf{r}_{-,\nu}|} & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix} + O(|\mathbf{r}-\mathbf{r}_{-,\nu}|).$$

Complete boundary data: $(Q, M, p, q, \vec{\mathbf{r}})$



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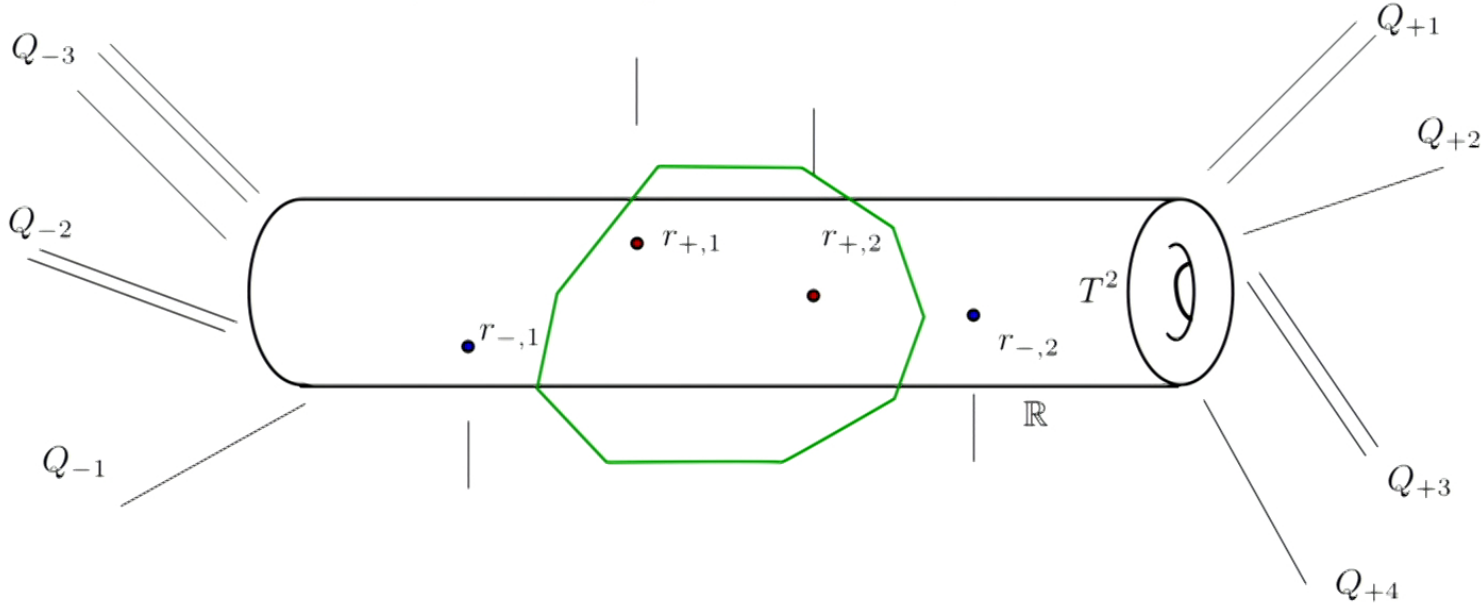


Spectral Description

Bogomolny equation $*D_A\Phi = -F$ can be written in the form

$$\begin{cases} [D_z - iD_y, D_x + i\Phi] = 0 & (1) \\ [D_z - iD_y, (D_z - iD_y)^\dagger] + [D_x + i\Phi, (D_x + i\Phi)^\dagger] = 0 & (2) \end{cases}$$

Eq. (1) implies that the monodromy $V_x(y,z)$ of $D_x + i\Phi$ is meromorphic in $s = e^{2\pi(z-iy)} \in \mathbb{C}^*$

Group-valued $V_x(y,z)$ is the analogue of the Higgs field of the Hitchin system.
Monowall = Hitchin system with the Higgs field valued in automorphisms instead of endomorphisms of the bundle.

$$F_x(s,t) = \det\left(V_x(y,z) - t\right)$$

is a degree n polynomial in t with coefficients being rational functions in s.

Spectral curve:

$$\Sigma_x = \{(s,t)|F_x(s,t) = 0\} \in \mathbb{C}^* \times \mathbb{C}^*$$

formed by the eigenvalues of the holonomy and equipped with a holomorphic eigenline bundle

$$M_x \to \Sigma_x$$

(Σ_x, M_x) form complete x-spectral data equivalent to the (A,Φ) solution.

Note: Analogously, we define the y-spectral data (Σ_y, M_y).

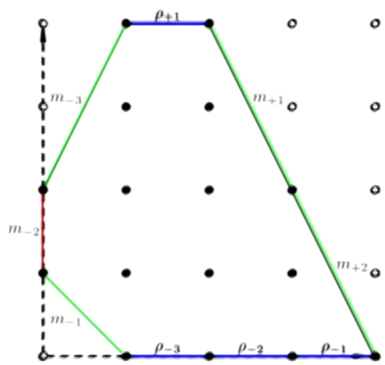
There is a 1-to-1 map $(\Sigma_x, M_x) \to (\Sigma_y, M_y)$

Let $G_x(s,t)=P(s)F_x(s,t)$ be a minimal monic polynomial in s and t, so that the spectral curve is given by a polynomial equation $G_x(s,t)=0$.

Newton Polygon

Newton polygon N_x of $G_x(s,t)$ is a minimal convex polygon containing all points (a,b) such that the monomial $s^a t^b$ is present in $G_x(s,t)$.

- Horizontal edges of Newton polygon correspond to the singularities:
- Northern edges to positive; and Southern edges, to negative.
- Vertical edges - constant eigenvalues of Φ as $z=\pm\infty$: $Q_{\pm,l}=0$.
- Slanted edges correspond to the cusps of the curve.
- Rank of the monowall gauge group is the hight of the Newton polygon.



Consider one edge of N_x

a single edge of N_x directed along (α,β) corresponds to $s^a t^b P(s^\alpha t^\beta)$ asymptotically satisfying $\alpha \log s+\beta \log t= 2\pi\beta (M+i\, p)$

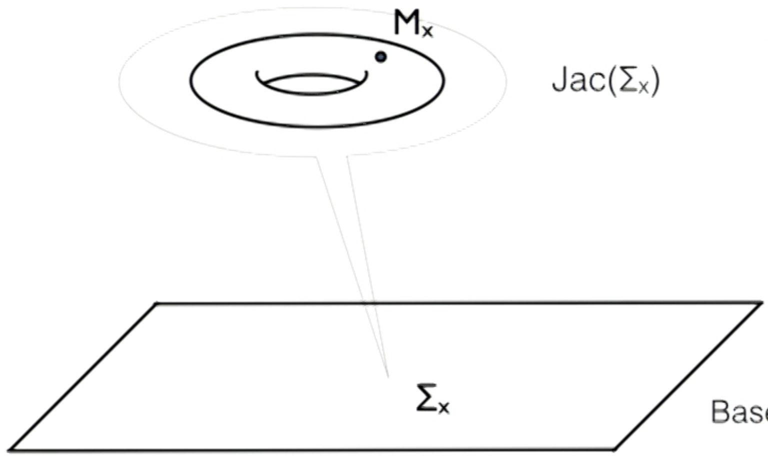
$$m_l^- = \exp\left[-2\pi\beta_{\pm,l}(M_{\pm,l}+ip_{\pm,l})\right]$$

Dressed Newton polygon N → Monowall Boundary data

Curve moduli = internal points

Monopowall Moduli = Moduli of Σ_x + Moduli of M_x

4g = 2g + 2g



Fiber = Jacobian of the Spectral Curve

Base = Moduli Space of Curves

Newton polygon determines the family of curves in $\mathbb{C}^* \times \mathbb{C}^*$, and its Jacobian fibration.

Genus g = Number of internal points on the Newton polygon. Khovanskii

This total space has hyperkähler metric.

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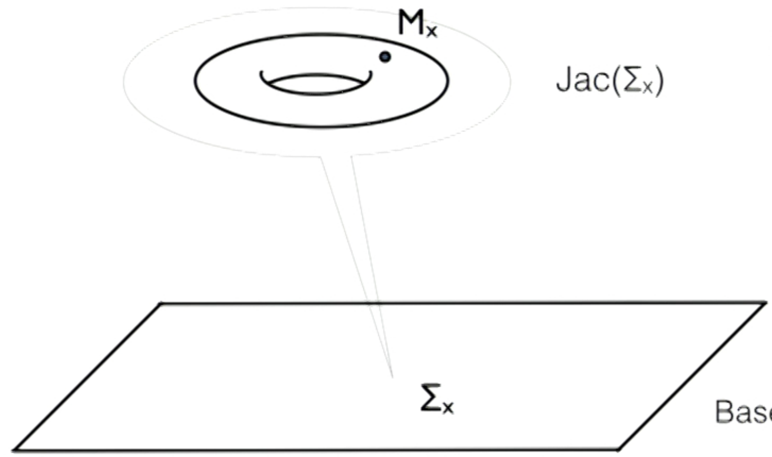

Jac(Σ_x)

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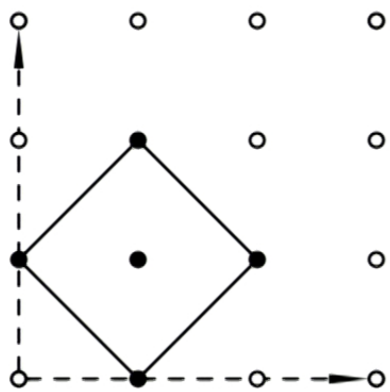

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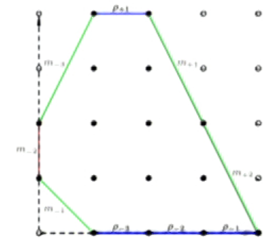
Parameter Count

Moduli Count:

Number of Moduli = 4 x (Number of internal points of N_x).

$$m_l^- = \exp\left[-2\pi\beta_{\pm,l}(M_{\pm,l} + ip_{\pm,l})\right]$$



Parameter Count:

Number of Parameters = 3 x (Number of perimeter points of N_x - 3).

G(s,t) ~ λG(μs,νt)

Asymptotic data is constrained:

Khovanskii '77

Charges Q=α/β satisfy:

$$\sum_{j=1}^{f_-} r_{-j}\beta_{-j} = \sum_{j=1}^{f_+} r_{+j}\beta_{+j} = n, \qquad r_{-0} + \sum_{j=1}^{f_-} r_{-j}\alpha_{-j} = r_{+0} + \sum_{j=1}^{f_+} r_{+j}\alpha_{+j},$$

$$\sum_{\nu=1}^{r_+} z_{+,\nu} - \sum_{\nu=1}^{r_-} z_{-,\nu} = \sum_{l=1}^{n} M_{+,l} - \sum_{l=1}^{n} M_{-,l},$$

Singularities and constant terms satisfy:
(Vietta theorem)

$$\sum_{\pm}\sum_{\nu=1}^{r_\pm} \pm y_{\pm,\nu} + \sum_{\pm}\sum_{l=1}^{n} \pm p_{\pm,l} + \frac{1}{2}\sum_{\substack{l_1,l_2=1\\ l_1<l_2}}^{2n} (Q_{,l_1} - Q_{,l_2}) \in \mathbb{Z},$$

$$\sum_{\pm}\sum_{\nu=1}^{r_\pm} \pm x_{\pm,\nu} + \sum_{\pm}\sum_{l=1}^{n} \pm q_{\pm,l} + \frac{1}{2}\sum_{\substack{l_1,l_2=1\\ l_1<l_2}}^{2n} (Q_{,l_1} - Q_{,l_2}) \in \mathbb{Z}.$$

GL(2,ℤ) Isometric Action

There is a natural GL(2,ℤ) action on C*xC*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (s,t) \mapsto (s^d t^c, s^b t^a),$$

Note: This is not the SL(2,ℤ) of the monopole torus in RxT².

it induces a map on the spectral data (Σ,M), under which

a monomial $s^{-\alpha} t^{\beta} \mapsto (s')^{-\alpha'} (t')^{\beta'}$ with $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$

and the resulting Newton polygon is $N' = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} N$

and the boundary data transforms as

$$g : (Q, M, p, q) \mapsto \left(\frac{aQ+b}{cQ+d}, \frac{M}{cQ+d}, \frac{p}{cQ+d}, \frac{q}{cQ+d} \right).$$

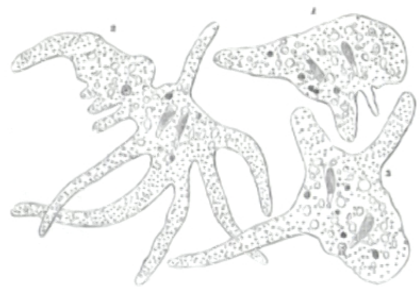
Meaning of GL(2,Z) generators:

- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ • Reflection (x,y,z) → (x,-y,-z)
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ • T element of SL(2,ℤ) is (A,Φ)→(A+2πydx, Φ+2πz).
- $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ • S element of SL(2,ℤ) is the Nahm transform

Question: What are the remnants of this symmetry when monowall degenerates to a Hitchin system?

Amoeba

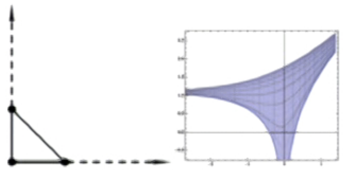
Gelfand-Kapranov-Zelevinski



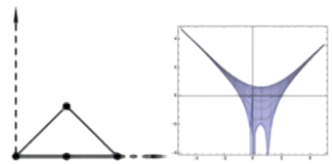
An amoeba A_x of $G_x(s,t)$ is a the image of Σ_x under the map

$$\mathbb{C}^* \times \mathbb{C}^* \to \mathbb{R}^2$$
$$(s,t) \mapsto (\log|s|, \log|t|).$$

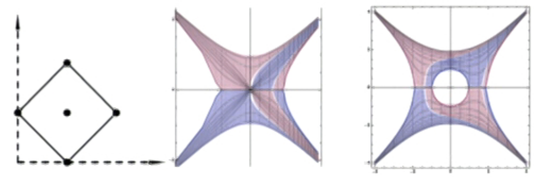
it gives the spread of eigenvalues of the Higgs field for a given z.



(0,1) U(1) monopole wall with one negative singularity.



(1,1) U(1) monopole wall with two negative singularities.



Reducible U(2) monopole with charges (Q_+; Q_-)=(1,-1; 1,-1)

Gelfand-Kapranov-Zelevinsky:

Each connected component of the compliment to the amoeba is convex.

It corresponds to a point of the Newton polygon.

The tentacles of the amoeba approach asymptotic lines exponentially fast.

Maslov Dequantization (Tropical Semiring)

$s = e^x, t = e^y \in \mathbb{R}_+$ $\qquad$ $x,\ y \in \mathbb{R}$

Conventional Operations:

$$s + t = e^x + e^y = e^{\ln(e^x + e^y)} \qquad x \oplus_1 y = \ln(e^x + e^y)$$

$$s \cdot t = e^{x+y} \qquad x \odot_1 y = x + y$$

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Rescaling does not change anything (semiring isomorphism):

$$s = e^{x/h}, t = e^{y/h} \in \mathbb{R}_+$$

$$s + t = e^{x/h} + e^{y/h} = e^{\ln(e^{x/h} + e^{y/h})} \qquad x \oplus_h y = h \ln(e^{x/h} + e^{y/h})$$

$$s \cdot t = e^{(x+y)/h} \qquad x \odot_h y = x + y$$

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$$s \cdot t = e^{(x+y)/h} \qquad x \odot_h y = x + y$$

Classical Limit h → 0

Tropical Operations:

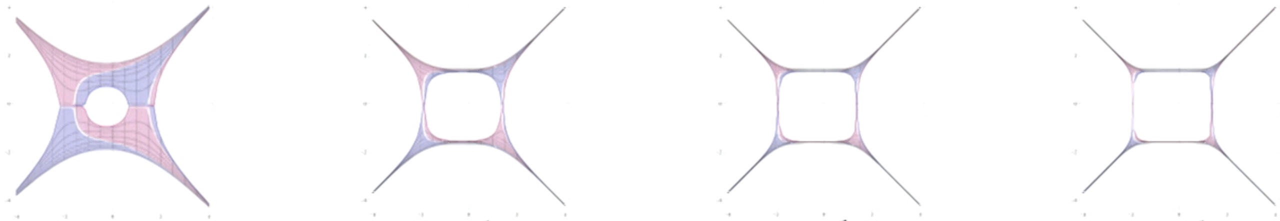
$$x \oplus y = \mathrm{Max}\,(x, y)$$

$$x \odot y = x + y$$

Tropical Geometry

Spectral curve F(s,t) in $\mathbb{C}^* \times \mathbb{C}^* = \mathbb{R} \times S^1_{\hat{x}} \times \mathbb{R} \times S^1_y$ if $\hat{x} \sim \hat{x} + 2\pi h$ and $y \sim y + 2\pi h$

then the natural coordinates are $s = e^{(a+i\hat{x})/h}$ and $t = e^{(b+iy)/h}$



$h = 1$ $h = \frac{1}{2}$ $h = \frac{1}{3}$ $h = \frac{1}{4}$

The rescaled Ronkin function r(a,b) is

$$r(a,b) := hR(a,b) = h \int \log |F(e^{(a+i\hat{x})/h}, e^{(b+iy)/h})| \frac{d\hat{x}}{h} \frac{dy}{h}$$

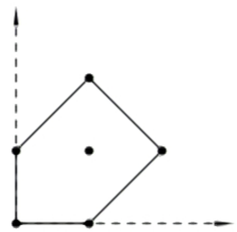
$$= h \int \log |\sum_{(m,n)\in N_x} C_{m,n} e^{m(a+i\hat{x})/h}, e^{n(b+iy)/h})| \frac{d\hat{x}}{h} \frac{dy}{h}$$

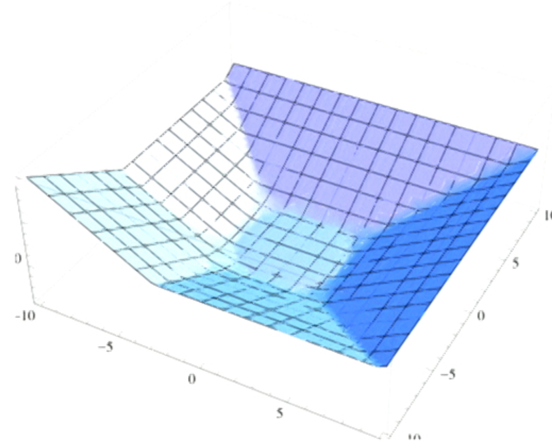
as h→0

$$= \mathrm{Max}_{(m,n)\in N_x} (ma + nb + c_{m,n}) \qquad |C_{m,n}| = e^{\frac{c_{m,n}}{h}}$$

Amoeba shrinks to a tropical variety = locus where r(a,b) is not linear.

Dual to a coherent triangulation of N!





Each point in a polygon determines a plane.

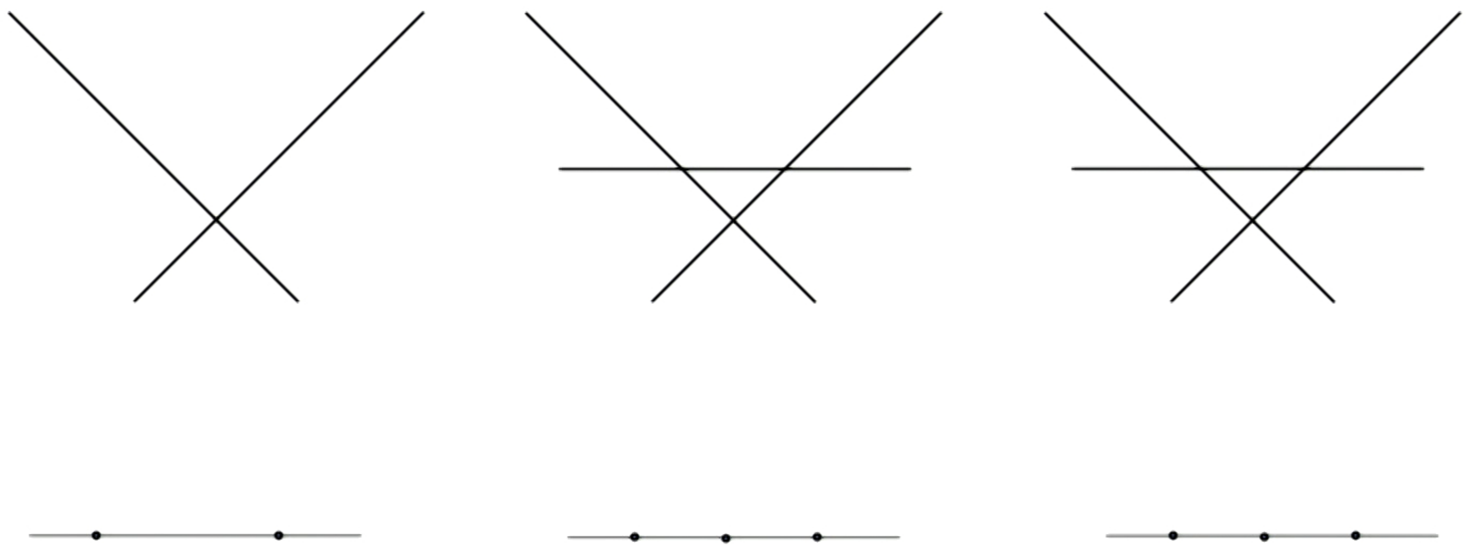
Point coordinates = plane slope

Point coefficient C in G(s,t) = plane height log|C|

Perimeter coefficients are fixed by monowall parameters, internal coefficient is varied.

Crystals

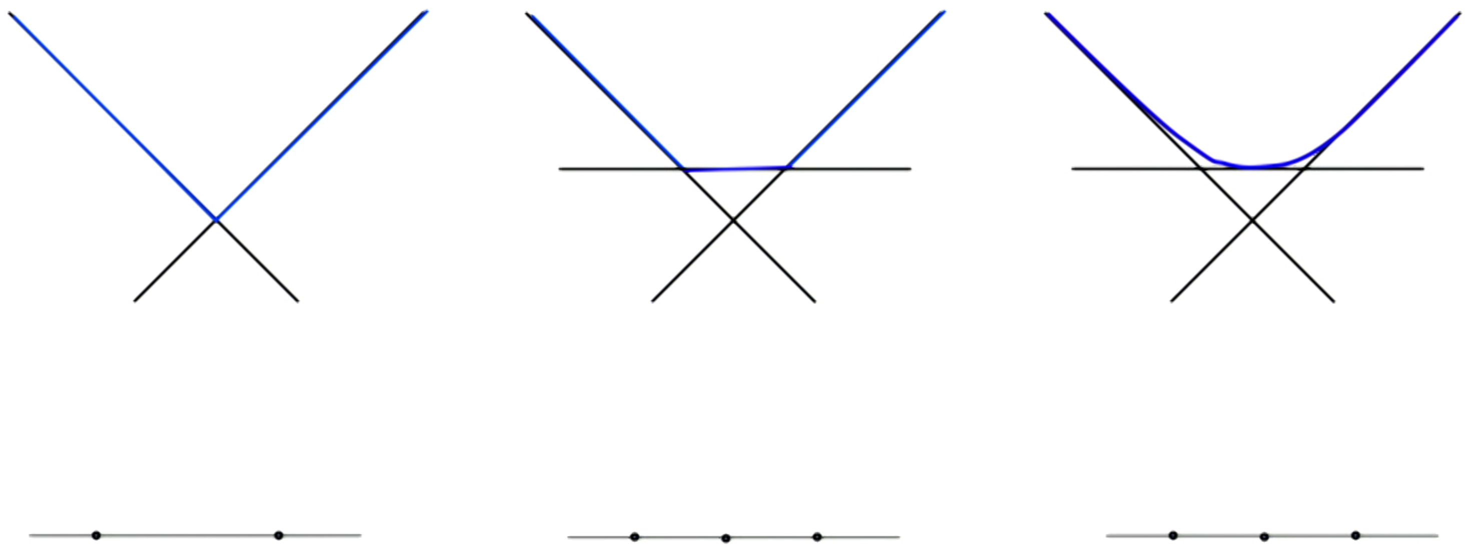
Max of External Planes | Max of all Planes | Ronkin Function



Crystal | Cut Crystal | Melted Crystal

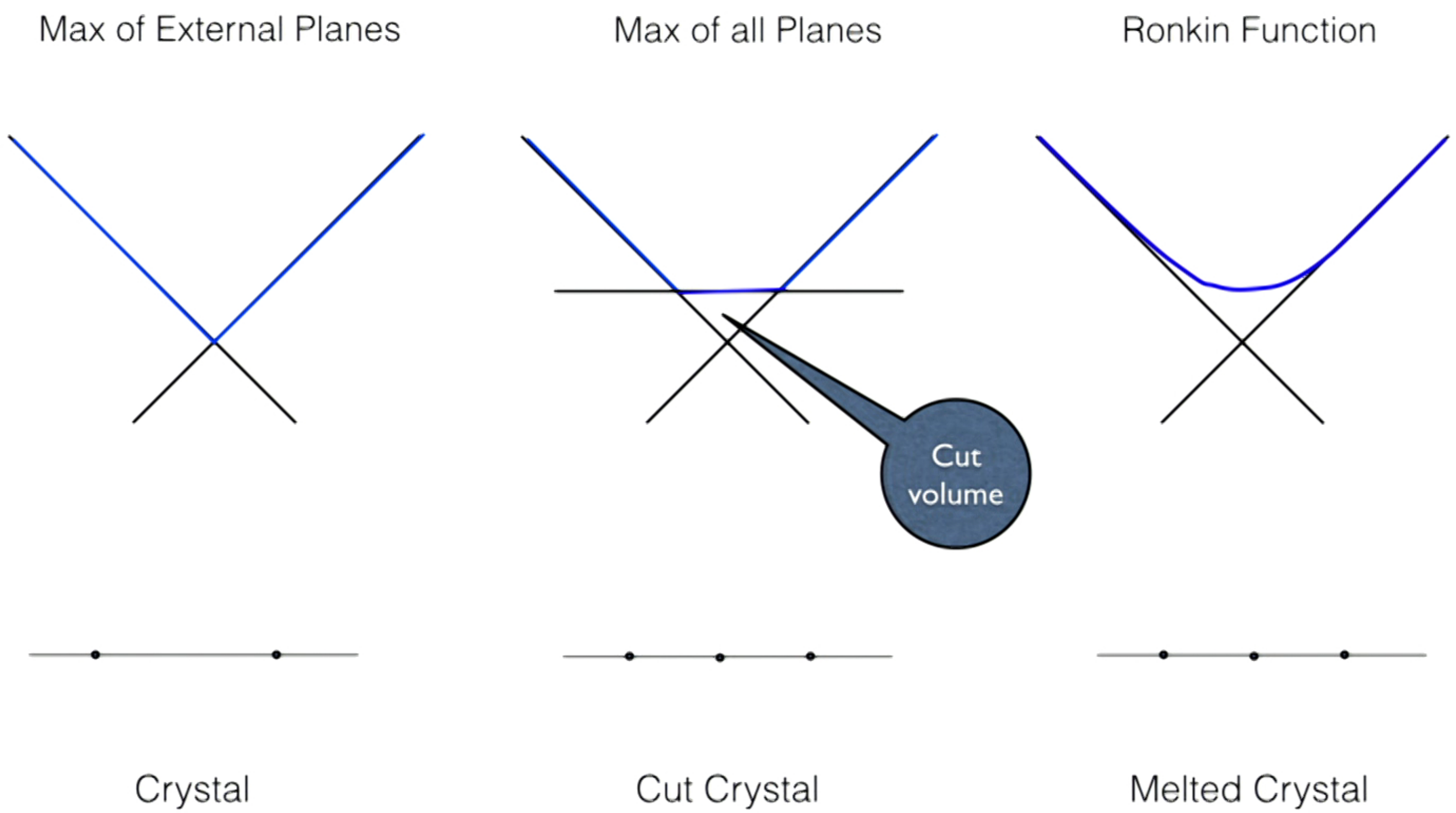
Crystals

Max of External Planes | Max of all Planes | Ronkin Function



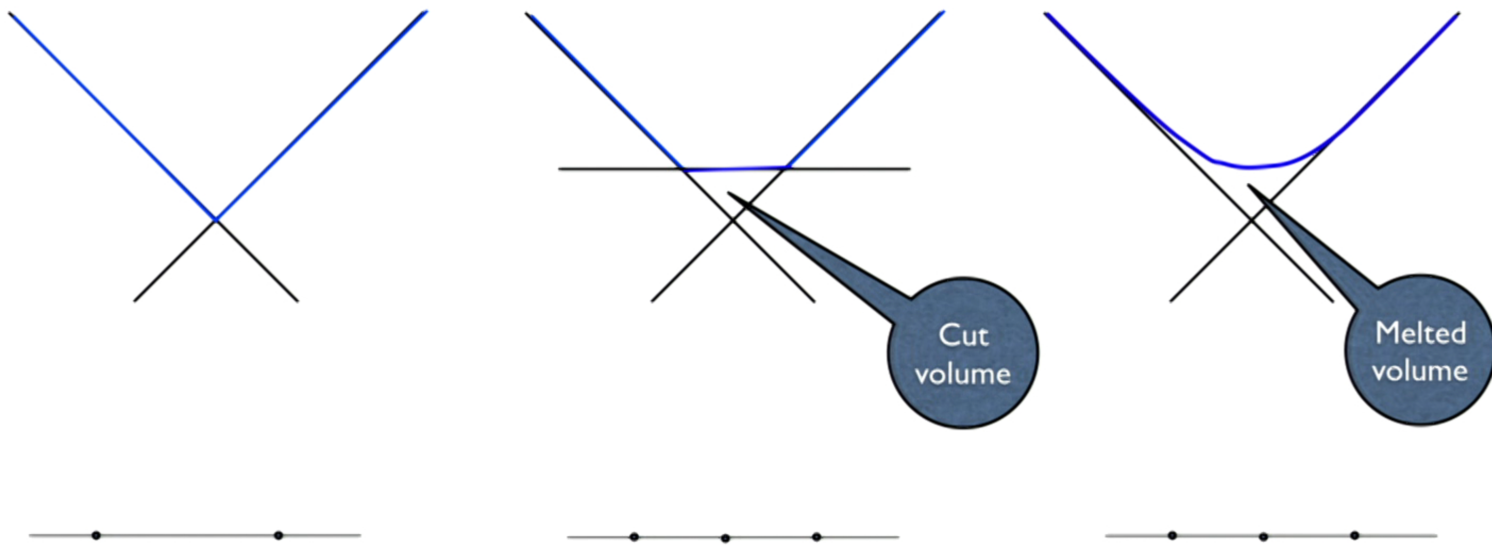
Crystal | Cut Crystal | Melted Crystal

Crystals


Max of External Planes
Max of all Planes
Ronkin Function
Cut volume
Crystal
Cut Crystal
Melted Crystal

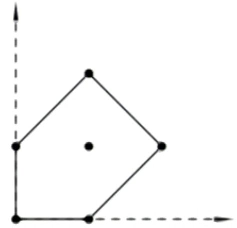
Crystals

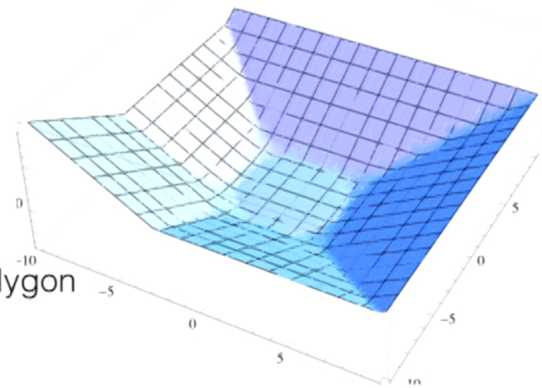
Max of External Planes | Max of all Planes | Ronkin Function



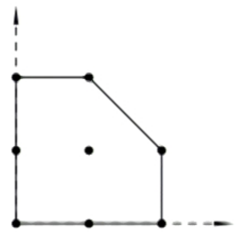
Crystal | Cut Crystal | Melted Crystal

1. For a given Newton polygon, in the limit of large internal coefficient the Ronkin function is well approximated by the tropical planes.





2. The cross section at high modus is the Polar Polygon $N^{\vee}=\{v \mid (v,w)>-1,\ \forall w\in N\}$



3. Volume = 1/3 $A_{Polar}\, x_1^3$+ lower terms

- Pick's formula Area=Int+Perim/2 -1
- $Perim_N + Prim_{Polar(N)} = 12$ for reflexive polygons

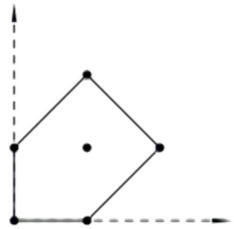
thus 8-N=12-Perim=2 Area(Polar)

(in perfect agreement with Seiberg's one loop computation!)
(with GKZ area normalization)

Conjecture:

The Kähler potential on the moduli space of a monowall = melted volume + o(r)

Moduli Space Family of a Given Newton Polygon

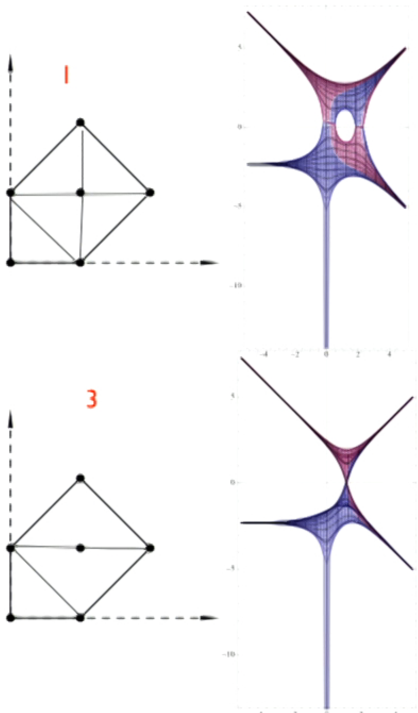


Real moduli: 1+3=4 Int
Parameters : 2 triplets = Perim -3

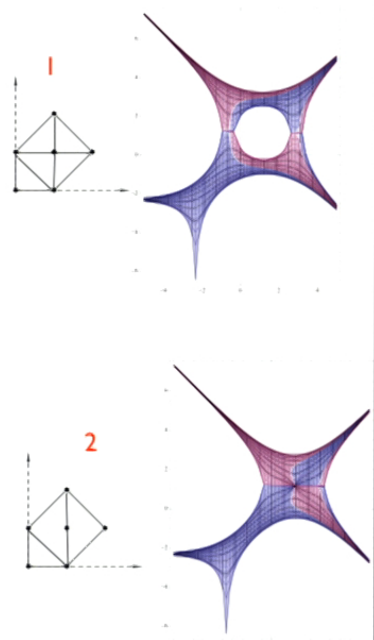
Each tropical degeneration ↔ coherent triangulation of N

Variation of modulus => collection of triangulations

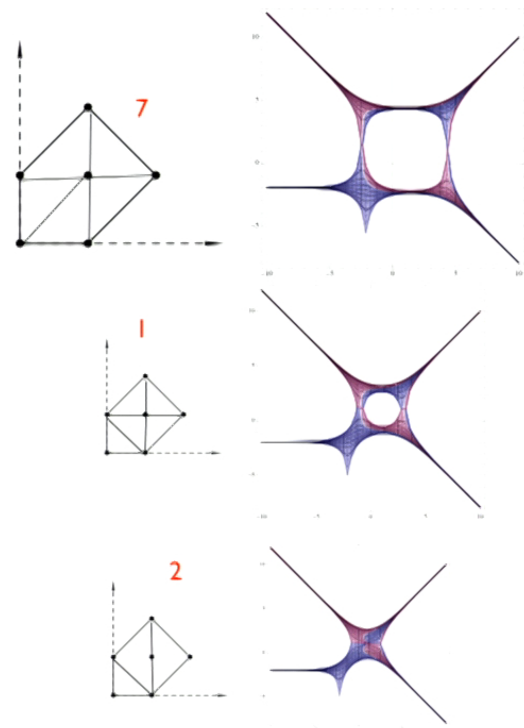
One Phase



Another Phase



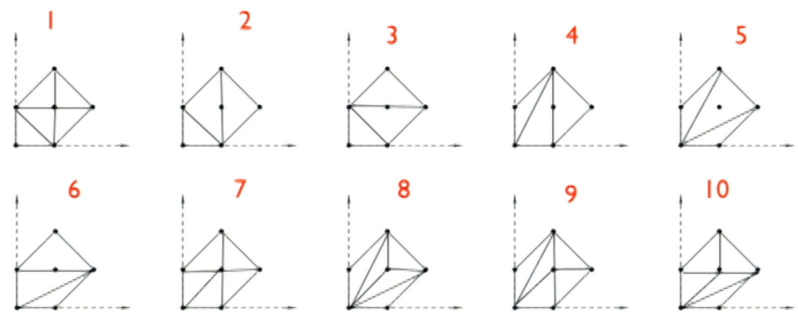
Another Phase



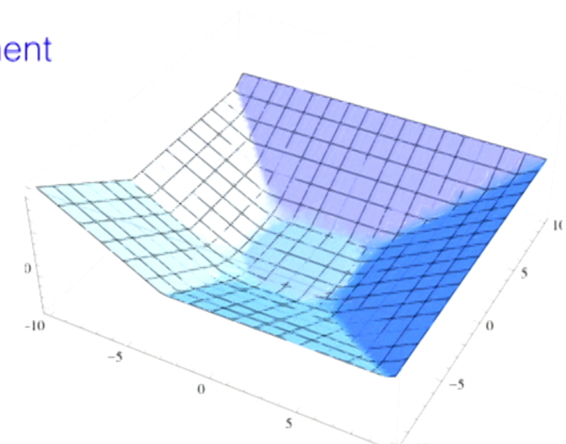
Phase Structure

The Skeleton of each amoeba is dual to some regular triangulation of the Newton polygon.

Regular Triangulations



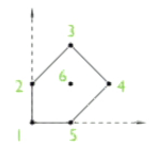
Plane Arrangement



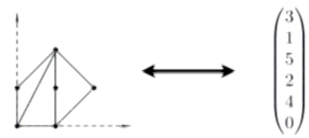
Secondary Polyhedron

Each triangulation of the Newton polygon N gives a vector of dimension #N:

- Number integer points in N



- For a given triangulation
  the i^{th} coordinate v^i is the sum of areas
  of triangles for which (n_i, m_i) is a vertex
  (area of unit simplex is 1)



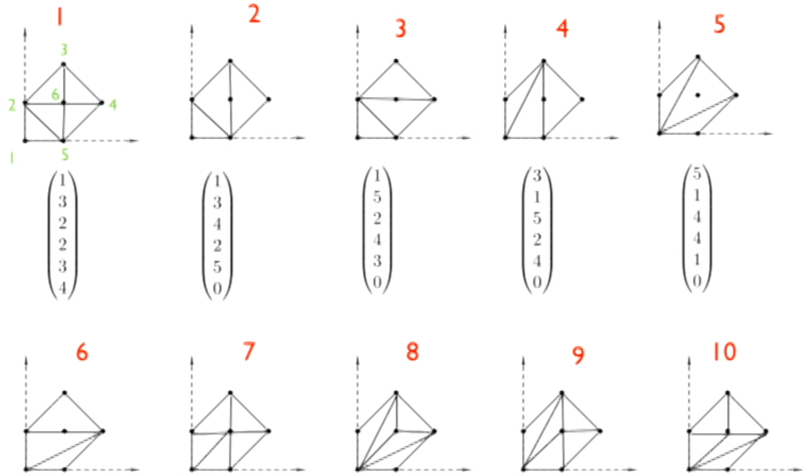
- These vectors lie is an (#N-3) - hyperspace since
  1. Sum of all coordinates in each vector = 3xArea(N)
  2. Sum of $(n_i, m_i)v^i$ is the center of mass of N.

Gelfand-Kapranov-Zelevinsky

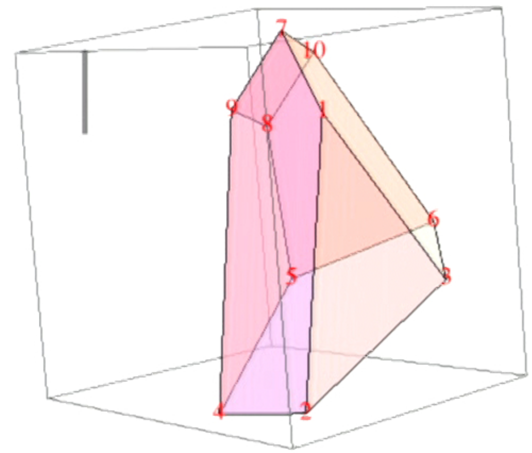
These vectors are vertices of a convex polyhedron, called the Secondary Polyhedron, S(N).

Regular Triangulations

Each regular triangulation gives a vertex of a convex polyhedron:



Secondary Polyhedron and Secondary Fan



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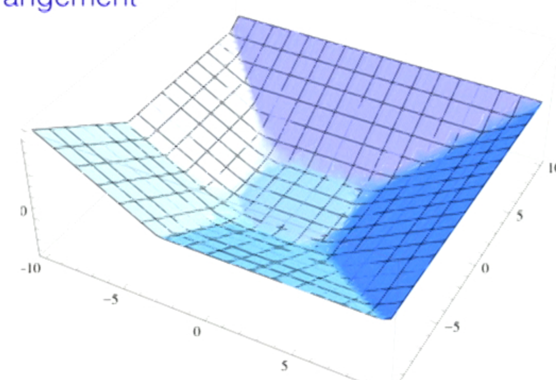

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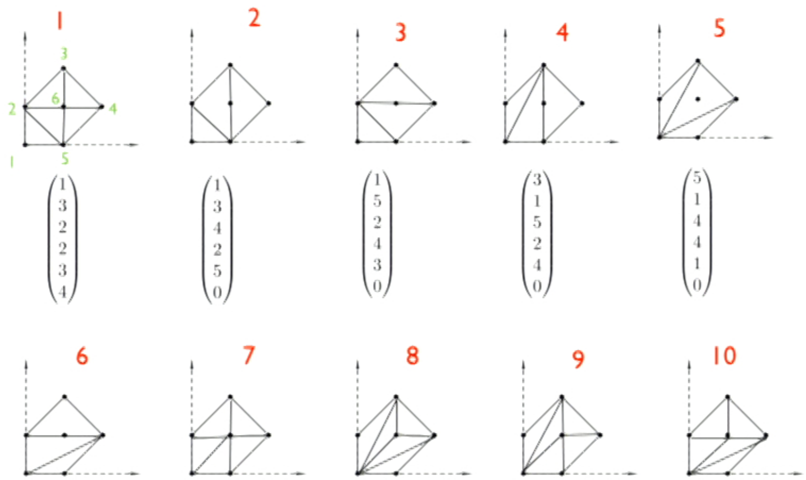

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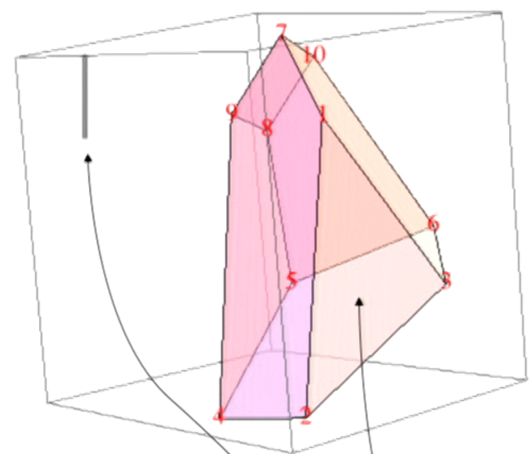

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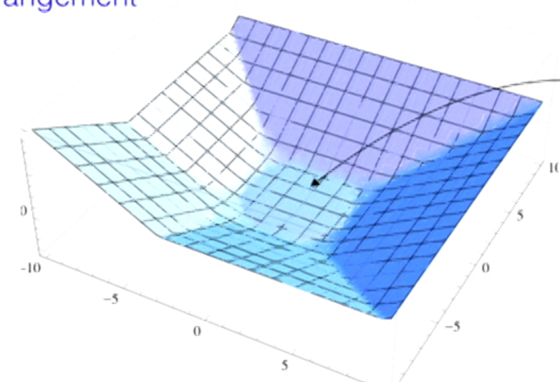
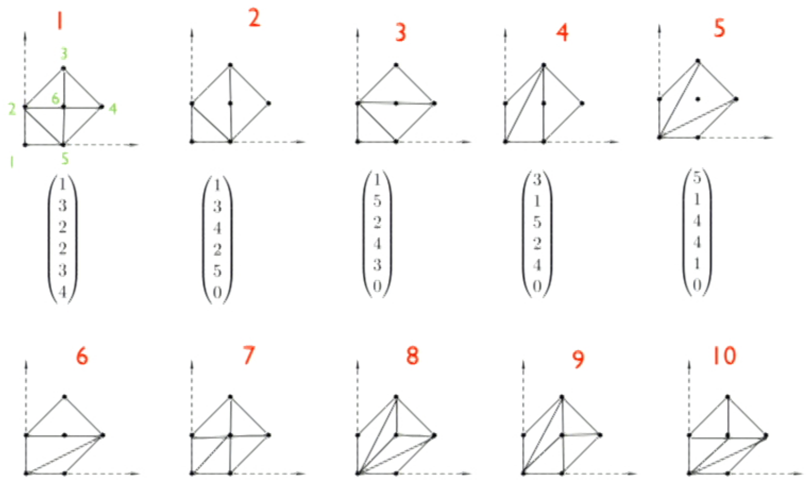


As we vary the modulus,
the internal plane moves and
we move along a given direction
through cones of the secondary fan.

Regular Triangulations

Each regular triangulation gives a vertex of a convex polyhedron:



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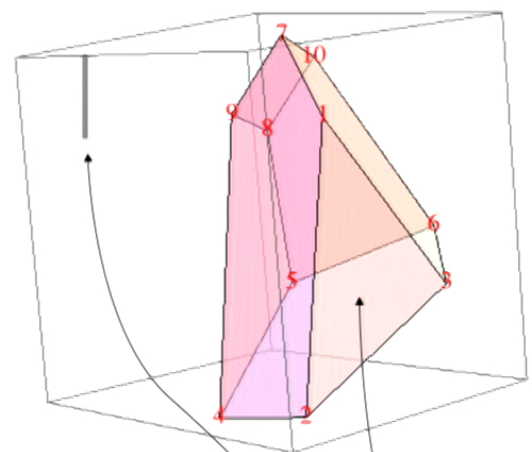
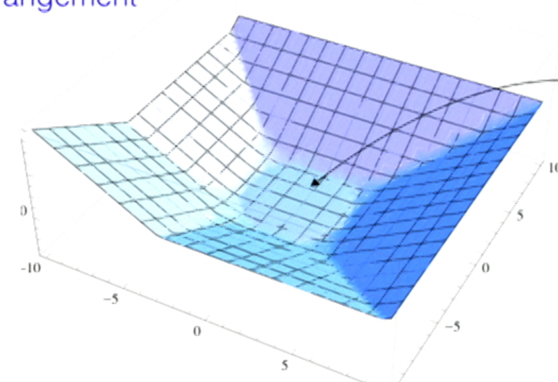


Plane Arrangement

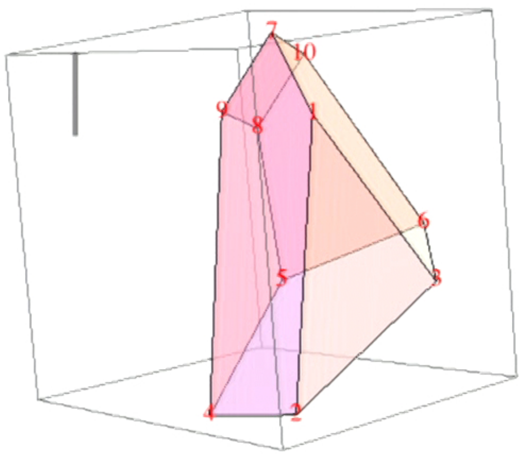


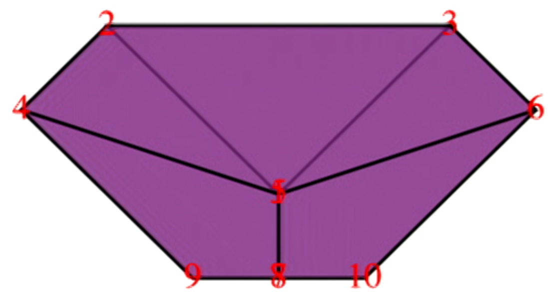
As we vary the modulus, the internal plane moves and we move along a given direction through cones of the secondary fan.

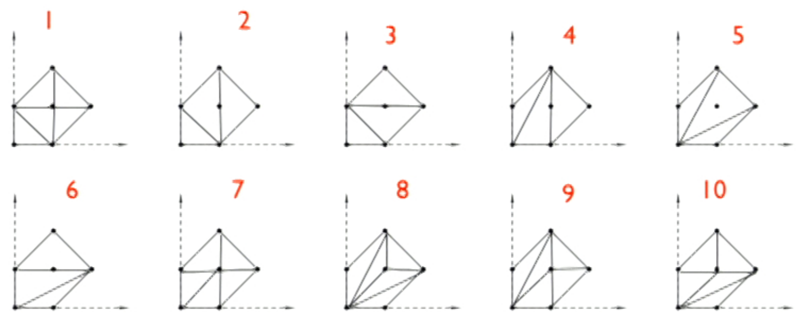
Associahedral Face

Phase Space of a Monowall

The phase structure of monowall moduli spaces corresponding to a given Newton polygon is given by the projection of the secondary fan onto the associahedral plane







Outlook

- To reach Slings we are studying the Dirac Index bundle for Instantons on ALG spaces. It was a 2D integrable system associated to it.
- Viewing the Monowall as a Hitchin System with the "group-valued Higgs field", a Hitchin system can be recovered in a limit.
  - What is the limiting significance, if any, of the secondary polygon and the secondary fan picture?
  - What happens with the monowall pair of spectral curves? (cf. Journey Between Two Curves)