Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 11

Date: Feb 08, 2017 10:30 AM

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Abstract:

Universality: Many microscopic details of the system make up the huge space of irrelevant operators.

$\{K^*\}$

eg) For the Ising model in a h-field. Two variables (T, h) must be adjusted to get to the critical point.

ie. while h is relevant, other odd order couplings (to $m^3, m^5, \ldots$) do not give relevant operators.

Last: scaling hypothesis:

$$r' = b^{-1} r \qquad N' = b^{-d} N$$

$$\xi' = b^{-1} \xi \qquad f' = b^{d} f$$

ng dimen

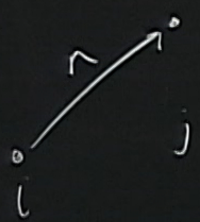
Scaling dimension x: $A' = b^{x} A$

e.g. for the Ising variable $\sigma' = b^{d-y_h}\,\sigma$

(same for $\varphi' = b^{d-y_h}\,\varphi$ in real space)

In particular. $G'_{ij} = b^{2d-2y_h}\, G_{ij}$

$u_t \sim t$



$$G(r,t,h=0) = b^{-2d+2y_h}\, G\left(b^{-1}r,\ b^{y_t}u_t\right)$$

Precisely at $T=T_c$ $\xi \to \infty$, no other length scale

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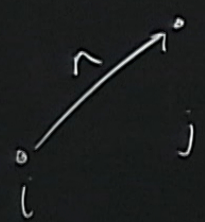

$$G(r,t,h=0) = b^{-2d+2y_h}\, G\left(b^{-1}r,\, b^{y_t}u_t\right)$$

<u>Precisely at $T=T_c$</u> $\xi \to \infty$, no other lengthscale (besides r)

if I set $b=r$ at $t=0$ precisely, then this would tell me

$$\boxed{G(r,0) \propto r^{-2d+2y_h}}$$

What exactly is this exponent? (involves η → not contained in Landau theory)

c.f. <u>Assignment 3.</u> : $G(r,0) \sim r^{-d+2}$

But this isn't exactly correct. In this case an <u>irrelevant field</u> affects the result: in this case the lattice constant $\underline{a}$.

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if you took a into account

imagine: $G(r,a) = b^{-d+2} G(b^{-1}r, b^{-1}a)$

if I set $b=r$ $= r^{-d+2} G(1, a/r)$

just because $a/r \ll 1$ in this case doesn't mean $G(1, \frac{a}{r}) \to 1$ or otherwise non-vanishing. Could be that $G(1,\frac{a}{r}) \xrightarrow[\frac{a}{r}\to 0]{} \left(\frac{a}{r}\right)^{\eta}$

$$\Rightarrow G(r,a) \propto a^{\eta} \boxed{r^{-d+2-\eta}}$$

This is the definition of the anomalous dimension.

so: at $T=T_c$ $\quad G(r,0) \propto r^{-2d+2y_h} \propto r^{-d+2-\eta}$

$$\Rightarrow \boxed{\eta = d - 2y_h + 2}$$

together with $\nu = \frac{1}{y_t}$ these give another set of scaling relationships

Hyperscaling relations

$$\alpha = 2 - d\nu, \quad \beta = \frac{\nu(d-2+\eta)}{2}, \quad \gamma = \nu(2-\eta), \quad \delta = \frac{d+2-\eta}{d-2+\eta}$$

Unlike the previous (thermodynamic) scaling relations, these in practice can be violated.

Finite size scaling: there are no phase transitions in finite size systems.

Imagine I have an L^d size system with Linear size L.

$$f_s(\{K\}, L^{-1}) = b^{-d} f_s(\{K'\}, bL^{-1})$$

So L^{-1} is like a relevant variable with $y=1$

- consider n (large, fixed) number of R.G. steps, and calculate any physical observable (e.g. $\chi = \frac{\partial^2 f}{\partial h^2}\Big|_{h=0}$)

$$\Rightarrow \quad f_s(u_t, u_h, L^{-1}) = b^{-nd} f_s(b^{ny_t} u_t, b^{ny_h} u_h, b^n L^{-1})$$

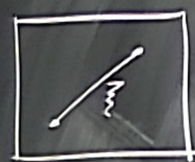
recall: $b^n = \left(\frac{K_t}{u_t}\right)^{\frac{1}{y_t}}$

$$f_s(u_t, u_h, L^{-1}) = \left(\frac{u_t}{K_t}\right)^{d/y_t} f_s\left(K_t, \left(\frac{K_t}{u_t}\right)^{\frac{y_h}{y_t}} u_h, \left(\frac{K_t}{u_t}\right)^{\frac{1}{y_t}} L^{-1}\right)$$

$$= \left|\frac{t}{t_0}\right|^{d/y_t} \Phi\left(\frac{h/h_0}{(t/t_0)^{y_h/y_t}}, |t|^{-\nu} L^{-1}\right)$$

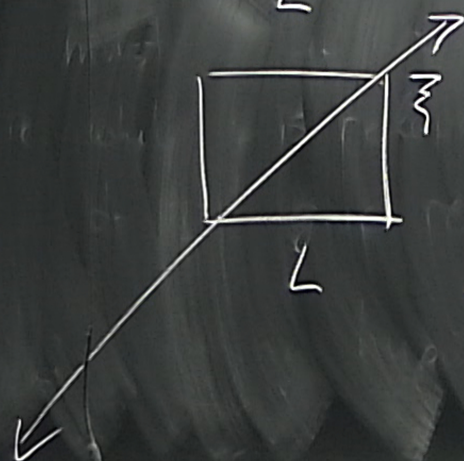
$$\left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} = \chi(t,L) \sim |t|^{-\gamma}\, \tilde{\Phi}\left(|t|^{-\nu} L^{-1}\right)$$

<u>Note:</u>



L

for $\xi \ll L$, your results will compare favourably to the true Thermodynamic Limit



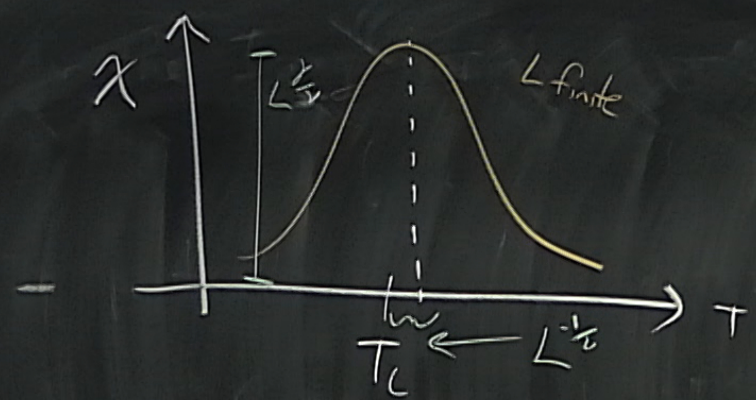
L

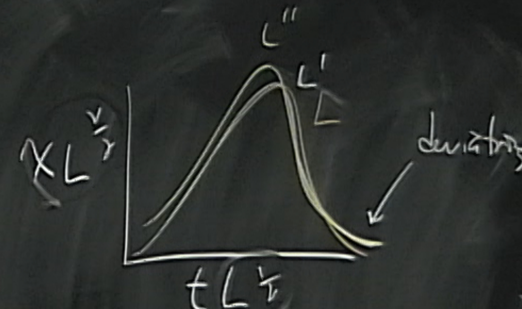
for $\xi \to L$, $|t|^{-\nu} \sim L$, $L^{-\frac{1}{\nu}} \simeq |t|$

therefore $\chi(t,L) \sim L^{\gamma/\nu}\, \bar{\Phi}\left(t L^{1/\nu}\right)$

$\chi(t,L) \sim L^{\gamma/\nu}\, \Phi(tL^{1/\nu})$

here that function Φ is analytic, but still universal.

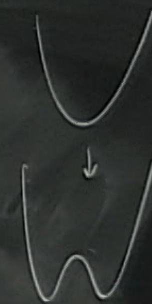




So a plot of $\chi L^{\nu/\gamma}$ versus $tL^{1/\nu}$ will collapse on to the universal function Φ

Wilsonian RG

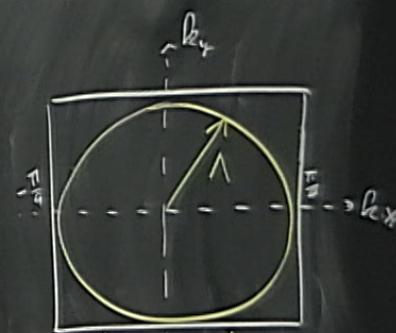
$$S[\varphi] = \int d^dx \left[\frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{r}{2}\varphi^2 + \frac{u}{4}\varphi^4\right]$$



RG: thin out short-wavelength D.O.F. from this functional, leaving what is importantant for critical phenomena.

We will work with the F.T. of the fields

$$\varphi(\vec{x}) = \int_0^\Lambda \frac{d^dk}{(2\pi)^d}\,\varphi(\vec{k})\,e^{i\vec{k}\cdot\vec{x}}$$



Λ is the "high-momentum cutoff"

in this case doesn't mean $G(1, \frac{a}{r}) \to 1$

Think back to real space

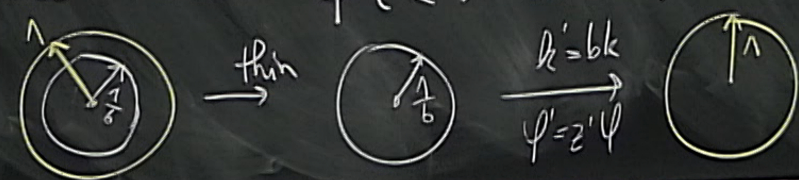
1) thin degrees of freedom $\left(N' = b^{-d} N\right)$
2) rescale lengths

This is done by ① tracing over all fields lying in the spherical shell

$$\Lambda/b < k < \Lambda \qquad (\text{like increasing } a' \to ba)$$

② Rescale lengths $k' = bk$ (since $x' = x/b$)

③ Rescale fields $\varphi'(k') = Z^{-1}\varphi(k)$ (Z^{-1} relates to a scaling dimension b^{x})



The first step in the RG is to introduce a s[...]
$b > 1$, separate out "fast" and "slow" modes

$$\varphi(\vec{x}) = \varphi_<(\vec{x}) + \varphi_>(\vec{x})$$

where
$$\varphi_<(\vec{x}) = \int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d}\, \varphi(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$
slow modes

$$\varphi_>(\vec{x}) = \int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\, \varphi(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$
fast modes

and rewrite the LGW functional $S[\varphi]$ in terms of these generally there are 3 terms we will calculate

$$S[\varphi_<, \varphi_>] = S_0[\varphi_<] + S_0[\varphi_>] + S_{int}[\varphi_<, \varphi_>]$$

Consider the quadratic parts first

$$S_0[\varphi] = \int d^d x \left[\frac{1}{2} (\vec{\nabla}\varphi)^2 + \frac{r}{2}\varphi^2 \right]$$

in momentum space

$$= \frac{1}{2}\int_0^\Lambda \frac{d^d k}{(2\pi)^d} \varphi(-\vec{k})\varphi(\vec{k})(k^2 + r)$$

$\sim t$

"mass" $\sim m$

$$= \frac{1}{2}\int_0^\Lambda \frac{d^d k}{(2\pi)^d} |\varphi(\vec{k})|^2 (k^2 + r)$$

$$S_0[\varphi] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d}\,(k^2+r)\,|\underbrace{\varphi_<(\vec{k})}|^2 + \frac{1}{2}\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\,(k^2+r)\,|\underbrace{\varphi_>(\vec{k})}|^2$$

$\varphi_<$: – $\varphi(\vec{k})$, $|k| < \frac{1}{b}$; – 0 otherwise (block spins)

$\varphi_>$: $\varphi(\vec{k})$ $\frac{\Lambda}{b} < |k| < \Lambda$; 0 otherwise (spins within a block)

$$S_0[\varphi_<] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^dk}{(2\pi)^d}\,(k^2+r)\,|\varphi_<(\vec{k})|^2$$

$$S_0[\varphi_>] = \frac{1}{2}\int_{\Lambda/b}^{\Lambda} \frac{d^dk}{(2\pi)^d}\,(k^2+r)\,|\varphi_>(\vec{k})|^2$$

So a plot of $\chi L^{\gamma/\nu}$ versus $tL^{1/\nu}$ will collapse

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The φ^4 term is more difficult

$$S_{int}[\varphi] = \frac{u}{4}\int d^d x \; \varphi^4(\vec{x})$$

$$= \frac{u}{4}\int d^d x \int_0^{\Lambda} \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_4}{(2\pi)^d} \; \varphi(\vec{k}_1)\varphi(\vec{k}_2)\varphi(\vec{k}_3)\varphi(\vec{k}_4) \, e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\cdot\vec{x}}$$

<u>note</u>: $$\int d^d x \, e^{i(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\cdot\vec{x}} = (2\pi)^d \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

<u>use</u> $$\varphi(\vec{k}) = \varphi_<(\vec{k}) + \varphi_>(\vec{k})$$

$$S_{int} = \frac{u}{4} \int_0^{\Lambda} \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_4}{(2\pi)^d} \cdot$$

$$\left[\varphi_<(\vec{k}_1) + \varphi_>(\vec{k}_1)\right]\left[\varphi_<(\vec{k}_2) + \varphi_>(\vec{k}_2)\right]\left[\varphi_<(\vec{k}_3) + \varphi_>(\vec{k}_3)\right]\left[\varphi_<(\vec{k}_4) + \varphi_>(\vec{k}_4)\right]$$

$$(2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

clearly there are coupling terms between slow & fast modes

$$S_{int}[\varphi_<, \varphi_>]$$