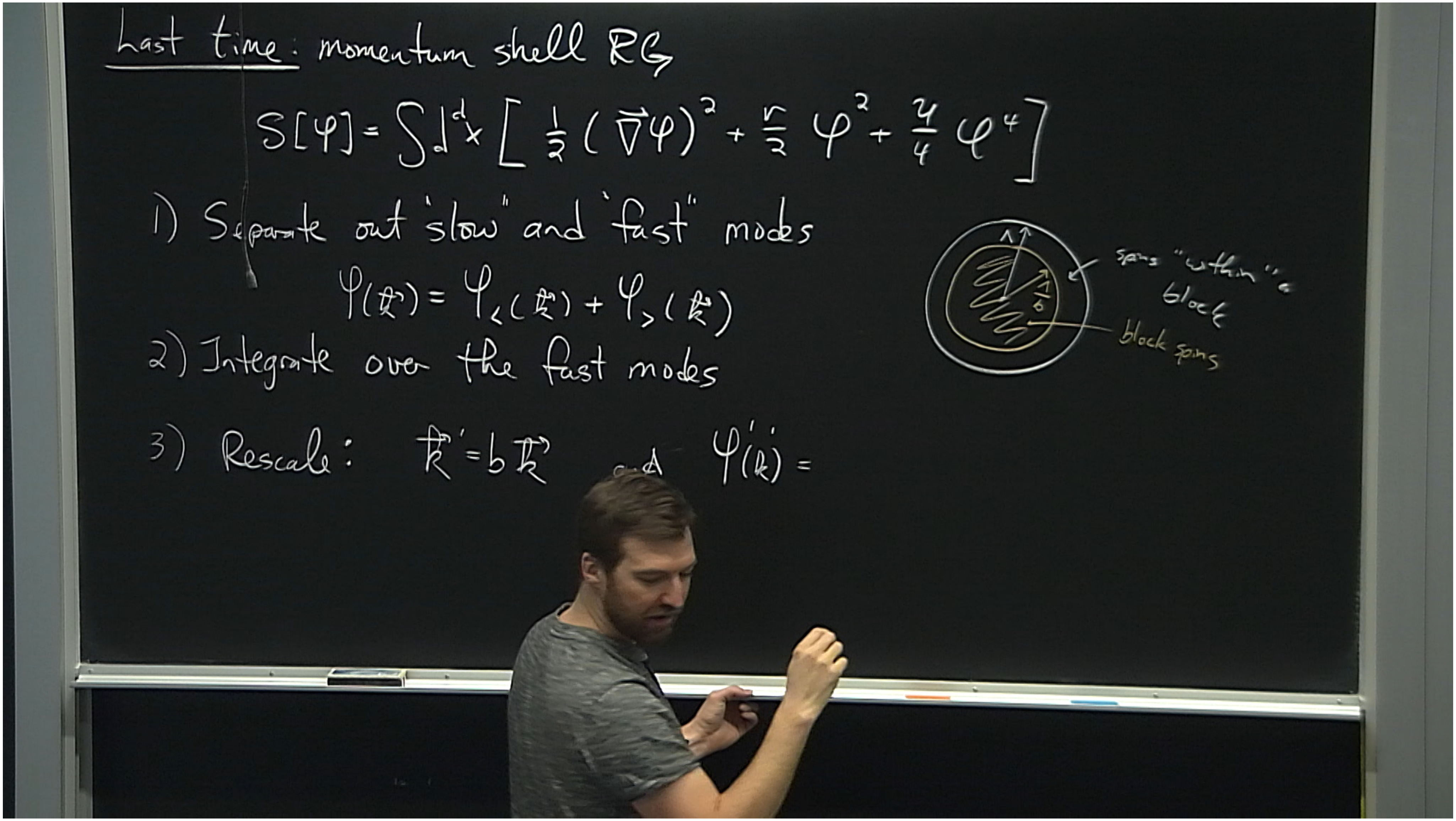
Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 12

Date: Feb 10, 2017 10:30 AM

URL: http://pirsa.org/17020005

Abstract:


Last time: momentum shell RG
$S[\varphi] = \int d^dx \left[\frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4 \right]$
1) Separate out "slow" and "fast" modes
$\varphi(\vec{k}) = \varphi_<(\vec{k}) + \varphi_>(\vec{k})$
2) Integrate over the fast modes
3) Rescale: $\vec{k}' = b\vec{k}$ and $\varphi'(k) =$
Λ
$\frac{\Lambda}{b}$
spins "within" a block
block spins

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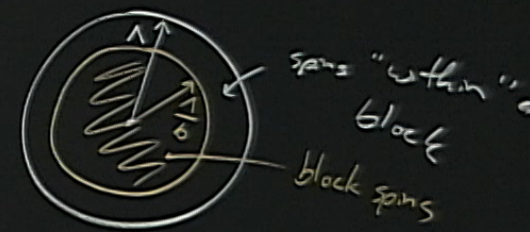
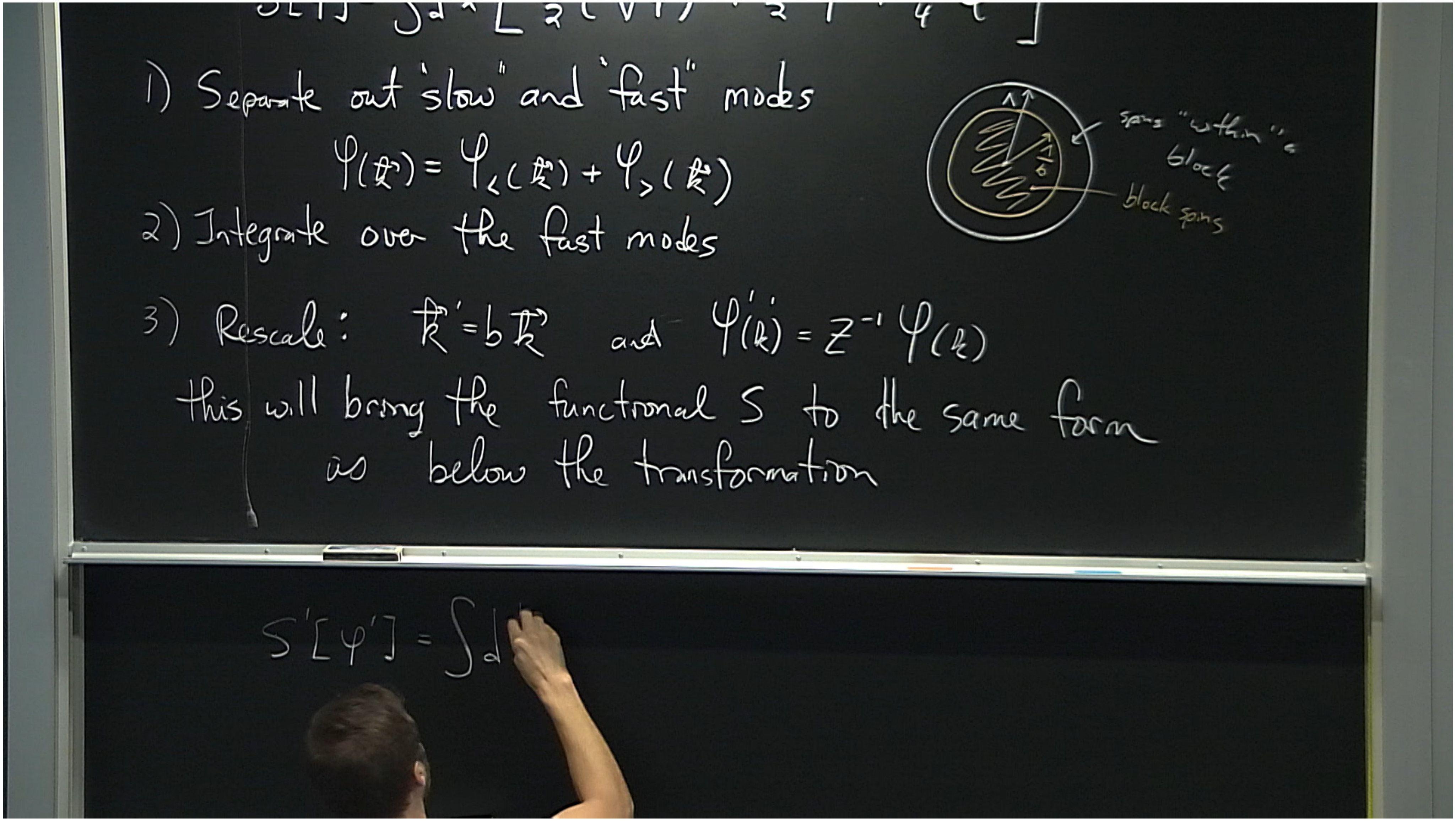


2) Integrate over the fast modes

3) Rescale: $\vec{k}' = b\vec{k}$ and $\varphi'(k) = z^{-1}\varphi(k)$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$\varphi(\vec{k}) = \varphi_<(\vec{k}) + \varphi_>(\vec{k})$
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3) Rescale: $\vec{k}' = b\vec{k}$ and $\varphi'(k) = Z^{-1}\varphi(k)$
this will bring the functional S to the same form as below the transformation
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Λ/b
spins "within" a block
block spins
$S'[\varphi'] = \int d$

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$$S'[\varphi'] = \int d^dx \left[\frac{1}{2}(\nabla\varphi')^2 + \frac{r'}{2}\varphi'^2 + \frac{u'}{4}\varphi'^4 \right]$$

Thus we represent each functional by a pair of variables (r, u)

the RG transformation is $(r, u) \to (r', u')$

Fixed points are when $(r^*, u^*) \to (r^*, u^*)$

is below the transformation

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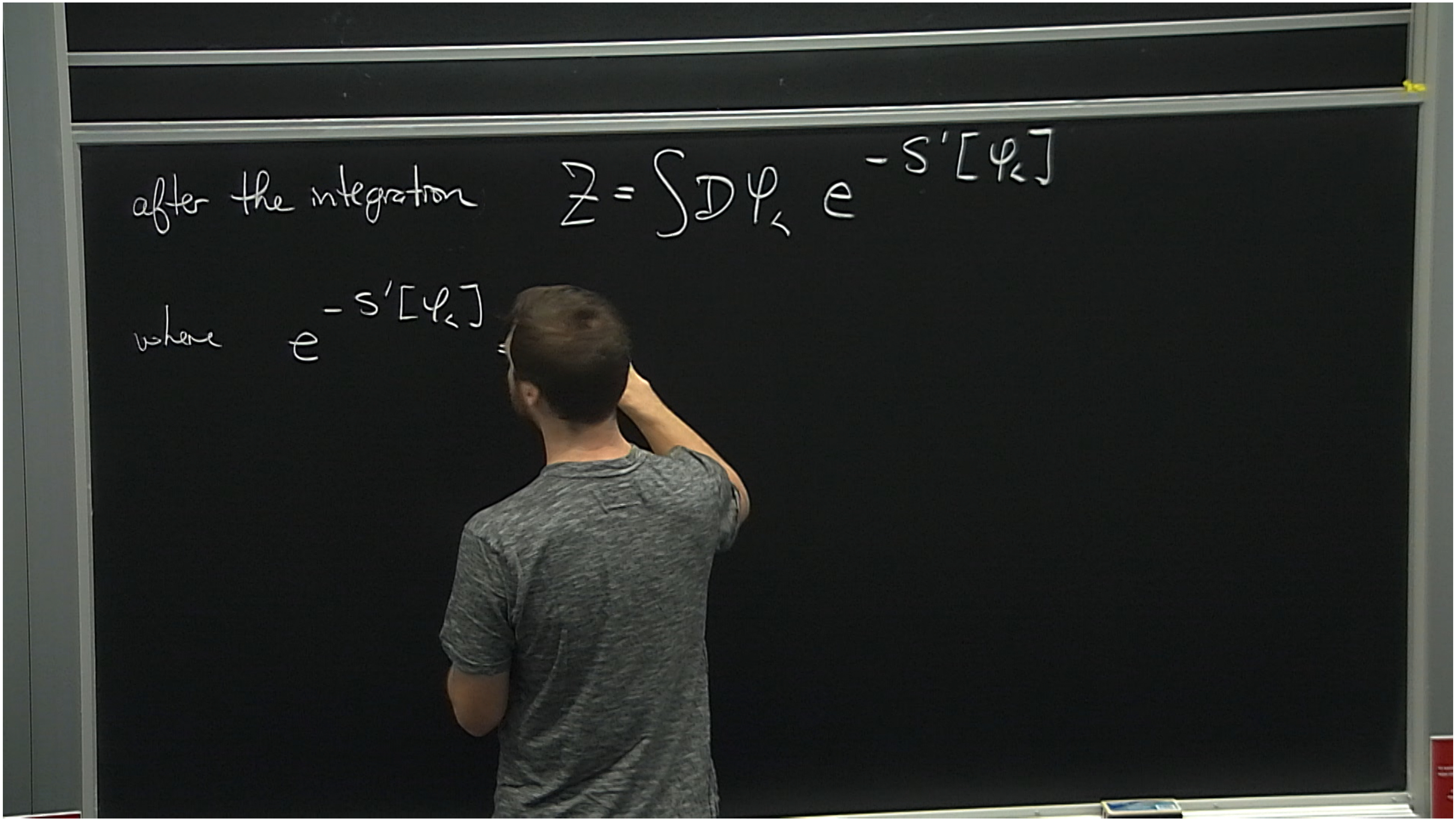
after the integration $Z = \int D\varphi_< \, e^{-S'[\varphi_<]}$
where $e^{-S'[\varphi_<]} =$

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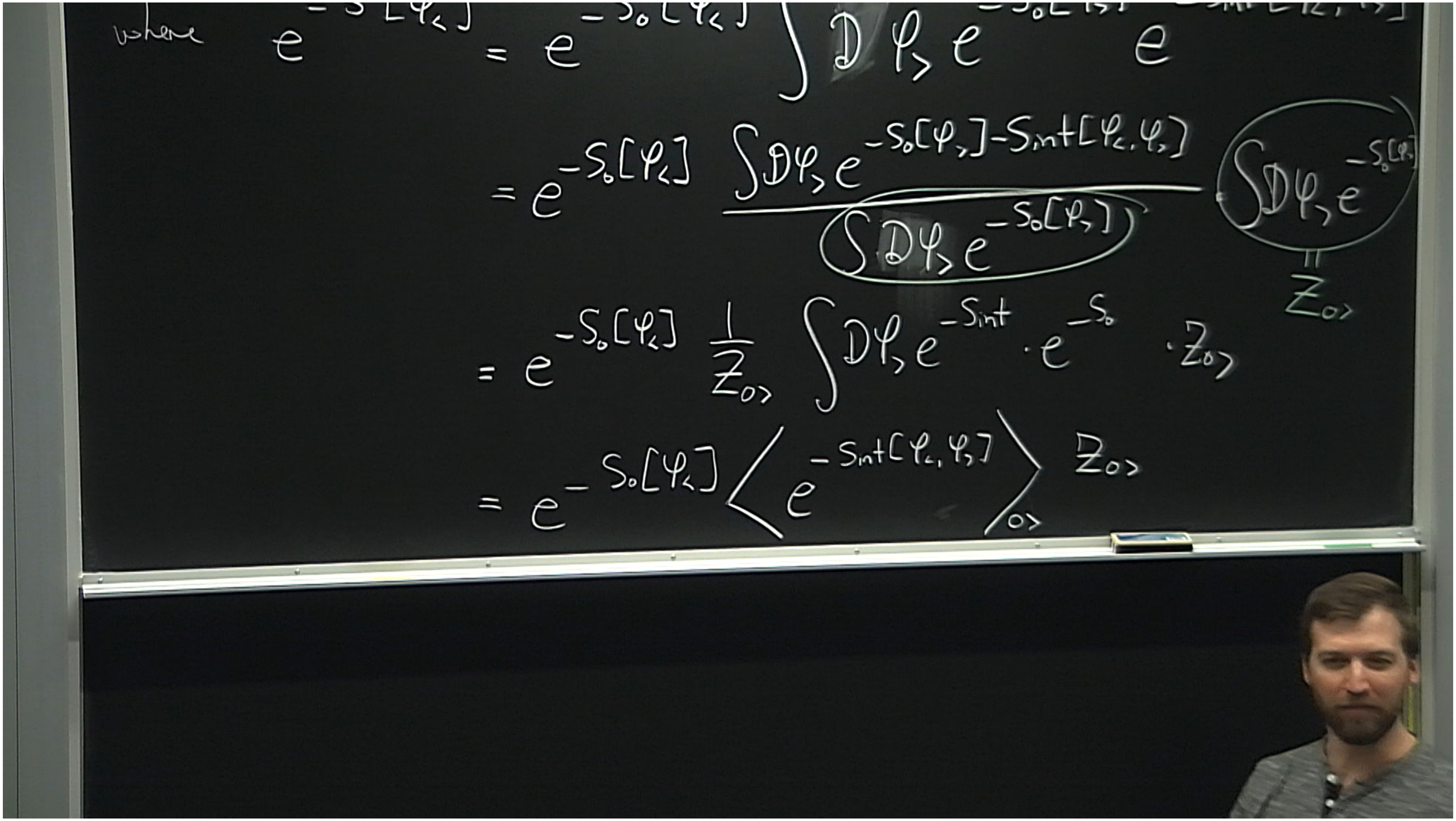
where $$e^{-S'[\varphi_<]} = e^{-S_0[\varphi_<]} \int \mathcal{D}\varphi_> \, e^{-S_0[\varphi_>]} \, e^{-S_{int}[\varphi_<, \varphi_>]}$$

$$= e^{-S_0[\varphi_<]} \frac{\int \mathcal{D}\varphi_> \, e^{-S_0[\varphi_>] - S_{int}[\varphi_<, \varphi_>]}}{\int \mathcal{D}\varphi_> \, e^{-S_0[\varphi_>]}} \int \mathcal{D}\varphi_> \, e^{-S_0[\varphi_>]}$$


where $e^{-S[\varphi_<]} = e^{-S_0[\varphi_<]} \int \mathcal{D}\varphi_> e^{-S_0[\varphi_>]} e^{-S_{int}[\varphi_<,\varphi_>]}$
$= e^{-S_0[\varphi_<]} \frac{\int \mathcal{D}\varphi_> e^{-S_0[\varphi_>]-S_{int}[\varphi_<,\varphi_>]}}{\int \mathcal{D}\varphi_> e^{-S_0[\varphi_>]}} \cdot \int \mathcal{D}\varphi_> e^{-S_0[\varphi_>]}$
$\int \mathcal{D}\varphi_> e^{-S_0[\varphi_>]} = Z_{0>}$
$= e^{-S_0[\varphi_<]} \frac{1}{Z_{0>}} \int \mathcal{D}\varphi_> e^{-S_{int}} \cdot e^{-S_0} \cdot Z_{0>}$
$= e^{-S_0[\varphi_<]} \left\langle e^{-S_{int}[\varphi_<,\varphi_>]} \right\rangle_{0>} Z_{0>}$

then $S'[\varphi_<] = S_0[\varphi_<] - \ln \left\langle e^{-S_{int}[\varphi_<, \varphi_>]} \right\rangle_{0>} \underbrace{- \ln Z_{0>}}_{\text{a constant (c.f. } g(K))}$

drop the last term: $S'[\varphi_<] = S_0[\varphi_<] - \ln \left\langle e^{-S_{int}} \right\rangle_{0>}$

Assume that S' and S are of similar form (will justify later)

Write:

$$S'[\varphi_<] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d} \left(\hat{r}_0 + \hat{r}_2 k^2 \right) \left| \varphi_<(\vec{k}) \right|^2$$

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$$+ \frac{\hat{u}}{4}\int_0^{\Lambda/b} \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_4}{(2\pi)^d}\, \varphi_<(\vec{k}_1)\varphi_<(\vec{k}_2)\varphi_<(\vec{k}_3)\varphi(\vec{k}_4)\,(2\pi)^d\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

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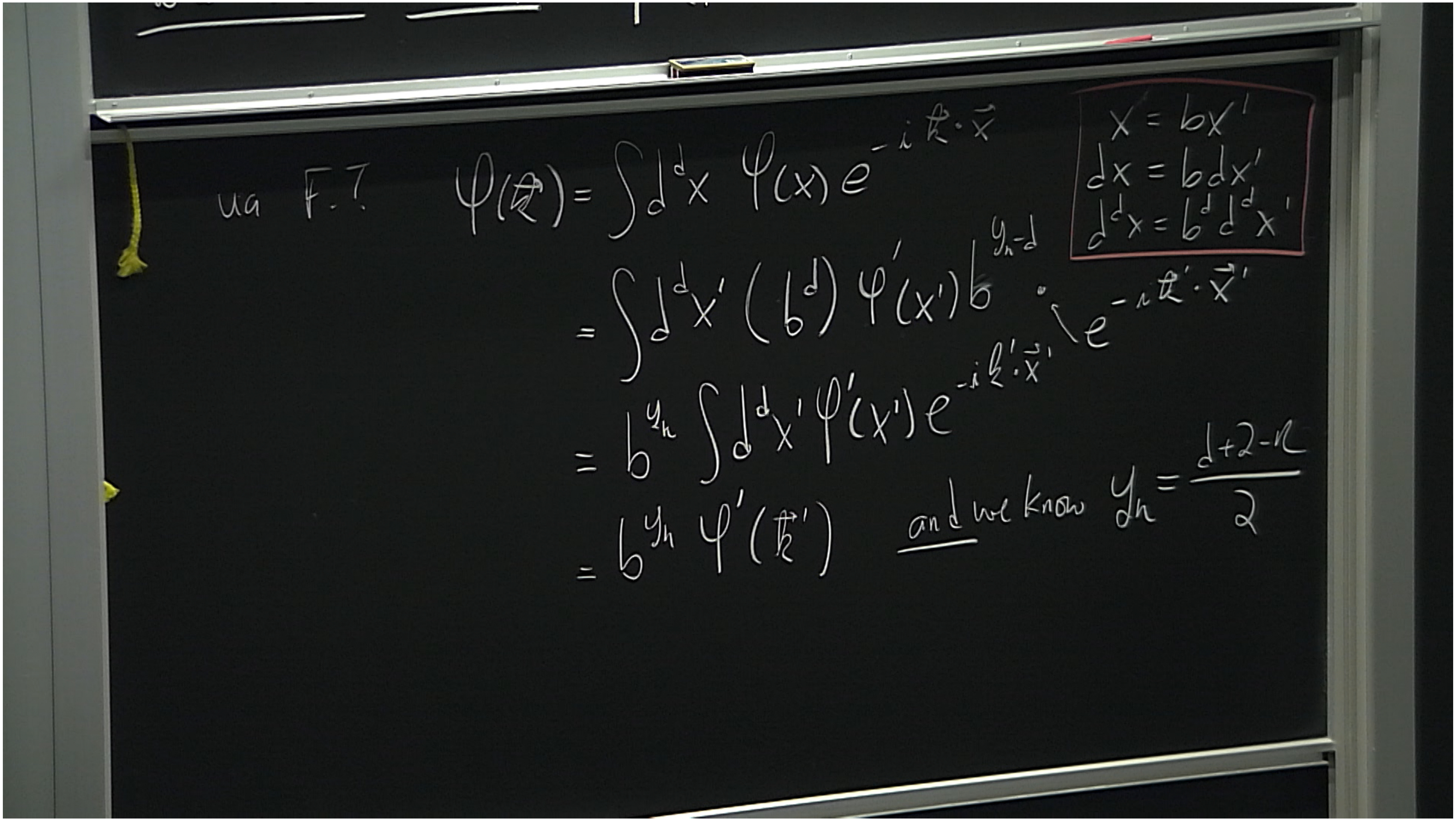
$$+ \frac{\tilde{u}}{4}\int_0^{\Lambda/b} \frac{d^dk_1}{(2\pi)^d} \cdots \frac{d^dk_4}{(2\pi)^d}\, \varphi_<(\vec{k}_1)\varphi_<(\vec{k}_2)\varphi_<(\vec{k}_3)\varphi_<(\vec{k}_4)\,(2\pi)^d \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

$$S'[\varphi'] = \frac{1}{2}\int_0 \frac{d^d k}{(2\pi)^d}\, b \,(r_0 + r_2 k^2 b \ldots) Z \ldots$$

$$+ \frac{\tilde{y}}{4}\int_0^{\Lambda} \frac{d^d k_1'}{(2\pi)^d} \cdots \frac{d^d k_4'}{(2\pi)^d}\, b^{-4d}\, Z^4\, \varphi'(\vec{k}_1') \cdots \varphi'(\vec{k}_4')\, (2\pi)^d\, \delta(\vec{k}_1 + \cdots + \vec{k}_4)\, b^d$$

What is Z? Recall $\varphi'(\vec{r}\,') = b^{d-y_h}\varphi(\vec{r})$



via F.T. $\varphi(\vec{k}) = \int d^d x \, \varphi(x) e^{-i\vec{k}\cdot\vec{x}}$
$= \int d^d x' \, (b^d) \, \varphi'(x') \, b^{y_h - d} \, e^{-i\vec{k}'\cdot\vec{x}'}$
$= b^{y_h} \int d^d x' \, \varphi'(x') e^{-i\vec{k}'\cdot\vec{x}'}$
$= b^{y_h} \varphi'(\vec{k}')$
and we know $y_h = \frac{d+2-\eta}{2}$
$x = bx'$
$dx = b\,dx'$
$d^d x = b^d d^d x'$

via F.T. $\varphi(\vec{k}) = \int d^dx\, \varphi(x)\, e^{-i\vec{k}\cdot\vec{x}}$

$$\boxed{\begin{aligned} x &= bx' \\ dx &= b\,dx' \\ d^dx &= b^d d^dx' \end{aligned}}$$

$$= \int d^dx' \,(b^d)\, \varphi'(x')\, b^{y_h-d} \cdot e^{-i\vec{k}'\cdot\vec{x}'}$$

$$= b^{y_h} \int d^dx'\, \varphi'(x')\, e^{-i\vec{k}'\cdot\vec{x}'}$$

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this means $Z = b^{\frac{d+2-\eta}{2}}$

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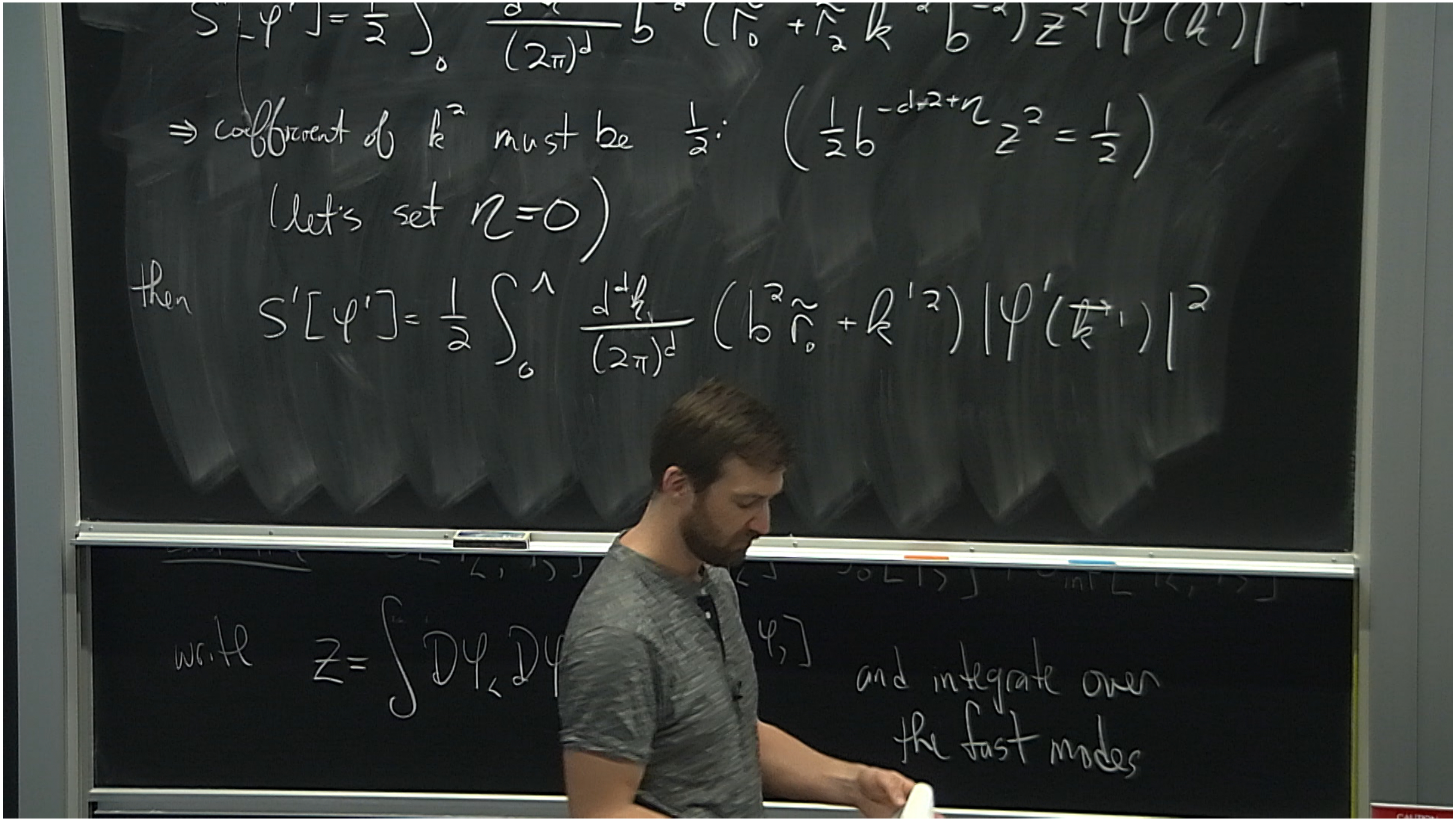
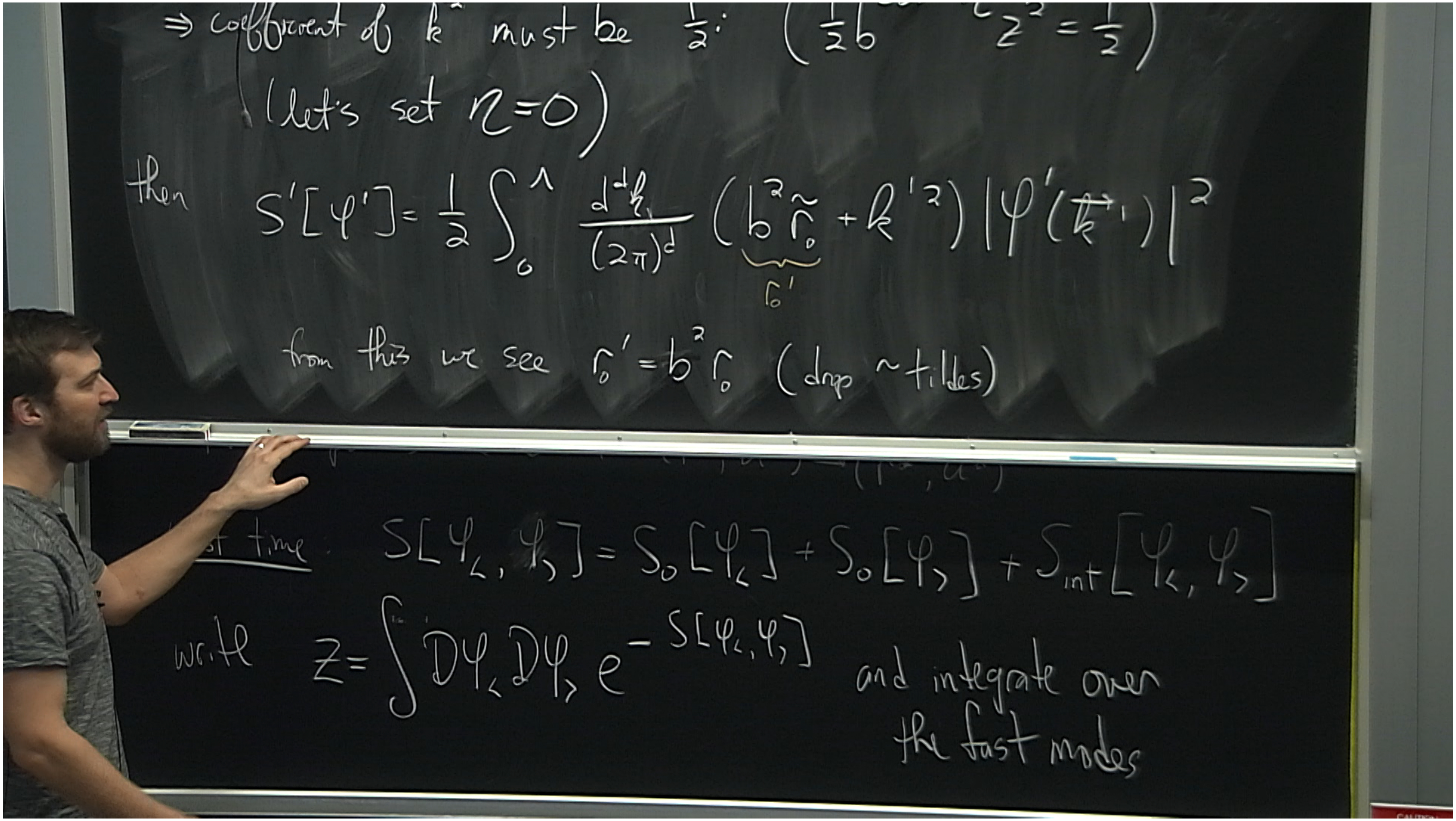
$= b^{y_h}\, \varphi'(\vec{k}')$ and we know $y_h = \frac{d+2-\eta}{2}$

this means $Z = b^{\frac{d+2-\eta}{2}}$, meaning $\tilde{r}_2 = b^{\eta}$

and remember η is often zero (e.g. in MFT)



$\Rightarrow$ coefficient of k'^2 must be $\frac{1}{2}$: $\left(\frac{1}{2} b^{-d+2+\eta} z^2 = \frac{1}{2}\right)$
(let's set $\eta = 0$)
then $S'[\varphi'] = \frac{1}{2}\int_0^\Lambda \frac{d^dk'}{(2\pi)^d}\left(b^2 \tilde{r}_0 + k'^2\right)|\varphi'(\vec{k}')|^2$
write $Z = \int D\varphi_< D\varphi$
and integrate over the fast modes



⇒ coefficient of k^2 must be $\frac{1}{2}$: $\left(\frac{1}{2} b \quad z^2 = \frac{1}{2}\right)$
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from this we see $r_0' = b^2 r_0$ (drop ~ tildes)
last time: $S[\varphi_<, \varphi_>] = S_0[\varphi_<] + S_0[\varphi_>] + S_{int}[\varphi_<, \varphi_>]$
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and in Landau theory $r \propto t$ $\quad \boxed{t' = b^2 t} \Rightarrow y_t = 2$

and we know (for example), $\nu = \frac{1}{y_t} \Rightarrow \boxed{\nu = \frac{1}{2}}$

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<u>Note:</u> the RG recursion relationship $r' = b^2 r$ is of

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Note: the RG recursion relationship $r' = b^2 r$ is often written in "differential" form:

$$b = e^{\Delta l} = e^{l} \quad (\text{defines } l = \Delta l)$$

so $r' = r b^2 = r e^{2l} = r(1+2l) = r + 2rl$

i.e. $\dfrac{r'-r}{\Delta l} = 2r$ and imagine r as a function of l which parameterizes the RG step.

$$\boxed{\frac{dr}{dl} = 2r}$$

And the fixed point is found by requiring that $\dfrac{dr}{dl} \to 0$

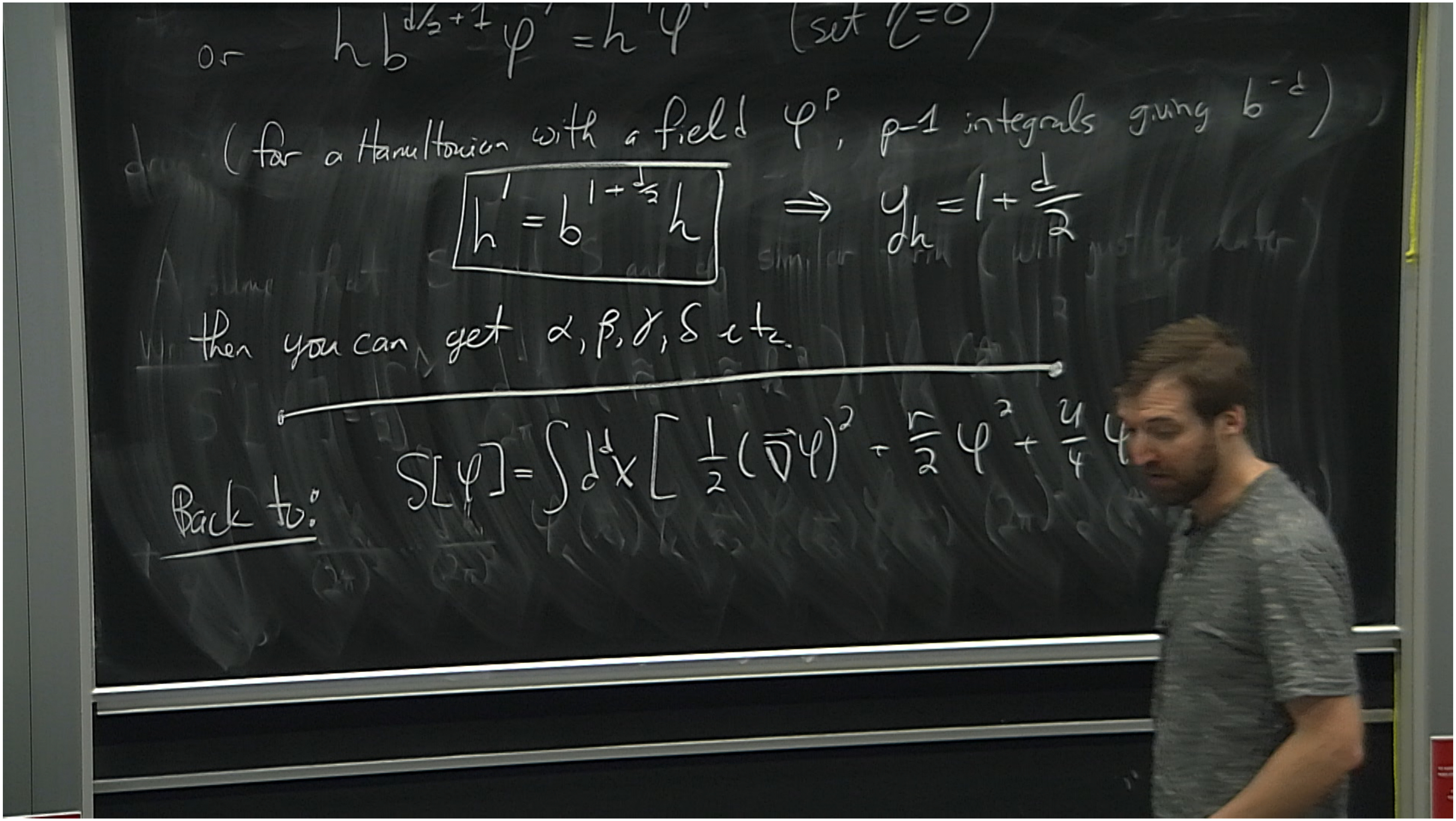
in this case $\boxed{r^* = 0}$.

or $h\, b^{d/2+1} \varphi' = h' \varphi'$ (set $\ell = 0$)

or $\quad h\, b^{d/2+1}\varphi' = h'\varphi'$ $\quad$ (set $\ell = 0$)

(for a Hamiltonian with a field φ^p, $p-1$ integrals giving b^{-d})

$$\boxed{h' = b^{1+\frac{d}{2}}\, h} \quad \Rightarrow \quad y_h = 1 + \frac{d}{2}$$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$\boxed{h' = b^{1+\frac{d}{2}} h}$ $\Rightarrow$ $y_h = 1 + \frac{d}{2}$
then you can get $\alpha, \beta, \gamma, \delta$ etc.
Back to: $S[\varphi] = \int d^dx \left[\frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi\right.$

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Back to: $S[\varphi] = \int d^d x \left[\frac{1}{2} (\vec{\nabla}\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4 \right]$

and $S = S_0[\varphi_<] + S_0[\varphi_>] + S_{int}[\varphi_<, \varphi_>]$

Recall: after integration over $\varphi_>$

$$e^{-S'[\varphi_<]} = e^{-S_0[\varphi_<]} \left\langle e^{-S_{int}[\varphi_<,\varphi_>]} \right\rangle_{0>}$$

$$\langle e^{-\Omega} \rangle = e^{-\langle \Omega \rangle + \frac{1}{2}\left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right] + \cdots}$$

"Proof" expand both sides:

LHS $\langle e^{-\Omega} \rangle = \langle 1 - \Omega + \frac{1}{2}\Omega^2 + \cdots \rangle = 1 - \langle \Omega \rangle + \frac{1}{2}\langle \Omega^2 \rangle + \cdots$

RHS $e^{-\langle \Omega \rangle + \frac{1}{2}\left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right] + \cdots}$

$$= 1 - \langle \Omega \rangle + \frac{1}{2}\left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right]$$

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$$= 1 - \langle \Omega \rangle + \frac{1}{2}\left[\langle \Omega^2 \rangle - \langle \Omega \rangle^2\right] + \frac{1}{2}\langle \Omega \rangle^2 + \cdots$$

$= \text{LHS}$ (up to second order)

$$\langle e^{-S_{int}} \rangle_{0>} = e^{-\langle S_{int} \rangle_{0>} + \frac{1}{2}\left[\langle S_{int}^2 \rangle_{0>} - \langle S_{int} \rangle_{0>}^2\right] + \cdots}$$

$\underline{\underline{\text{or}}}$

$$S'[\varphi_<] = S_0[\varphi_<] + \langle S_{int} \rangle_{0>} - \frac{1}{2}\left[\langle S_{int}^2 \rangle_{0>} - \langle S_{int} \rangle_{0>}^2\right]$$

<u>Recall:</u> $S_{int}[\varphi_<, \varphi_>] = \frac{u}{4} \int_0^\Lambda \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_4}{(2\pi)^d} (2\pi)^d \delta(\vec{k}_1 + \cdots + \vec{k}_4)$

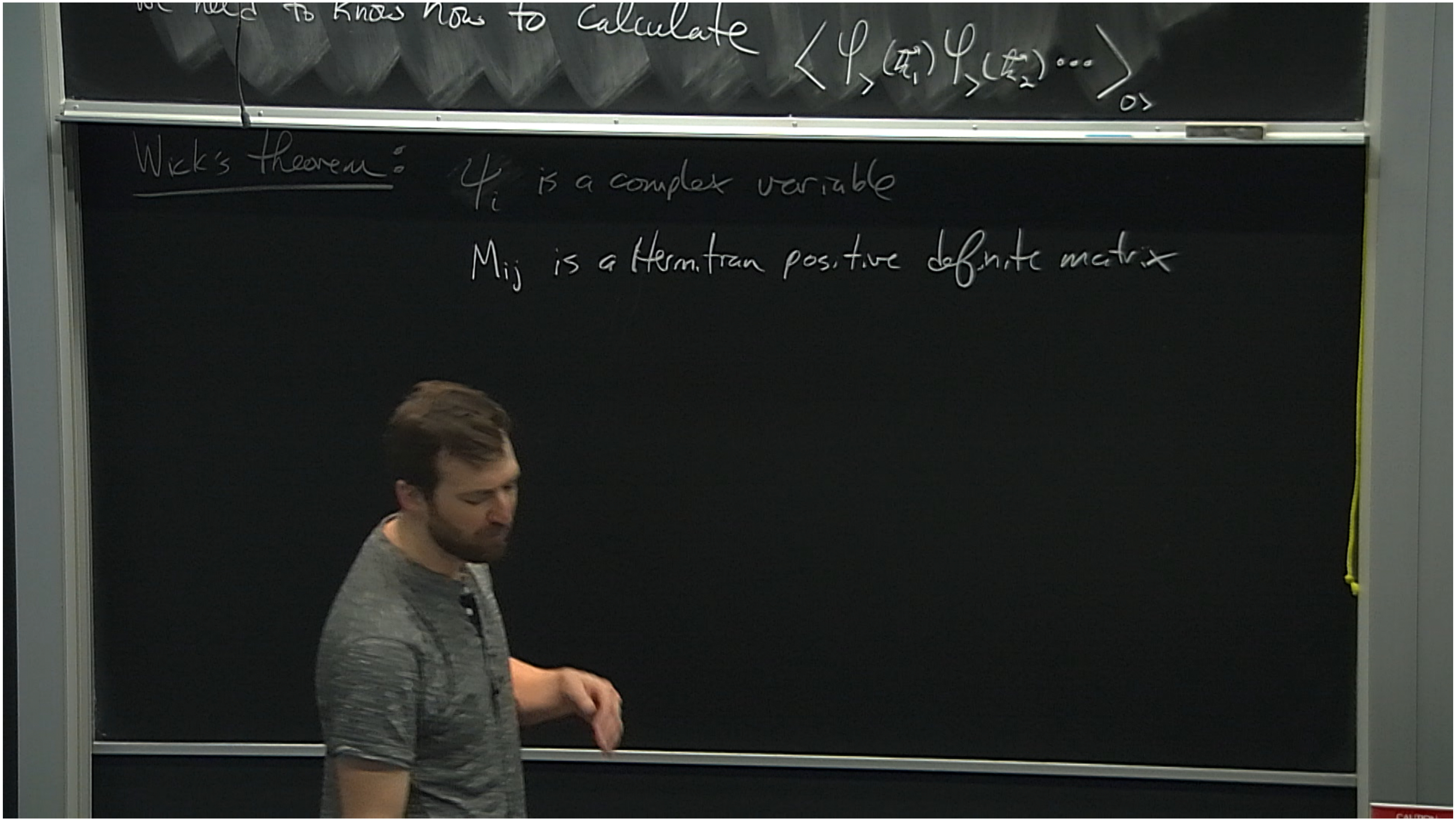
$[$

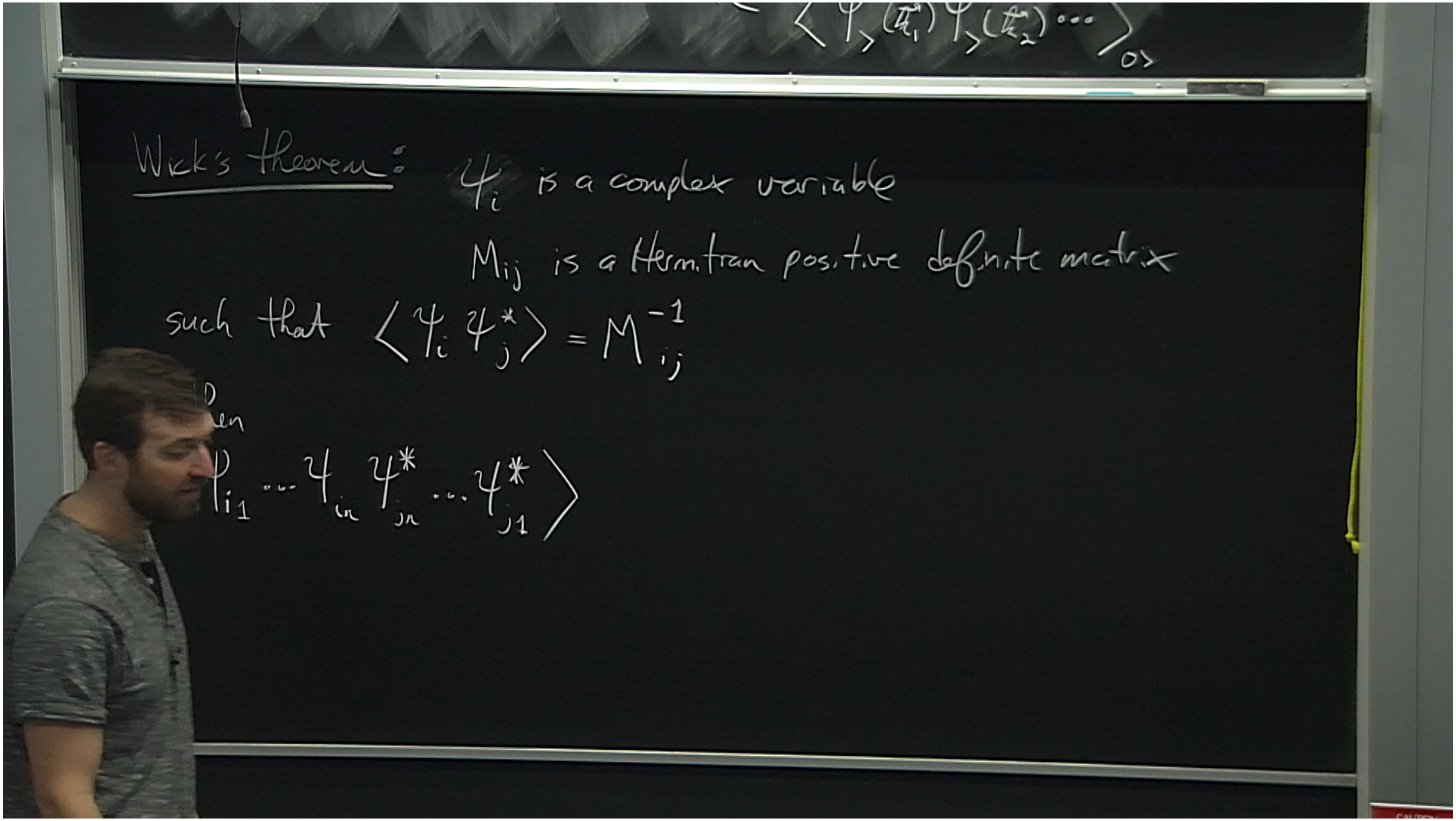
or: $S'[\varphi_<] = S_0[\varphi_<] + \langle S_{int}\rangle_{0>} - \frac{1}{2}\left[\langle S_{int}^2\rangle_{0>} - \langle S_{int}\rangle_{0>}^2\right]$

Recall: $S_{int}[\varphi_<, \varphi_>] = \frac{u}{4}\int_0^\Lambda \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dk_4}{(2\pi)^d}(2\pi)^d\,\delta(\vec{k}_1 + \cdots + \vec{k}_4)$

$$\left[\varphi_<(\vec{k}_1) + \varphi_>(\vec{k}_1)\right]\cdots\left[\varphi_<(\vec{k}_4) + \varphi_>(\vec{k}_4)\right]$$

so $r' = rb = re^{2l} = r(1+2l) = r + 2rl$


to know how to calculate
Wick's theorem:
is a complex variable
Mij is a Hermitian positive definite matrix


Wick's theorem:
ψ_i is a complex variable
M_ij is a Hermitian positive definite matrix
such that ⟨ψ_i ψ_j^*⟩ = M^{-1}_{ij}
ψ_{i1} … ψ_{in} ψ^*_{jn} … ψ^*_{j1} ⟩

example:

$$\langle \psi_1 \psi_2 \psi_3^* \psi_4^* \rangle = M^{-1}_{13} M^{-1}_{24} + M^{-1}_{14} M^{-1}_{23}$$

We're going to apply Wick's theorem to

$$S'[\varphi_L] = S_0[\varphi_L] + \langle S_{int} \rangle_{0>} - \frac{1}{2}\left[\langle S_{int}^2 \rangle_{0>} - \langle S_{int} \rangle_{0>}^2 \right]$$

example:

$$\langle \psi_1 \psi_2 \psi_3^* \psi_4^* \rangle = M^{-1}_{13} M^{-1}_{24} + M^{-1}_{14} M^{-1}_{23}$$

We're going to apply Wick's theorem to

$$S'[\varphi_L] = S_0[\varphi_L] + \langle S_{int} \rangle_{0>} - \frac{1}{2}\left[\langle S^2_{int} \rangle_{0>} - \langle S_{int} \rangle^2_{0>} \right]$$