Title: 2016/2017 Statistical Mechanics 2 - Roger Melko - Lecture 10

Date: Feb 03, 2017 10:30 AM

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Abstract:

Generalization to a set of variables $\{K\}$

$N' = b^{-d} N \Rightarrow \quad f_s(\{K\}) = b^{-d} f(\{K'\})$

Relevant, irrelevant & marginal

$K' = R(K) \xrightarrow{\text{generalize}} \{K'\} = R(\{K\})$

Let's assume the starting Hamiltonian is near the fixed point.

$K_n = K_n^* + \delta K_n$ after the RG step $K'_n = K_n^* + \delta K_n'$

defines δK_n defines $\delta K_n'$

↑ defines δK_n

defines δK_n

Taylor expand

$$K_n' = K_n^* + \sum_m \left.\frac{\partial K_n'}{\partial K_m}\right|_{K_m = K_m^*} \delta K_m + \cdots$$

"linearized RG Equations"

or

$$(K_n' - K_n^*) = \delta K_n' = \sum_m T_{mn} (K_m - K_m^*)$$

↑ defines this matrix

Let the eigenvalues of T be λ^i

eigenvectors of T are e^i

$$\sum_m e^i_m T_{nm} = \lambda^i e^i_n$$

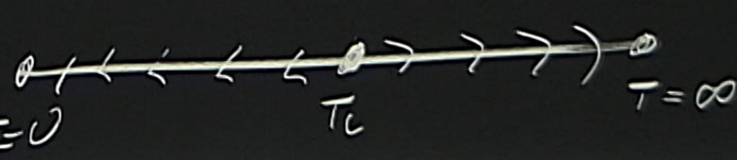
Define a "scaling variable" $u_i \equiv \sum_n e^i_n (K_n - K_n^*)$

$$u'_i = \sum_n e^i_n (K'_n - K_n^*) = \sum_{nm} e^i_n T_{nm} (K_m - K_m^*)$$

$$= \sum_m \lambda^i e^i_m (K_m - K_m^*) = \lambda^i u_i = b^{y_i} u_i$$

This defines the y_i in general. These tell you what is relevant & irrelevant

Recall:



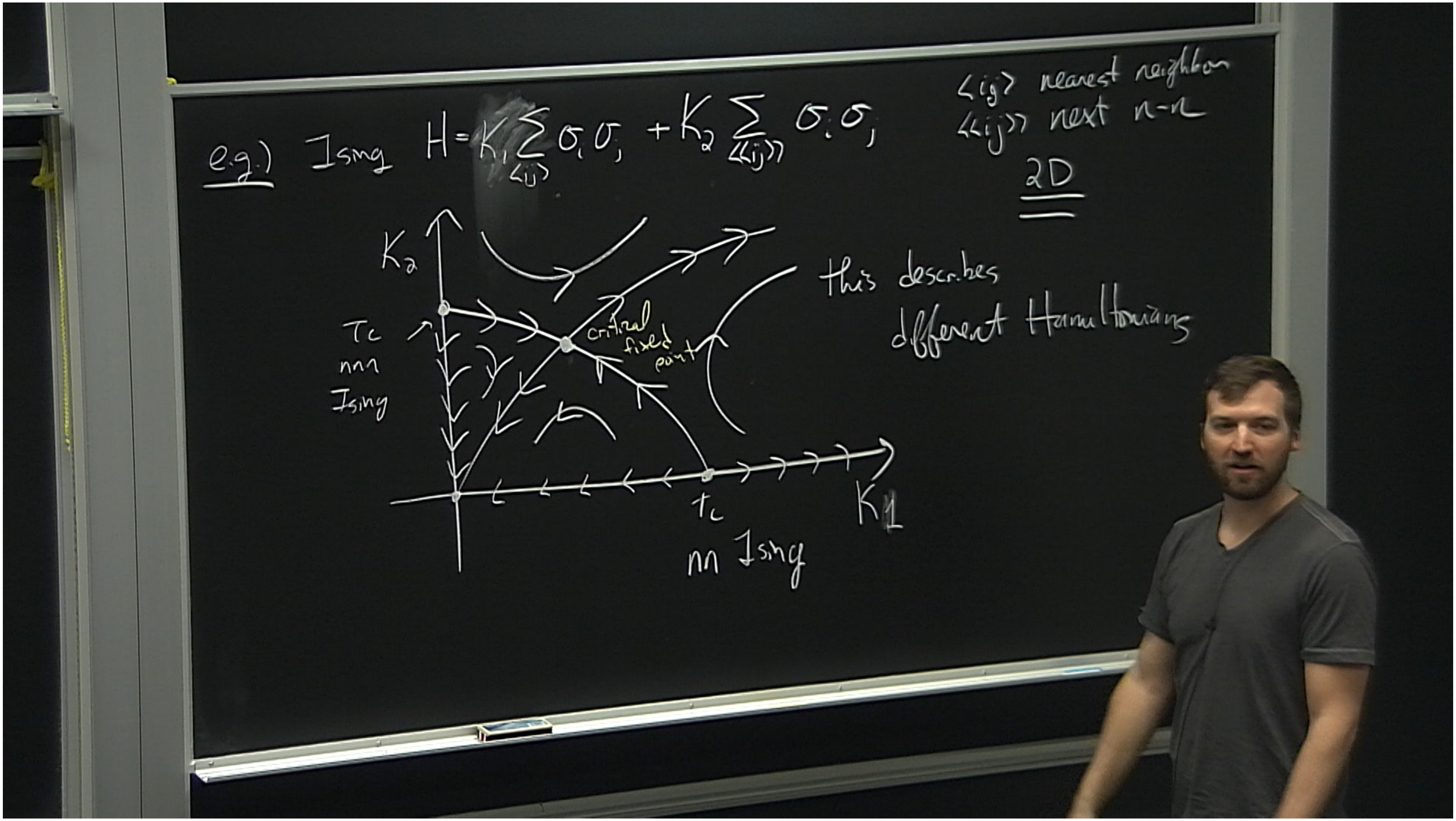
1D

Distinguish 3 cases:

1) $y_i > 0$ u_i is relevant. Repeated RG iterations drive u_i away from the fixed point.

2) $y_i < 0$ u_i is irrelevant. u_i will iterate to zero (assuming we start close enough to the fixed point).

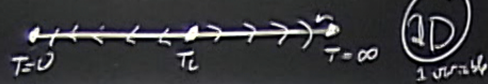
3) $y_i = 0$ u_i is marginal. Need higher order terms

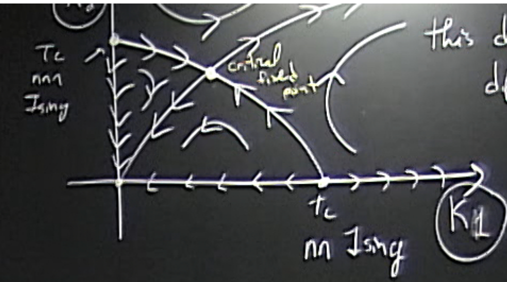

e.g.) Ising $H = K_1 \sum_{\langle ij \rangle} \sigma_i \sigma_j + K_2 \sum_{\langle\langle ij \rangle\rangle} \sigma_i \sigma_j$
⟨ij⟩ nearest neighbor
⟨⟨ij⟩⟩ next n-n
2D
K_2
T_c
nnn
Ising
critical
fixed
point
this describes
different Hamiltonians
T_c
nn Ising
K_1

$$u_i = \sum_n e_n (K_n - K_n^*) = \sum_{nm} e_n \ldots (K_m - K_m^*)$$

$$= \sum_m \lambda^i e_m^i (K_m - K_m^*) = \lambda^i u_i = b^{y_i} u_i$$

This defines the y_i in general. These tell you what is relevant & irrelevant

Recall: 



this describes different Hamiltonians

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Back to $f_s(\{K\}) = b^{-d} f_s(\{K'\})$

Ignore (for now) irrelevant variables – assume two relevant variables: u_t and u_h (correspond to physical variables)

Near the critical point these vanish – write

assume: $u_t = \frac{t}{t_0} + O(t^2, h^2)$, $u_h = \frac{h}{h_0} + O(t^2, h^2)$

$$f_s(u_t, u_h) = b^{-d} f_s(b^{y_t} u_t, b^{y_h} u_h)$$

Back to $f_s(\{K\}) = b^{-d}\underbrace{f_s(\{K'\})}$, $f' = b^d f$

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$$\sum_m e^i_m T_{nm} = \lambda^i e^i_n$$

absorb K_t into the definition of constant t_0

$$\boxed{f_s(t,h) = \left|\frac{t}{t_0}\right|^{d/y_t} \Phi\left(\frac{h/h_0}{|t/t_0|^{y_h/y_t}}\right)}$$

– where Φ is a universal scaling function.

From this we can construct all of the critical exponents:

- Specific heat $(h=0)$, $\left.\frac{\partial^2 f}{\partial t^2}\right|_{h=0} \propto |t|^{-\alpha}$ that is how we define α

so $\partial t^2 \big|_{h=0} \propto$ … y_t

- magnetization

$$\frac{\partial f}{\partial h}\bigg|_{h=0} \text{ defines } m \sim |t|^{\beta}$$

$$\longrightarrow (t)^{\frac{d-y_h}{y_t}} \Rightarrow \boxed{\beta = \frac{d-y_h}{y_t}}$$

- susceptibility

$$\frac{\partial^2 f}{\partial h^2}\bigg|_{h=0} \propto |t|^{\frac{(d-2y_h)}{y_t}}$$

$$\boxed{\gamma = \frac{2y_h-d}{y_t}}$$

Since thermodynamic exponents are given in terms of y_t and y_h they have scaling relations:

$$\boxed{\alpha + 2\beta + \gamma = 2} \qquad \boxed{\alpha + \beta(1+\delta) = 2}$$

Note. since the correlation length scales as

$$\xi(u_t, u_h) = b^n \, \xi(b^{n y_t} u_t, b^{n y_h} u_h)$$

$$= \left(\frac{K_t}{u_t}\right)^{1/y_t} \xi\left(K_t,\ \left(\frac{K_t}{u_t}\right)^{\frac{y_h}{y_t}} u_h\right) \Rightarrow \boxed{\nu = \frac{1}{y_t}}$$

Correlation function

Recall the definition of the scaling dimension $A' = b^x A$

The microscopic variable σ has a scaling dimension

$$\sigma' = b^x \sigma$$

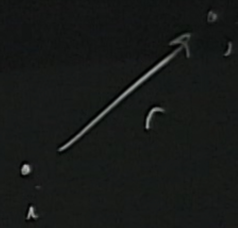
$$\sigma_I' = b^x \sigma_i$$

$\vec{r}_i$ associated with.

G is define

G is defined $G_{ij} = G(|\vec{r}_i - \vec{r}_j|) = G(r)$

$$= \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$$



expect: $m' = b^x m$ for the magnetization

$$G'(r) = b^{2x} G(r)$$

After one step of RG

$$G(r, t, h=0) = b^{-2x} G\left(b^{-1} r, b^{y_t} \frac{t}{t_0}\right)$$

$$m(t, h=0) = b^{-x} m\left(b^{y_t} \frac{t}{t_0}, 0\right)$$

after 1-iterations $\quad f_s(u_t, u_h) = b^{-d} f_s(b^{y_t} u_t, b^{y_h} u_h) \quad u_h = \frac{h}{h_0}$

$$\Rightarrow \frac{\partial f}{\partial h}\Big|_{h=0} = m(t, h=0) = b^{-d}\, b^{y_h} \frac{1}{h_0} \underbrace{\frac{\partial}{\partial h} f_s}_{m}\left(b^{y_t} \frac{t}{t_0}, 0\right)$$

$$\Rightarrow \quad m(t) = b^{-d+y_h}\, m\left(b^{y_t} \frac{t}{t_0}, 0\right)$$

$$\text{or} \quad b^{-x} = b^{-d+y_h} \qquad \boxed{x = d - y_h}$$

So

$$G(r,t) = b^{-2d+2y_n} G\left(b^{-1} r, b^{y_t} \frac{t}{t_0}\right) \quad (*)$$

as previous → iterate to some fixed n $\quad b^{n y_t} u_t = K_t = \text{fixed}$

$$G(r,t) = b^{2n(y_n - d)} G\left(b^{-n} r, b^{n y_t} t/t_0\right)$$

$$\Rightarrow \boxed{G(r,t) = t^{\frac{2(d-y_n)}{y_t}} \Phi\left(r t^{\frac{1}{y_t}}\right)}$$

know $e^{-\frac{r}{\xi}}$

$\Rightarrow$ again $\boxed{\nu = \frac{1}{y_t}}$