Title: Critical points and spectral curves

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Abstract: Critical values of the integrable system correspond to singular spectral curves. In this talk we shall discuss critical points, points in the moduli space where one of the Hamiltonian vector fields vanishes. These involve torsion-free sheaves on the spectral curve instead of line bundles and a lifting to a 3-manifold which fibres over the cotangent bundle. The case of rank 2 will be described in more detail.

Critical points and spectral curves

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- compact Riemann surface Σ, genus > 1

- holomorphic vector bundle V rank n

- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

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- stability $\Rightarrow$ moduli space $\mathcal{M}$

- $\mathcal{M}$ is symplectic

- fibration $h : \mathcal{M} \to \mathcal{A} = \bigoplus_{k=1}^{n} H^0(\Sigma, K^k)$

$$h(V, \Phi) = (b_1, \dots, b_n) = \left(\operatorname{tr}\Phi, \frac{1}{2}\operatorname{tr}\Phi^2, \dots, \frac{1}{n}\operatorname{tr}\Phi^n \right).$$

- $f \in \mathcal{A}^*$ defines a function on $\mathcal{M}$

- $\Rightarrow$ Hamiltonian vector field X_f

- $[X_f, X_g] = 0$: integrable system

- h is proper
- Dh is generically surjective
- $h^{-1}(b) =$ torus for b a regular value

Question: What are the critical points of h?

($\sim$ what are the zeros of X_f for some $f \in \mathcal{A}^*$?)

SPECTRAL CURVES

- characteristic polynomial $\det(x - \Phi) = 0$

- coefficients a_i universal polynomials in $(b_1, \ldots, b_n) \in \mathcal{A}$

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- x the tautological section of $\pi^* K$ on $\pi : K \to \Sigma$

  (= canonical 1-form on $T^*\Sigma$)
- $x^n + \pi^* a_1 x^{n-1} + \cdots + \pi^* a_n = 0$

  equation of spectral curve $S \subset K$.
- $x \sim$ eigenvalue, eigenspace $\sim$ line bundle

- torus $h^{-1}(b)$ = Jacobian of S

- line bundle L on S defines direct image $V = \pi_* L$

- $U \subset \Sigma$ open $H^0(U, V) = H^0(\pi^{-1}(U), L)$

- torus $h^{-1}(b)$ = Jacobian of S
- line bundle L on S defines direct image $V = \pi_* L$
- $U \subset \Sigma$ open $H^0(U, V) = H^0(\pi^{-1}(U), L)$
- $x : L \to L\pi^* K$ on S
- $\Phi = \pi_* x : V \to V \otimes K$

- Serre duality $\mathcal{A}^* = \bigoplus_{k=1}^{n} H^1(\Sigma, K^{1-k})$

- $\mu_k \in H^1(\Sigma, K^{1-k})$ defines $x^{k-1}\pi^*\mu_k \in H^1(S, \mathcal{O})$

- $H^1(S, \mathcal{O})$ = tangent space to Jacobian $H^1(S, \mathcal{O}^*)$

- $f = (\mu_1, \ldots, \mu_n) \in \mathcal{A}^*$, Hamiltonian vector field is

$$X_f = \sum_{k=1}^{n} \pi^*\mu_k \left(x^{k-1} - \frac{(k-1)}{n} \pi^* b_{k-1} \right).$$

D.Baraglia, *Classification of the automorphism and isometry groups of Higgs bundle moduli spaces*, Proc. London Math. Soc. **112** (2016) 827–854.

- regular value b of $h : \mathcal{M}^{2m} \to \mathbf{C}^m$

- Hamiltonian vector fields $X_1, \ldots, X_m$ everywhere linearly independent

- tangent to $h^{-1}(b) \Rightarrow m$-dimensional torus

- linear dependence at a point $\Rightarrow$ singular S

- critical *values* $\sim$ singular curves

- $x^{k-1}\pi^*\mu_k \in H^1(K, \mathcal{O})$

- restrict to $S \subset K$

- **Prop**: Let $S \subset K$ be the divisor of a section of $\pi^* K^n$. Then each element $c \in H^1(S, \mathcal{O})$ can be written uniquely in the form
$$c = \sum_{k=1}^{n-1} x^k \pi^* c_k$$
where $c_k \in H^1(\Sigma, K^{-k})$.

- ... even when S is singular

- Hamiltonian vector field

$$X_f = \sum_{k=1}^{n} \pi^* \mu_k \left(x^{k-1} - \frac{(k-1)}{n} \pi^* b_{k-1} \right).$$

- is non-zero on $H^1(S, \mathcal{O}^*)$

- ... even when S is singular

- Where are the zeros?

- $h^{-1}(b)$ compact

- S singular $H^1(S, \mathcal{O}^*)$ non-compact

- Higgs fibre = compactification by rank one stable torsion-free sheaves

C.T.Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*, Pub. math. IHES (**79**) (1994) 47–129..

$SL(2,\mathbf{C})$-Higgs bundles

- holomorphic vector bundle V rank 2, $\Lambda^2 V \cong \mathcal{O}$
- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}_0\, V \otimes K) \sim \mathrm{tr}\,\Phi = 0$
- $\mathrm{tr}\,\Phi^2 = q \in H^0(\Sigma, K^2)$ quadratic differential
- spectral curve S: $x^2 - q = 0$
- $\mathcal{A} = H^0(\Sigma, K^2) \qquad \mathcal{A}^* = H^1(\Sigma, K^{-1})$

- S singular if q has zeros of multiplicity > 1

  local form of equation $x^2 = z^m$

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- $x(f_0(z) + x f_1(z)) = z^m f_1(z) + x f_0(z) \Rightarrow$

$$\text{Higgs field} \quad \Phi = \begin{pmatrix} 0 & z^m \\ 1 & 0 \end{pmatrix}.$$

Fact: If S is reduced, a rank one torsion free sheaf L is the direct image $\nu_* L'$ of a line bundle on a partial normalization $\nu : S' \to S$.

- $x^2 = z^{2m}$ normalized by two disjoint components $x = \pm z^m$

  z local coordinate on each

Higgs field $\Phi = \begin{pmatrix} z^m & 0 \\ 0 & -z^m \end{pmatrix}$.

- $\Phi \sim z^m \times$ semisimple

- $x^2 = z^{2m+1}$ put $x = t^{2m+1}, z = t^2$, t local coordinate

- $f(t) = f_0(z) + tf_1(z)$ and

  $x(f_0(z) + tf_1(z)) = z^{m+1}f_1(z) + tz^m f_0(z)$

- Higgs field $\Phi = \begin{pmatrix} 0 & z^{m+1} \\ z^m & 0 \end{pmatrix}$.

- $\Phi \sim z^m \times$ nilpotent

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- $f(t) = f_0(z) + tf_1(z)$ and
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- Higgs field $\Phi = \begin{pmatrix} 0 & z^{m+1} \\ z^m & 0 \end{pmatrix}.$

- **Hence**:
  critical point (V, Φ) requires Φ to vanish at some point(s)

DERIVATIVE OF $h : \mathcal{M} \to \mathcal{A}$

- holomorphic structure on C^∞ bundle V: $\bar{\partial}$-operator

  $\bar{\partial}_A : \Omega^0(\Sigma, V) \to \Omega^{01}(\Sigma, V)$

- Higgs bundle $\bar{\partial}_A \Phi = 0$

- infinitesimal deformation $(\dot{A}, \dot{\Phi})$

- tangent space to $\mathcal{M}$ at $(A, \Phi) =$

  $\{(\dot{A}, \dot{\Phi}) : \bar{\partial}_A \dot{\Phi} + [\dot{A}, \Phi] = 0\}$ modulo

  $\{(\dot{A}, \dot{\Phi}) : \dot{A} = \bar{\partial}_A \psi, \dot{\Phi} = [\psi, \Phi]\}$

- $[\dot{A}] \in H^1(\Sigma, \mathrm{End}_0\, V)$

- $\to H^0(\Sigma, \mathrm{End}_0\, V \otimes K) \to T\mathcal{M}_{(A,\Phi)} \xrightarrow{\alpha} H^1(\Sigma, \mathrm{End}_0\, V) \to$

- if V is stable $T^*\mathcal{N} \subset \mathcal{M}$ open

- $\alpha =$ derivative of projection $T^*\mathcal{N} \to \mathcal{N}$

- complex of sheaves $\mathcal{O}(\mathrm{End}_0 V) \stackrel{[\Phi,-]}{\rightarrow} \mathcal{O}(\mathrm{End}_0 V \otimes K)$

- $\mathbb{H}^1$ = tangent space to $\mathcal{M}$

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- $0 \to H^1(\Sigma, \ker[\Phi, -]) \to \mathbb{H}^1 \to H^0(\Sigma, \mathrm{coker}[\Phi, -]) \to 0.$

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- in $\mathfrak{sl}(2,\mathbf{C})$ if $A \neq 0$ and $[B,A]=0$ then $B=\lambda A$

- $\Rightarrow$ if $\Phi \neq 0$ everywhere then

  $\Phi : \mathcal{O}(K^{-1}) \to \ker[\Phi,-]$ is an isomorphism

- if $\Phi \neq 0$ then

- $0 \to H^1(\Sigma, K^{-1}) \to \mathbb{H}^1 \xrightarrow{\beta} H^0(\Sigma, K^2) \to 0.$

- β is the derivative of h and is surjective

- $V = \pi_* L$ for a line bundle L

- if Φ vanishes on a divisor D, zeros of a section s of $\mathcal{O}(D)$, then

- $0 \to H^1(\Sigma, K^{-1}(-D)) \to \mathbb{H}^1 \stackrel{\beta}{\to} H^0(\Sigma, K^2(-D)) \to 0.$

- and $Dh = s\beta$ where $H^0(\Sigma, K^2(-D)) \stackrel{s}{\to} H^0(\Sigma, K^2)$

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- and $Dh = s\beta$ where $H^0(\Sigma, K^2(-D)) \xrightarrow{s} H^0(\Sigma, K^2)$

- $a \in \mathcal{A}^* = H^1(\Sigma, K^{-1})$ annihilates $sH^0(\Sigma, K^2(-D))$ if

$$sa = 0 \in H^1(\Sigma, K^{-1}(D))$$

- $\to H^0(\Sigma, K^{-1}(D)) \to H^0(D, K^{-1}(D)) \to H^1(\Sigma, K^{-1}) \xrightarrow{s} .$
  $\quad\quad = 0$

- $sa = 0 \Rightarrow$ Dolbeault representative of a of the form

$$\frac{\bar{\partial}\alpha}{s}$$

  where α is holomorphic in a neighbourhood of the zeros of s and supported in a slightly bigger one

THE HESSIAN

- $A(t) = A + t\dot{A} + t^2\ddot{A}, \qquad \Phi(t) = \Phi + t\dot{\Phi} + t^2\ddot{\Phi}$

- $\bar{\partial}_A\dot{\Phi} + [\dot{A}, \Phi] = 0 \qquad \bar{\partial}_A\ddot{\Phi} + [\dot{A}, \dot{\Phi}] + [\ddot{A}, \Phi] = 0$

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$$\int_\Sigma \mathrm{tr}\,\Phi(t)^2\frac{\bar{\partial}\alpha}{s} = \int_\Sigma (\mathrm{tr}\,\Phi^2 + 2t\,\mathrm{tr}\,\Phi\dot{\Phi} + t^2(\mathrm{tr}\,\dot{\Phi}^2 + 2\Phi\ddot{\Phi}))\frac{\bar{\partial}\alpha}{s} + \ldots$$

- second variation:

$$\int_\Sigma (\mathrm{tr}\,\dot{\Phi}^2 + 2\,\mathrm{tr}\,\Phi\ddot{\Phi})\frac{\bar{\partial}\alpha}{s}.$$

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- ... Φ/s is smooth and holomorphic

- $$\mathrm{tr}(\Phi\bar{\partial}_A\ddot{\Phi}) = -\,\mathrm{tr}(\Phi([\dot{A},\dot{\Phi}] + [\ddot{A},\Phi])) =$$

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- Stokes' theorem:

$$\int_\Sigma \mathrm{tr}\,\dot{\Phi}^2\frac{\bar{\partial}\alpha}{s} + \int_\Sigma \bar{\partial}(\mathrm{tr}\,\dot{\Phi}^2)\frac{\alpha}{s}$$

- Hessian:

$$2\pi i \sum_{s(z_k)=0} \mathrm{Res}_{z=z_k}\frac{\alpha\,\mathrm{tr}\,\dot{\Phi}^2}{s}.$$

THE NILPOTENT CONE

- $h^{-1}(0)$ = nilpotent cone

- Φ nilpotent $\Rightarrow \ker \Phi \sim L \subset V$

- $0 \to L \to V \to L^* \to 0$

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- Higgs field $b \in H^0(\Sigma, \mathrm{Hom}(L^*, LK) = H^0(\Sigma, L^2K)$

- For stability $b \neq 0$ and so $\deg L^2K = 2g - 2 - 2d \geq 0$.

- $h^{-1}(0)$ = union of (Lagrangian) components
- $\mathcal{N}$ = (poly) stable bundles $\Phi \equiv 0$
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- Higgs field $b \in H^0(\Sigma, L^2K)$ has zeros if deg $L^2K > 0$

- $\Rightarrow$ all components except L^2K trivial are critical points for h

- $h^{-1}(0)$ homologous to smooth fibre

- homology class of closure of each component has a multiplicity

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- Hessians at a general point are linearly independent

- $\Rightarrow$ multiplicity $= 2^{\deg L^2 K}$

- multiplicity of $\mathcal{N} = 2^{3g-3}$

LIFTING THE SPECTRAL CURVE

- torsion-free sheaf L on $S \sim \nu_* L'$ for a line bundle on a partial normalization S'

- $f \in \mathcal{A}^*, a_f \in H^1(S, \mathcal{O})$

- flow of Hamiltonian vector field

  $L \mapsto \exp(ta_f)L$ on $H^1(S, \mathcal{O}^*)$

- $L' \mapsto \exp(t\nu^* a_f)L'$ on $H^1(S', \mathcal{O}^*)$

- fixed point of flow $\Rightarrow \nu^* a_f = 0 \in H^1(S', \mathcal{O})$
- $a_f \in H^1(S, \mathcal{O})$ is the restriction of a class in $H^1(K, \mathcal{O})$
- ... which defines a principal $\mathbf{C}$-bundle Z over K

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Note: Z is a Calabi-Yau threefold

- *Example*: (V, Φ) $SL(2, \mathbf{C})$-Higgs bundle

- $a \in H^1(\Sigma, K^{-1})$ defines a rank 2 bundle E

$$0 \to \mathcal{O} \to E \to K \to 0.$$

- $\mathcal{O} \subset E$ non-vanishing section

- translation $\Rightarrow$ free $\mathbf{C}$-action

- quotient $= K$ and $Z = E$

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$$\bar{\partial}_E(f,\varphi) = (\bar{\partial} f + (\bar{\partial}\alpha/s)\varphi, \bar{\partial}\varphi)$$

- $x, z, y = (y, 0)$ local holomorphic coordinates

- $x^2 = z^{2m}$ lifts to $y = -\alpha x/s \sim \pm\alpha$

- $x^2 = z^{2m+1}$, normalized by $x = t^{2m+1}, z = t^2$
  lifts to $y = -\alpha x/s \sim -\alpha t^{2m+1}/t^{2m} = -\alpha t$

- smooth curves

- critical point $a \operatorname{tr} \Phi\dot{\Phi} = 0$ all $\dot{\Phi}$

- $\Rightarrow [a\Phi] = 0 \in H^1(\Sigma, \operatorname{End}_0 V)$

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- $(\bar{\partial}\alpha/s)\Phi = \bar{\partial}(\alpha\Phi/s) = \bar{\partial}\psi$

- then

$$\bar{\partial}_E(-\psi, \Phi) = (-\bar{\partial}\psi + (\bar{\partial}\alpha/s)\Phi, \bar{\partial}\Phi) = 0$$

- $\tilde{\Phi} = (-\psi, \Phi)$ holomorphic section of $\mathrm{End}_0 V \otimes E$

- $\psi = \alpha\Phi/s \Rightarrow [\psi, \Phi] = 0$

$$\Rightarrow \tilde{\Phi} \wedge \tilde{\Phi} = 0 \in H^0(\Sigma, \mathrm{End}_0\, V \otimes \Lambda^2 E).$$

- *cf.* higher-dimensional Higgs bundles $\Phi \in H^0(M, \mathrm{End}_0\, V \otimes T^*)$ and $\Phi \wedge \Phi = 0$