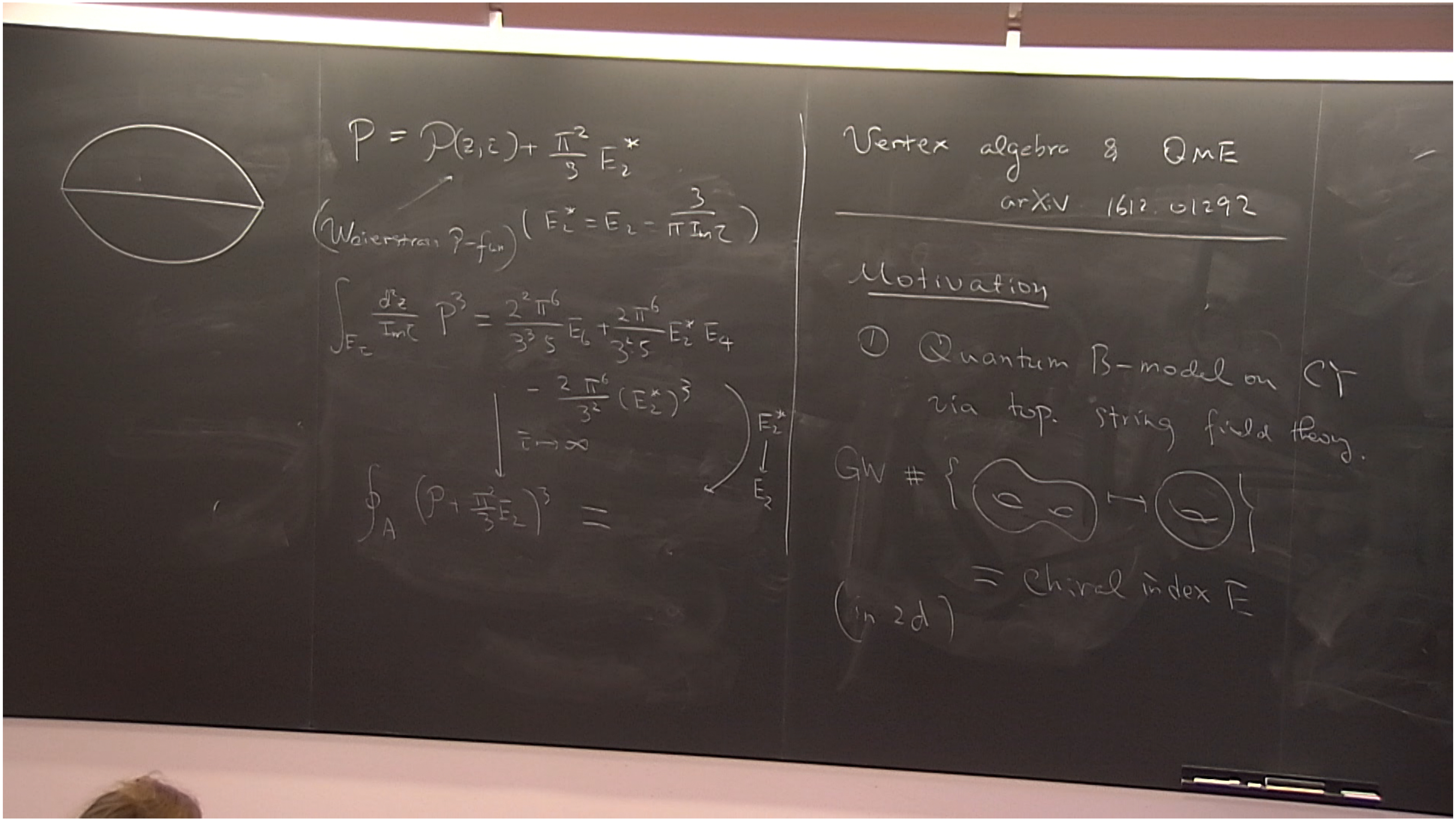
Title: Vertex algebras and quantum master equation.

Date: Jan 18, 2017 02:00 PM

URL: http://pirsa.org/17010071

Abstract: <p>We study the effective BV quantization theory for chiral deformation of two dimensional conformal field theories. We establish an exact correspondence between renormalized quantum master equations for effective functionals and Maurer-Cartan equations for chiral vertex operators. The generating functions are proven to be quasi-modular forms. As an application, we construct an exact solution of quantum B-model (BCOV theory) in complex one dimension that solves the higher genus mirror symmetry conjecture on elliptic curves. The talk is based on arXiv: 1612.01292[math.QA]</p>



$$P = \wp(z,\tau) + \frac{\pi^2}{3} E_2^*$$

(Weierstrass $\wp$-fun) $\left(E_2^* = E_2 - \frac{3}{\pi \operatorname{Im}\tau} \right)$

$$\int_{E_\tau} \frac{d^2 z}{\operatorname{Im}\tau} P^3 = \frac{2^2 \pi^6}{3^3 5} E_6 + \frac{2\pi^6}{3^2 5} E_2^* E_4 - \frac{2\pi^6}{3^2} (E_2^*)^3$$

$\bar\tau \mapsto \infty$

$E_2^* \to E_2$

$$\oint_A \left(\wp + \frac{\pi^2}{3} E_2 \right)^3 =$$

Vertex algebra & QME

arXiv 1612.01292

Motivation

① Quantum B-model on CY via top. string field theory.

$GW\ \#\ \{ \ \mapsto\ \}$

$=$ Chiral index E

(in 2d)

on CY

ld theory.

E

② Why integrable hierarchy appears naturally in B-model?

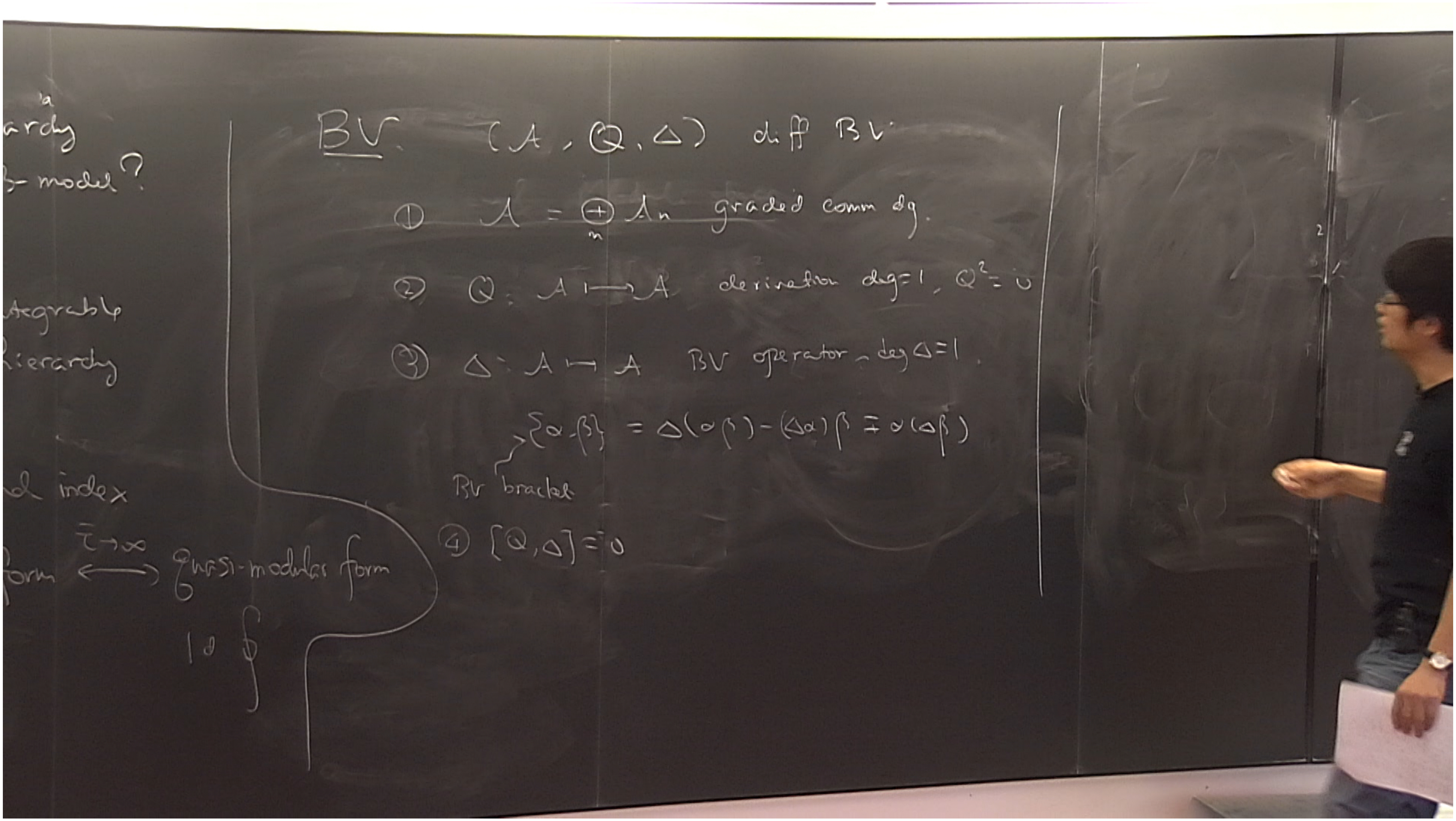
Zwiebach's string vertex $\overset{?}{\Longrightarrow}$ integrable hierarchy

③ Chiral deformation and index

④ almost hol. modular form $\xleftrightarrow{\bar{\tau}\to\infty}$ quasi-modular form

2d $\int_E$

1d $\oint$



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-model?

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d index

$\bar{\tau} \to \infty$
form $\longleftrightarrow$ quasi-modular form

1∂ ∫

BV. $(\mathcal{A}, Q, \Delta)$ diff BV

① $\mathcal{A} = \bigoplus_n \mathcal{A}_n$ graded comm alg.

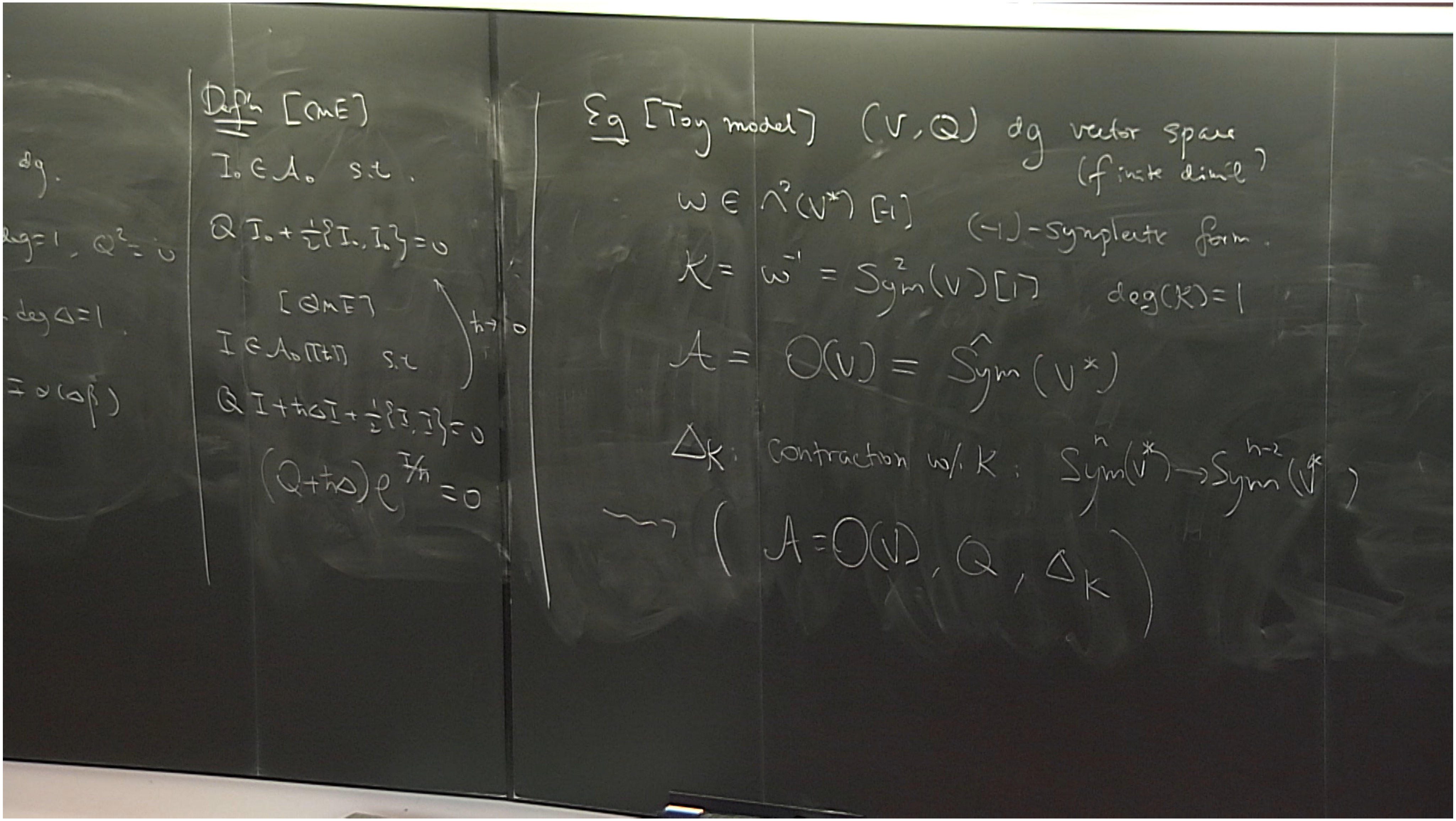
② $Q : \mathcal{A} \longmapsto \mathcal{A}$ derivation $\deg = 1$, $Q^2 = 0$

③ $\Delta : \mathcal{A} \longmapsto \mathcal{A}$ BV operator, $\deg \Delta = 1$.

$$\{\alpha, \beta\} = \Delta(\alpha\beta) - (\Delta\alpha)\beta \mp \alpha(\Delta\beta)$$

BV bracket

④ $[Q, \Delta] = 0$



dg.

deg = 1, $Q^2 = 0$

deg $\Delta = 1$.

$\mp \alpha(\Delta\beta)$

Def'n [CME]

$I_0 \in \mathcal{A}_0$ s.t.

$$Q I_0 + \frac{1}{2}\{I_0, I_0\} = 0$$

[QME]

$I \in \mathcal{A}_0[[\hbar]]$ s.t.

$$Q I + \hbar \Delta I + \frac{1}{2}\{I, I\} = 0$$

$$(Q + \hbar\Delta) e^{I/\hbar} = 0$$

$\hbar \to 0$

Eg [Toy model] (V, Q) dg vector space (finite dim'l)

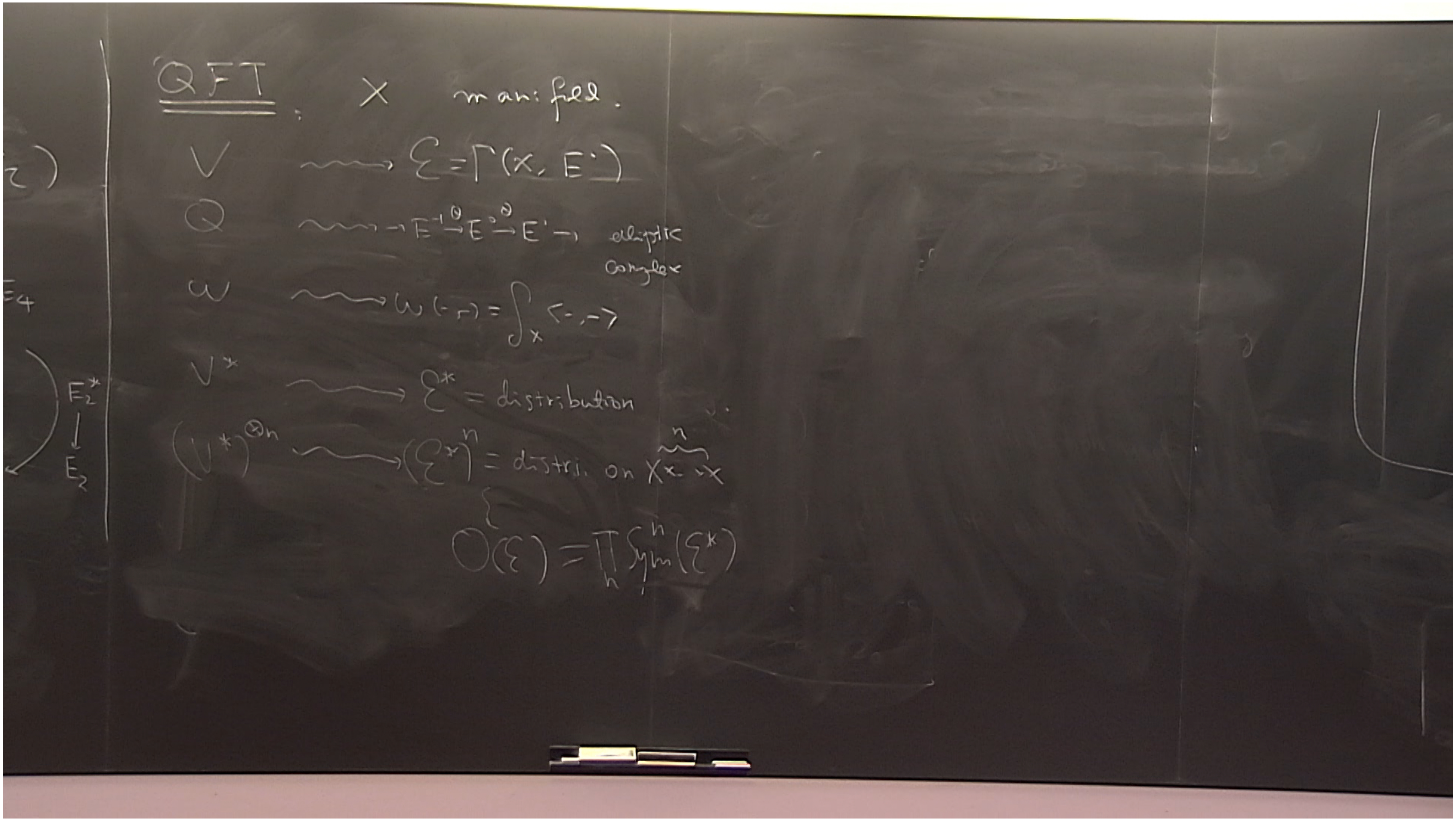
$\omega \in \wedge^2(V^*)[-1]$ (-1)-symplectic form.

$K = \omega^{-1} = \mathrm{Sym}^2(V)[1]$ $\deg(K) = 1$

$$\mathcal{A} = \mathcal{O}(V) = \widehat{\mathrm{Sym}}(V^*)$$

Δ_K: contraction w/ K: $\mathrm{Sym}^n(V^*) \to \mathrm{Sym}^{n-2}(V^*)$

$$\rightsquigarrow \left(\mathcal{A} = \mathcal{O}(V), Q, \Delta_K\right)$$



$$E_2^* \xrightarrow{} E_2$$

QFT : X manifold.

$$V \rightsquigarrow \mathcal{E} = \Gamma(X, E^\cdot)$$

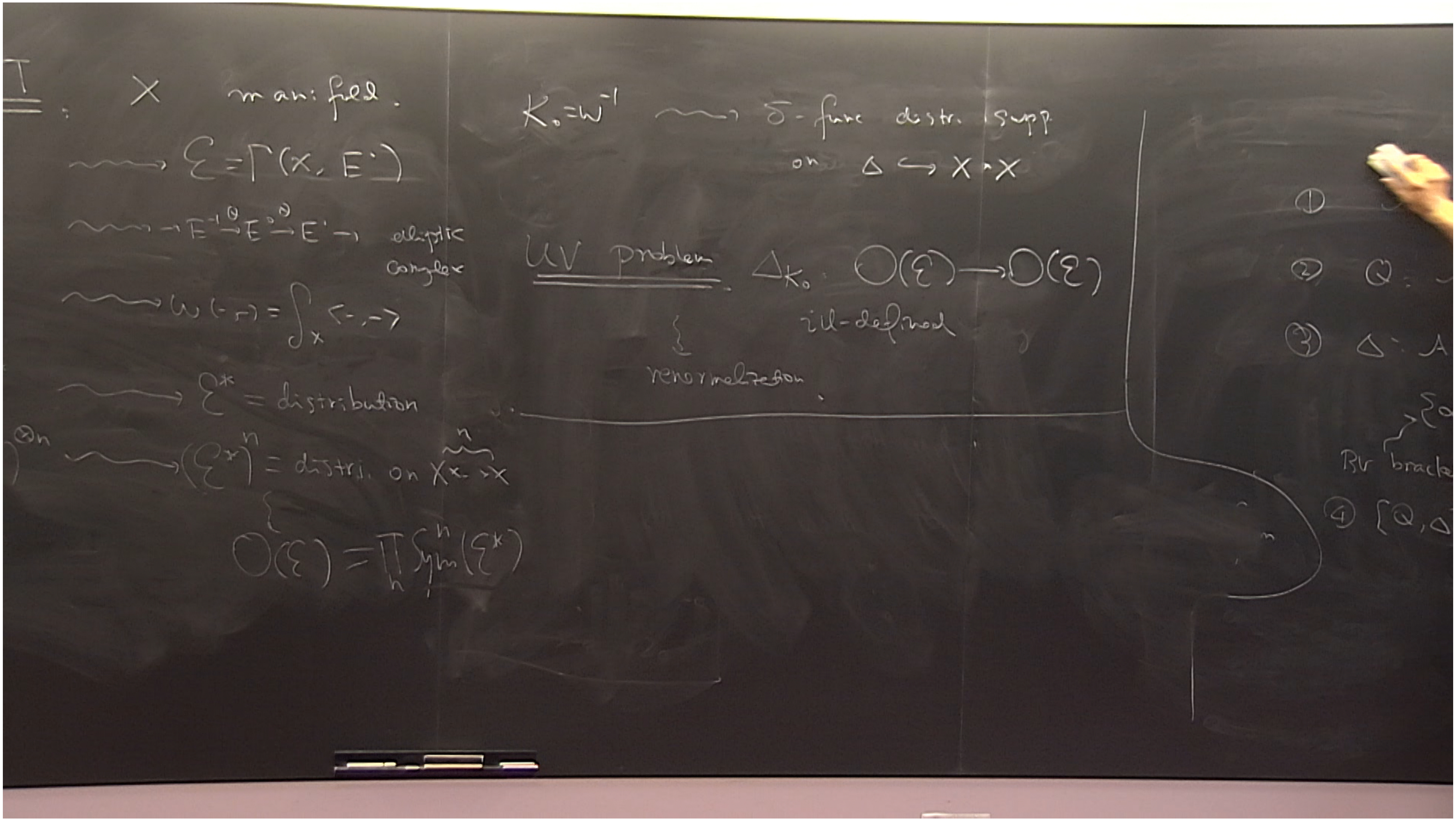
$$Q \rightsquigarrow \to E^{-1} \xrightarrow{\partial} E^0 \xrightarrow{\partial} E^1 \to$$ elliptic complex

$$\omega \rightsquigarrow \omega(-,-) = \int_X \langle -,- \rangle$$

$$V^* \rightsquigarrow \mathcal{E}^* = \text{distribution}$$

$$(V^*)^{\otimes n} \rightsquigarrow (\mathcal{E}^*)^n = \text{distri. on } \overbrace{X \times \cdots \times X}^{n}$$

$$\mathcal{O}(\mathcal{E}) = \prod_n \mathrm{Sym}^n(\mathcal{E}^*)$$



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$)^{\otimes n} \rightsquigarrow (\mathcal{E}^*)^{n} =$ distri. on $\overbrace{X \times \cdots \times X}^{n}$

$\mathcal{O}(\mathcal{E}) = \prod_n \mathrm{Sym}^n(\mathcal{E}^*)$

$K_0 = \omega^{-1} \rightsquigarrow \delta$-func distr. supp. on $\Delta \hookrightarrow X \times X$

UV problem. $\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$

ill-defined

renormalization.

①

② Q :

③ Δ : A

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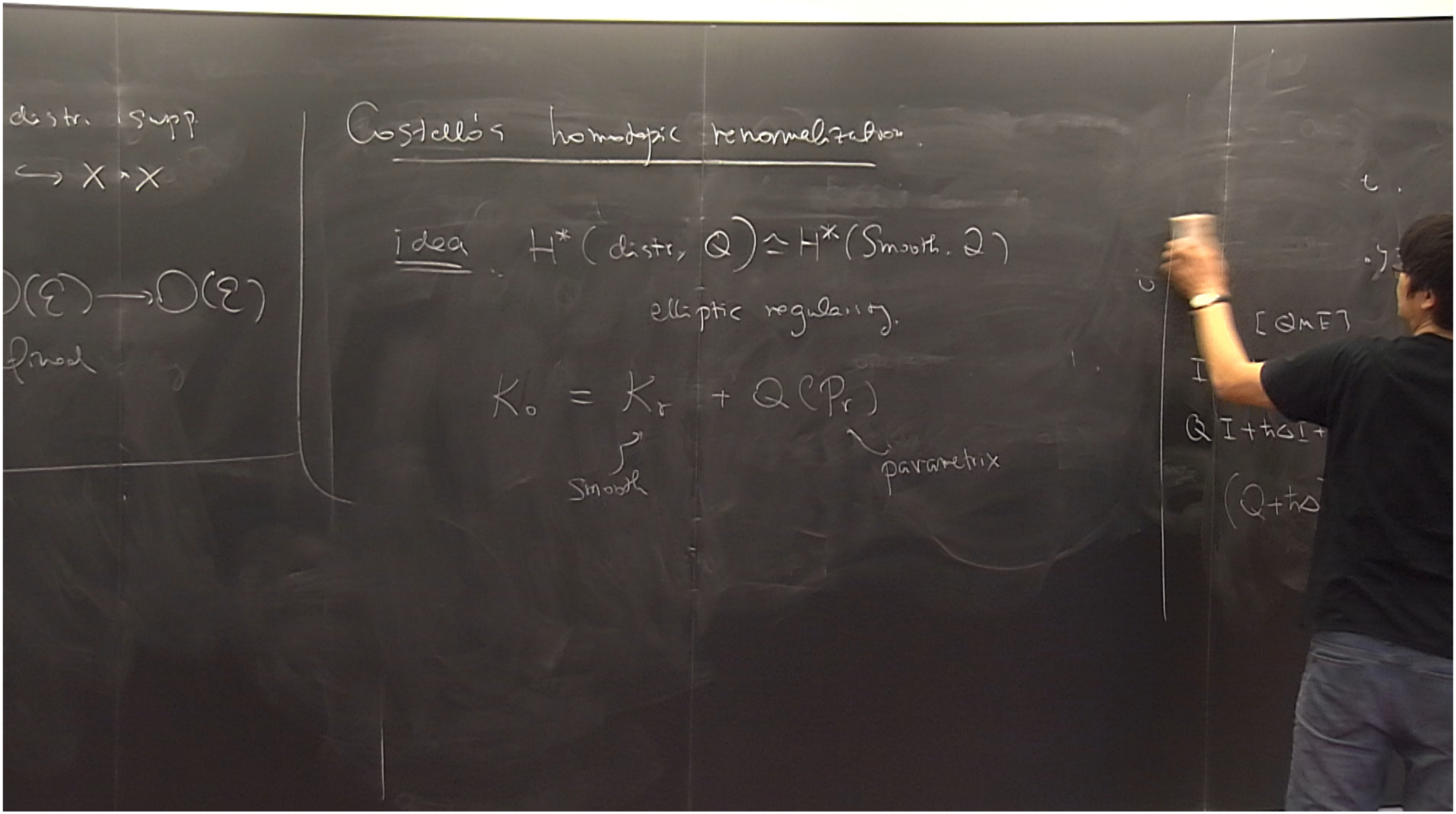
BV bracke

④ $[Q, \Delta$



distr. supp

$\hookrightarrow X \times X$

$)(\mathcal{E}) \longrightarrow \mathcal{O}(\mathcal{E})$

fined

Costello's homotopic renormalization.

idea: $H^*(\text{distr}, Q) \simeq H^*(\text{Smooth}, Q)$

elliptic regularity.

$$K_0 = \underbrace{K_r}_{\text{smooth}} + \underbrace{Q(P_r)}_{\text{paravnetrix}}$$

t .

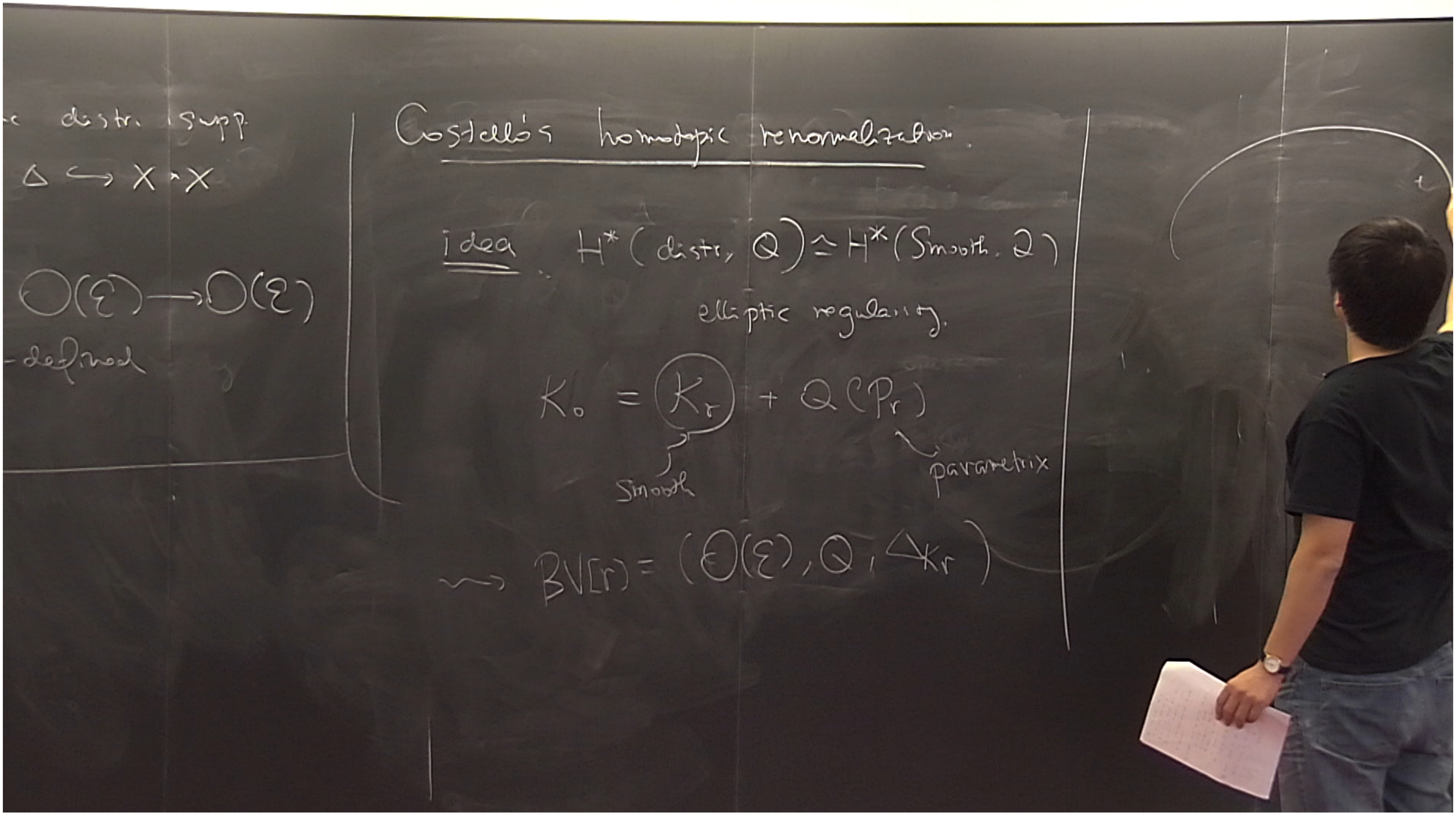
$\cdot) =$

[QME]

I

$Q I + \hbar \Delta I +$

$(Q + \hbar\Delta$



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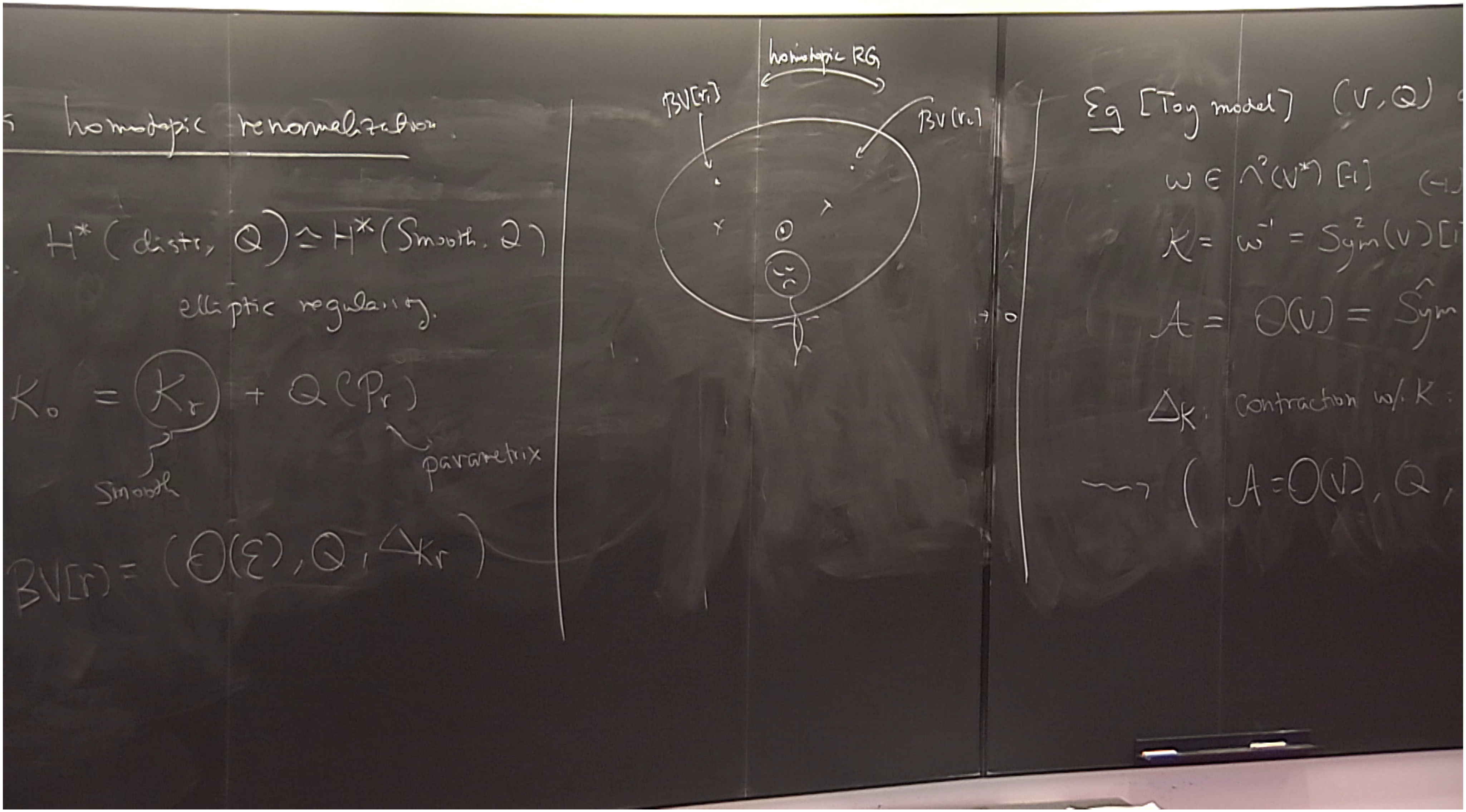
$$\rightsquigarrow BV[r] = (\mathcal{O}(\mathcal{E}), Q, \Delta_{K_r})$$

t



homotopic renormalization

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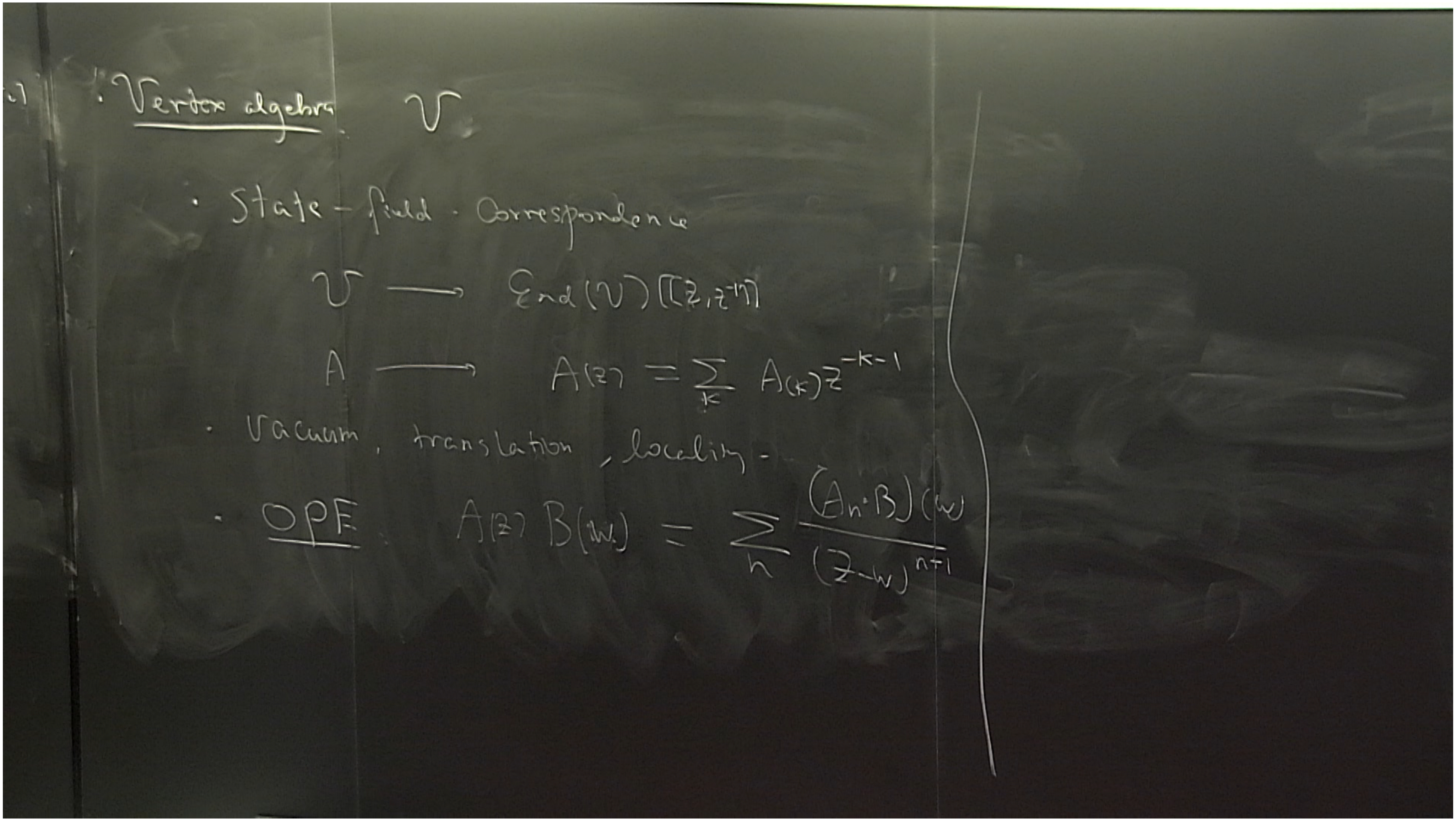
regularity.
Q (P_r)
parametrix
), Q, Δ_{K_r})
Q + ħΔ_{K_r}
homotopic RG
BV[r_1]
BV[r_2]
Counter terms
+ quantum corrections



Vertex algebra V:

- State-field-correspondence

$$V \longrightarrow \mathrm{End}(V)[[z,z^{-1}]]$$

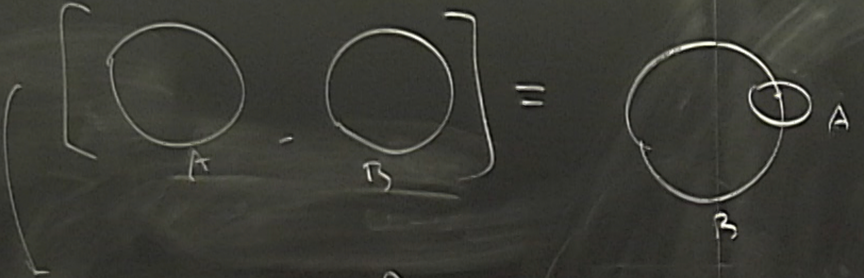
$$A \longrightarrow A(z) = \sum_{k} A_{(k)} z^{-k-1}$$

- vacuum, translation, locality -
- OPE: $A(z)\,B(w) = \sum_{n} \frac{(A_n \cdot B)(w)}{(z-w)^{n+1}}$

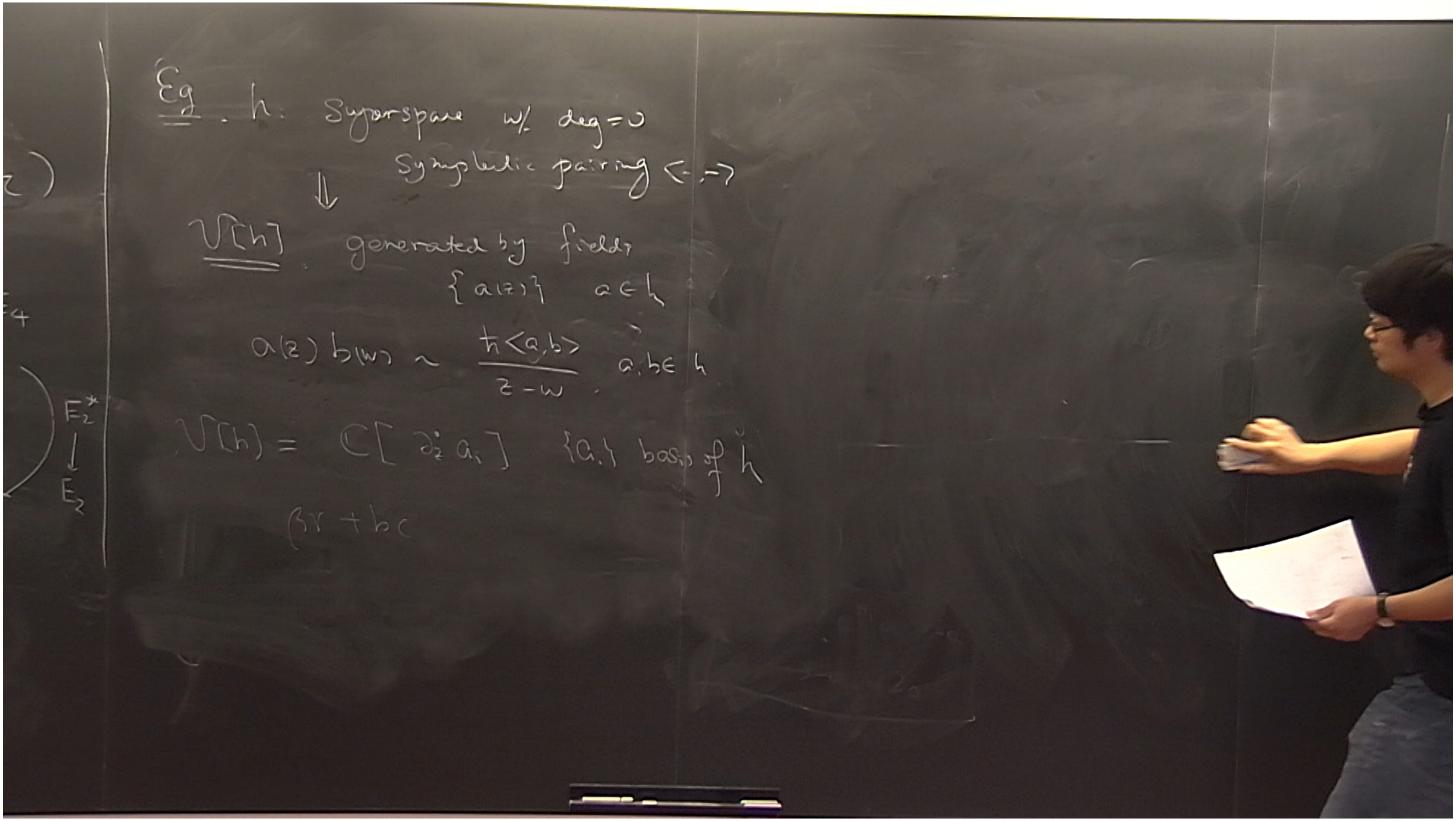
$-k-1$

$$\frac{A_n \cdot B)(w)}{(z-w)^{n+1}}$$

$$\rightsquigarrow \oint V = \mathrm{Span}_{\mathbb{C}} \left\{ \oint z^m A(z) := A_{(m)} \right\}_{\substack{A \in V \\ m \in \mathbb{Z}}} .$$



Lie algebra. $\oint V$



Eg . h: superspace w/ deg $= 0$
symplectic pairing $\langle -, - \rangle$

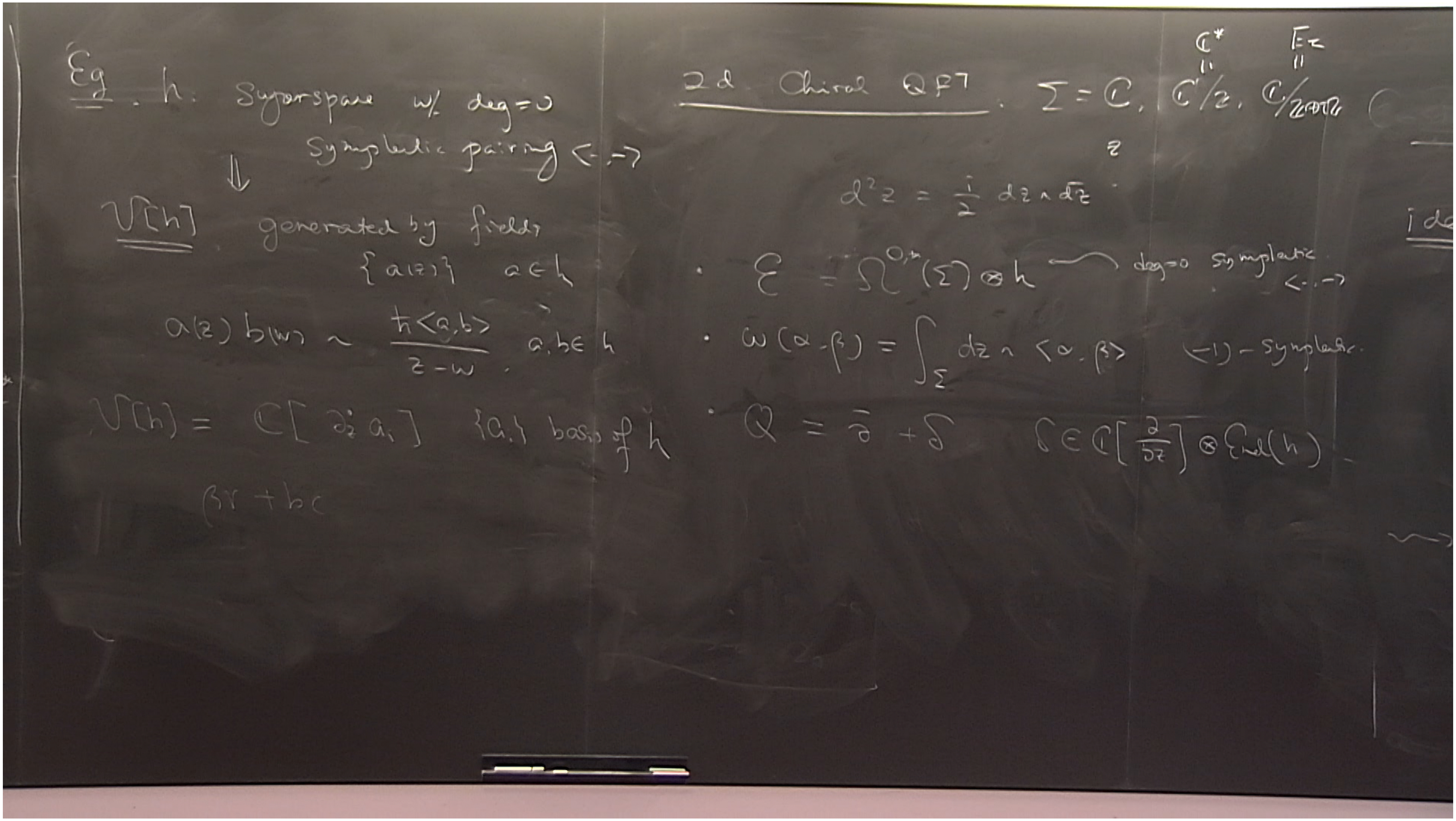
$\Downarrow$

$V[h]$. generated by fields
$\{a(z)\}$ $a \in h$

$$a(z)\, b(w) \sim \frac{\hbar \langle a, b \rangle}{z - w} . \quad a, b \in h$$

$$V[h) = \mathbb{C}[\, \partial_z^i a_i \,] \quad \{a_i\} \text{ basis of } h$$

$\beta\gamma + bc$



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$\beta\gamma + bc$

2d Chiral QFT

$\Sigma = \underset{z}{\mathbb{C}},\ \mathbb{C}/\mathbb{Z} = \mathbb{C}^*,\ \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z} = E_\tau$

$$d^2 z = \frac{i}{2}\, dz \wedge d\bar{z}.$$

- $\mathcal{E} = \Omega^{0,*}(\Sigma) \otimes h \rightsquigarrow$ deg = 0 symplectic $\langle -,- \rangle$
- $\omega(\alpha, \beta) = \int_\Sigma dz \wedge \langle \alpha, \beta \rangle$ (-1)-symplectic.
- $Q = \bar{\partial} + \delta$ $\delta \in \mathbb{C}\left[\frac{\partial}{\partial z}\right] \otimes \mathrm{End}(h)$.

i d

$\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$

symplectic $\langle -,- \rangle$

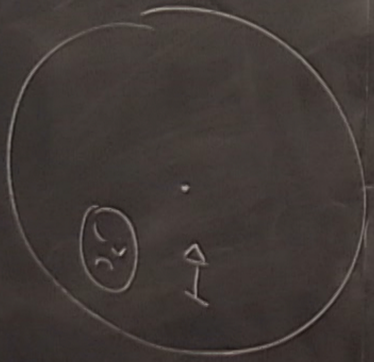
(-1) - symplectic.

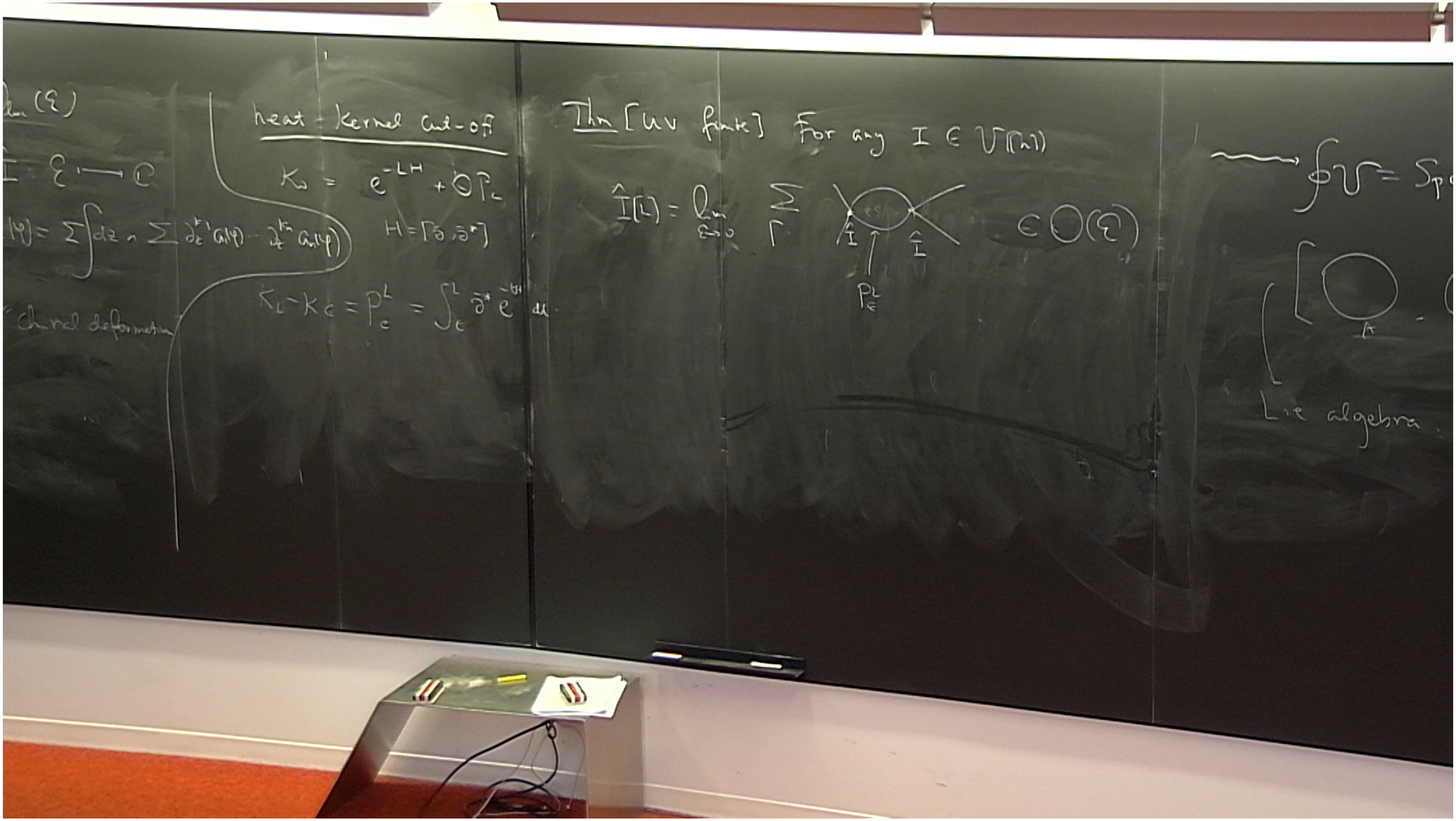
$\otimes \mathrm{End}(h)$

$$\begin{tikzcd}
V[h^*] \arrow[r] & \mathcal{O}_{loc}(\mathcal{E}) \\
\ni I \arrow[r] & \hat{I} : \mathcal{E} \mapsto \mathbb{C}
\end{tikzcd}$$

$$I = \sum (\partial_z^{k_1} a_1) \cdots (\partial_z^{k_n} a_n)$$

$$\hat{I}(\varphi) = \sum \int dz \wedge \sum \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_n} a_n(\varphi)$$





$\mathcal{O}_{loc}(\mathcal{E})$

$I : \mathcal{E} \mapsto \mathbb{C}$

$I(\varphi) = \sum \int dz \wedge \sum \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_n} a_n(\varphi)$

"chiral deformation"

heat-kernel cut-off

$K_L = e^{-LH} + \Theta P_L$

$H = [\bar\partial, \bar\partial^*]$

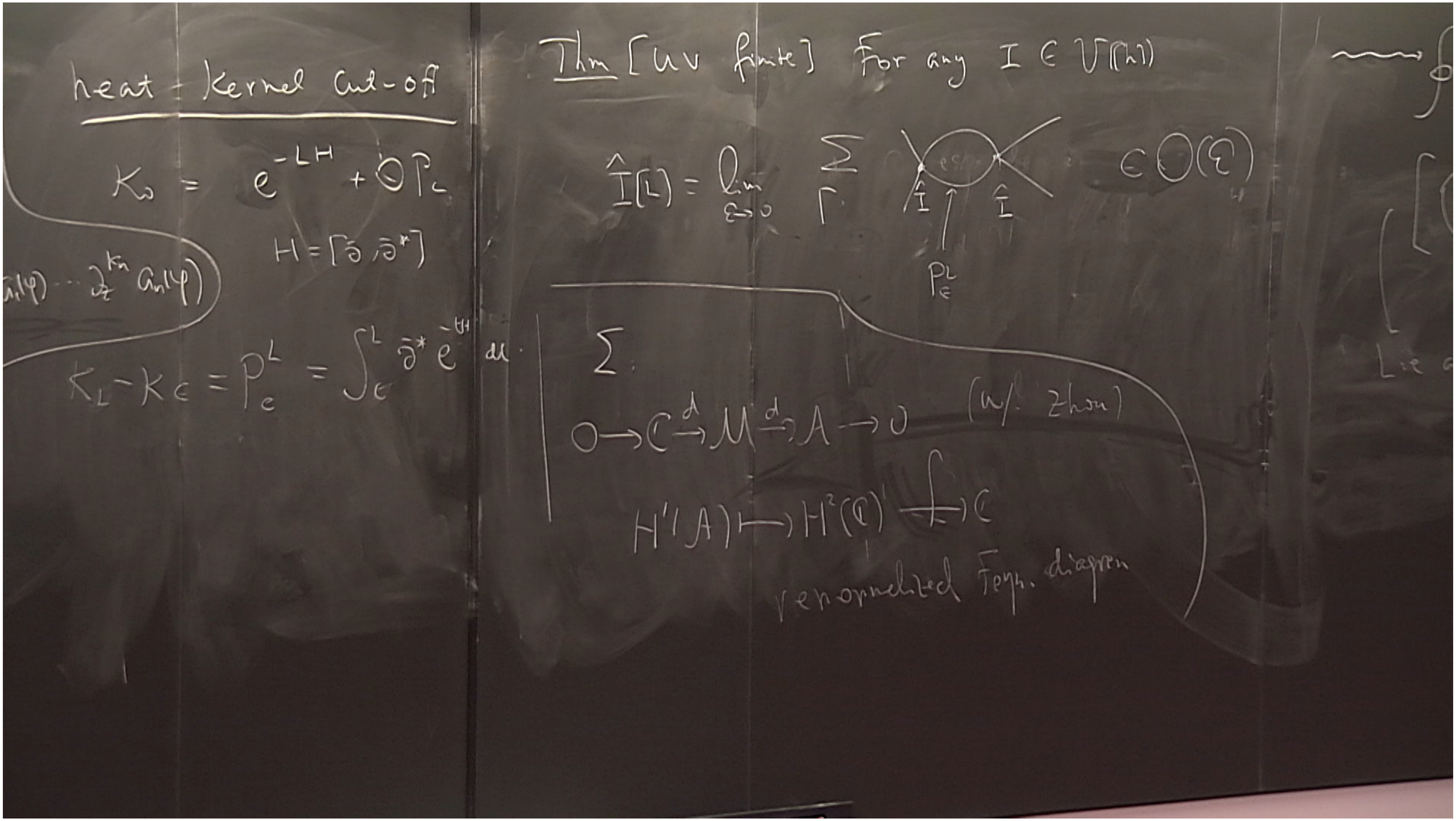
$K_L - K_\epsilon = P_\epsilon^L = \int_\epsilon^L \bar\partial^* e^{-uH} du$

Thm [UV finite] For any $I \in \mathcal{V}[[\hbar]]$

$\hat{I}[L] = \lim_{\epsilon \to 0} \sum_\Gamma$ $\in \mathcal{O}(\mathcal{E})$

$\rightsquigarrow \oint \mathcal{V} = S_p$

Lie algebra



heat-kernel cut-off

$$K_\omega = e^{-LH} + \Theta P_L$$

$$H = [\bar\partial, \bar\partial^*]$$

$$a_1(\varphi) \cdots \partial_z^{k_n} a_n(\varphi))$$

$$K_L - K_\epsilon = P_\epsilon^L = \int_\epsilon^L \bar\partial^* e^{-uH}\, du$$

Thm [UV finite] For any $I \in V[[\hbar]]$

$$\hat{I}[L] = \lim_{\varepsilon \to 0} \sum_\Gamma \quad \in \mathcal{O}(\mathcal{E})$$

$\rightsquigarrow$

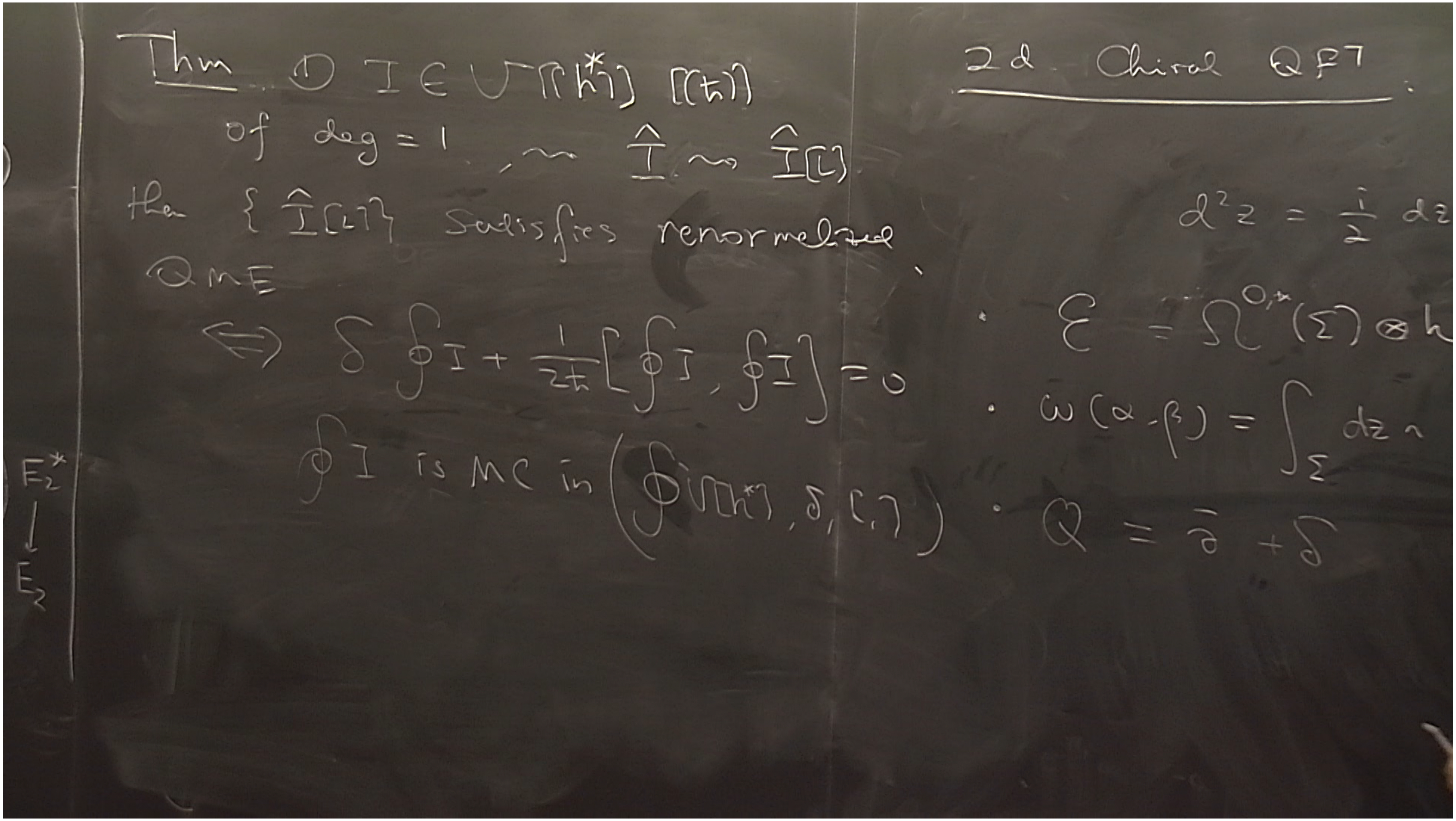
Lie a

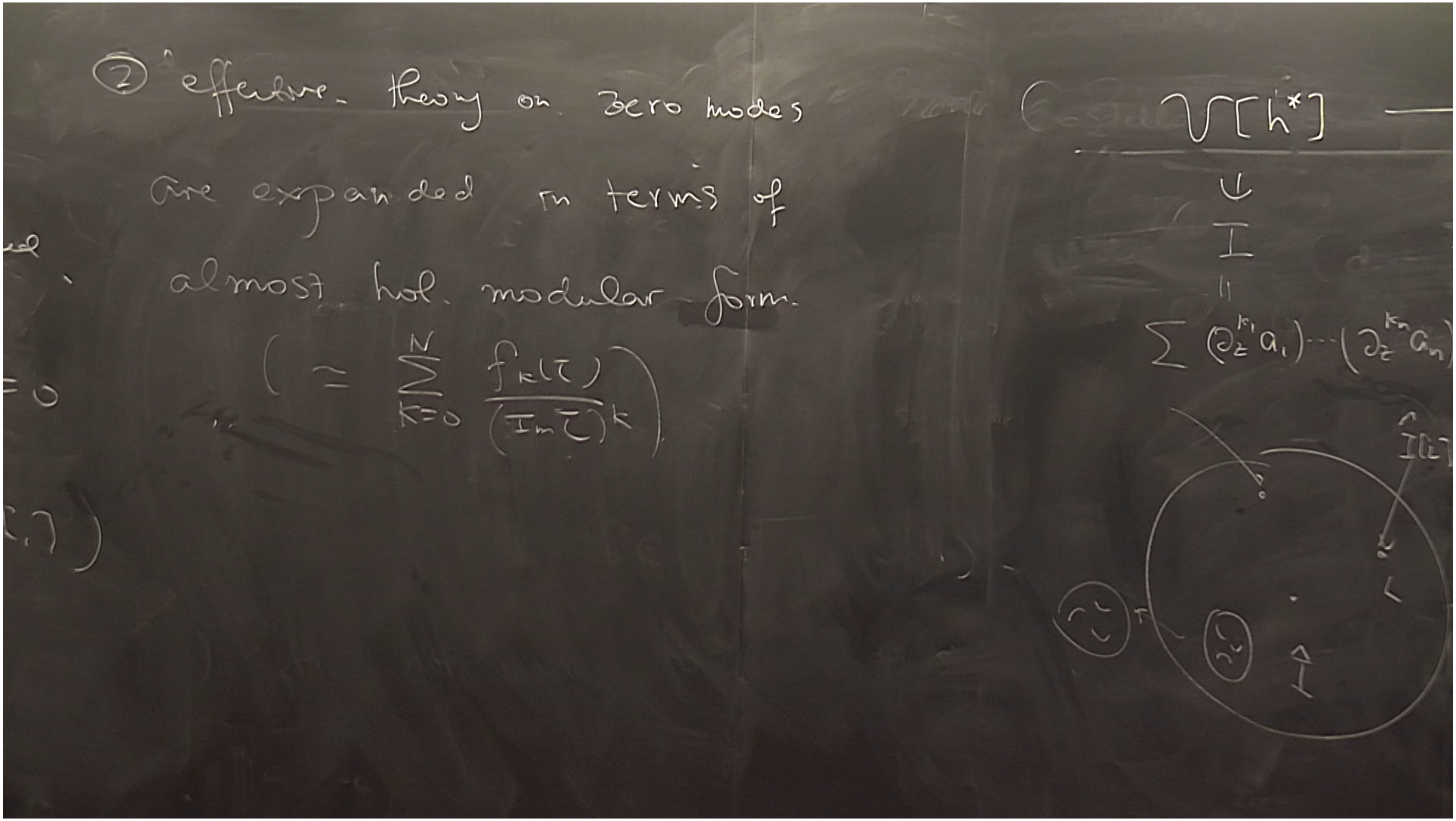
Σ:

$$0 \to \mathbb{C} \xrightarrow{d} \mathcal{M} \xrightarrow{d} \mathcal{A} \to 0 \quad \text{(w/ Zhou)}$$

$$H^1(\mathcal{A}) \hookrightarrow H^2(\mathbb{C})' \xrightarrow{\int} \mathbb{C}$$

renormalized Feyn. diagram





② effective theory on zero modes

are expanded in terms of

almost hol. modular form.

$$\left(\simeq \sum_{k=0}^{N} \frac{f_k(\tau)}{(\mathrm{Im}\,\tau)^k} \right)$$

$$\mathcal{V}[h^*] \ni I = \sum (\partial_z^{k_1} a_1) \cdots (\partial_z^{k_n} a_n)$$

ed

$= 0$

$[\,,\,])$

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$$\Big\downarrow \ \bar\tau \to \infty$$

$f_0(\tau)$ is quasi-modular.

via $\oint_A \oint_A - -$

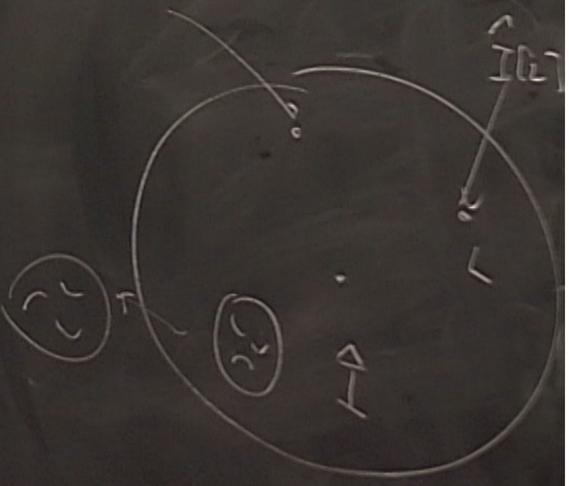
$$\mathcal{V}[h^*]$$

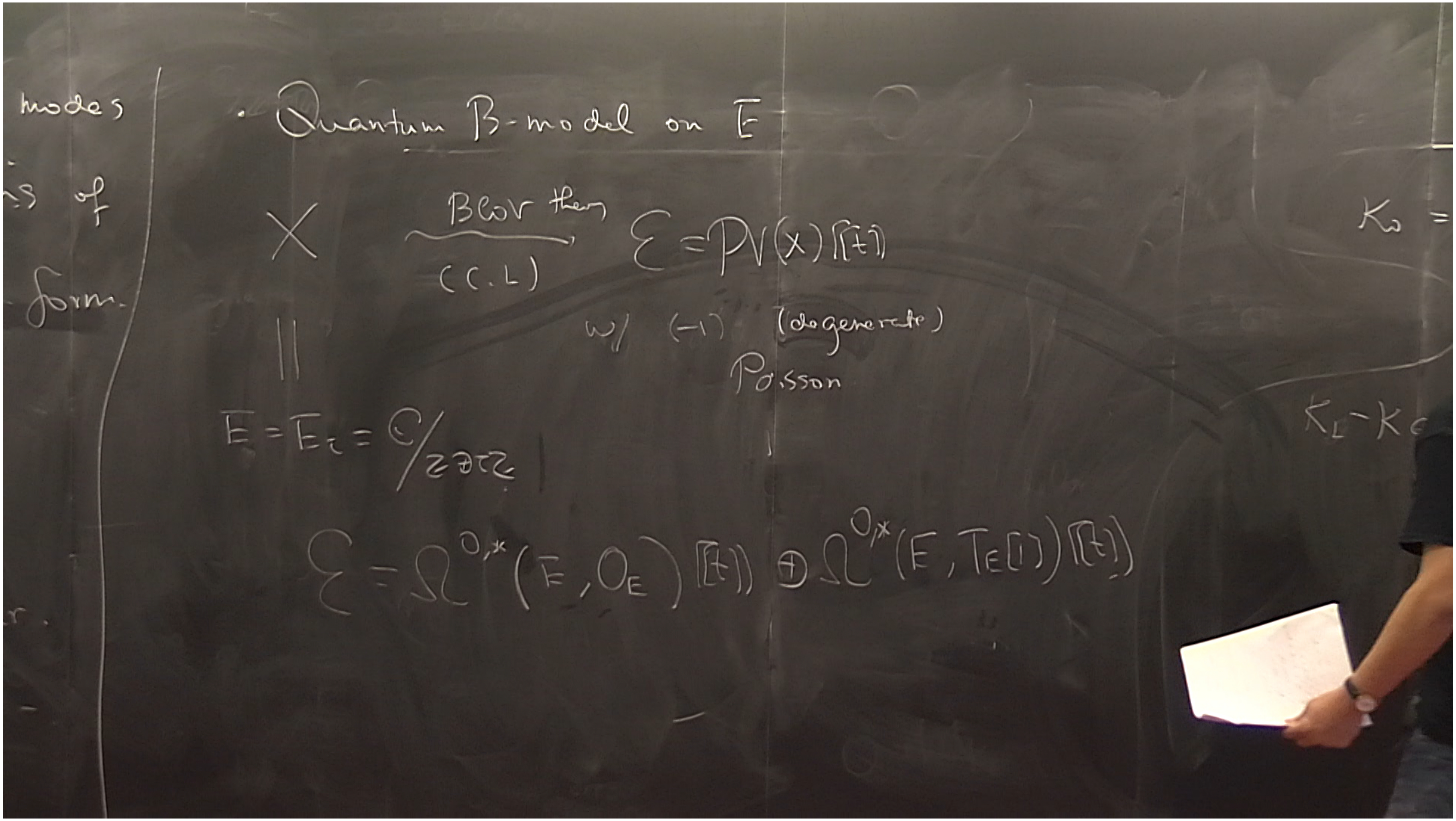
$$\psi$$

$$I$$

$$\|$$

$$\sum (\partial_z^{k_1} a_1)\cdots(\partial_z^{k_n} a_n)$$





modes

s of

form.

r.

· Quantum B-model on E

$$X \xrightarrow[(C.L)]{\text{BCOV theory}} \mathcal{E} = PV(X)[[t]]$$

w/ (-1) (degenerate) Poisson

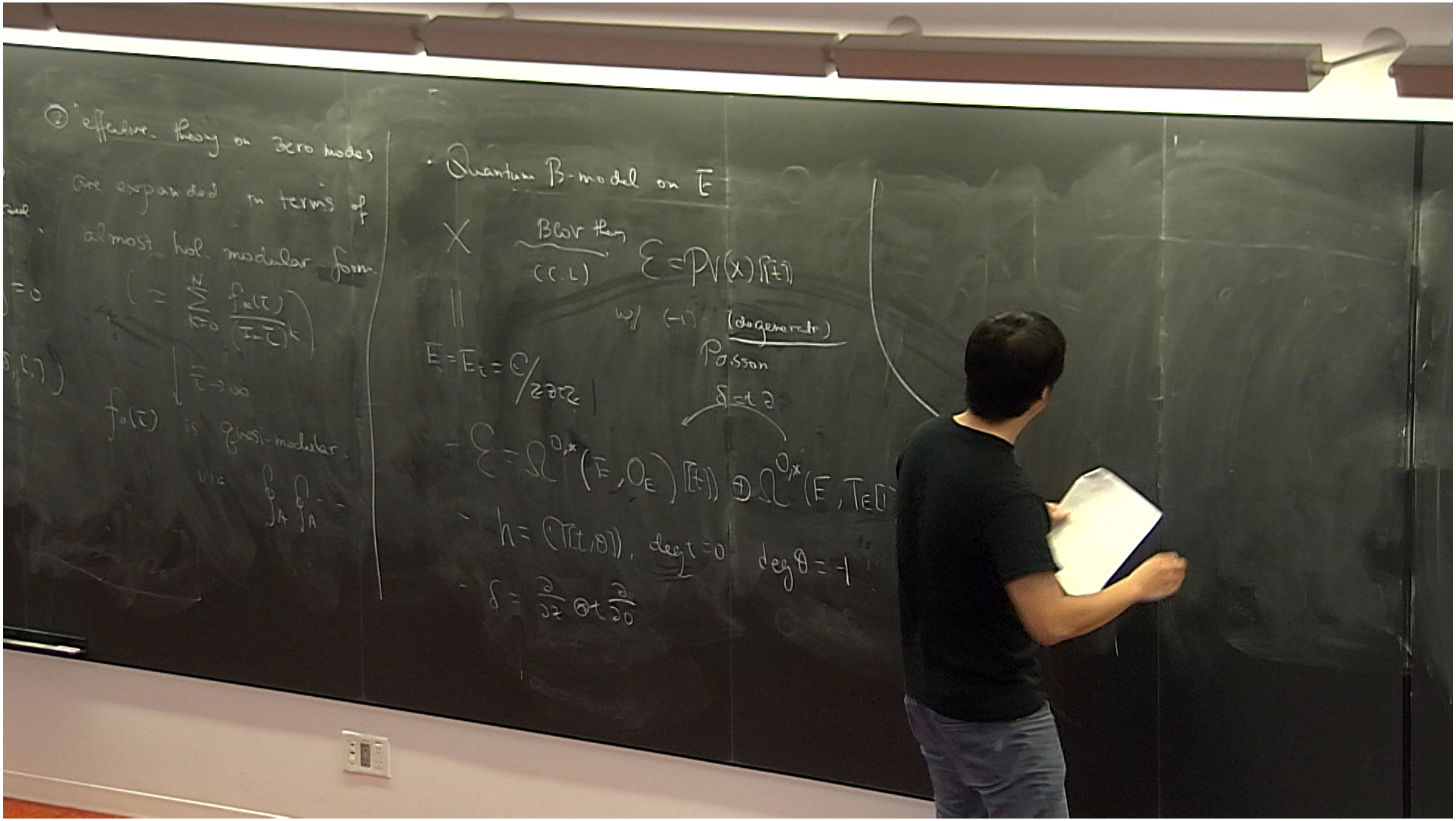
$$\|$$

$$E = E_\tau = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$$

$$\mathcal{E} = \Omega^{0,*}(E, \mathcal{O}_E)[[t]] \oplus \Omega^{0,*}(E, T_E[1])[[t]]$$

$K_0 =$

$K_L - K_\epsilon$



② "effective" theory on zero modes are expanded in terms of almost hol. modular form.

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$$\Big\downarrow \ \bar{\tau} \to \infty$$

$f_0(\tau)$ is quasi-modular.

via $\oint_A \oint_A \cdots$

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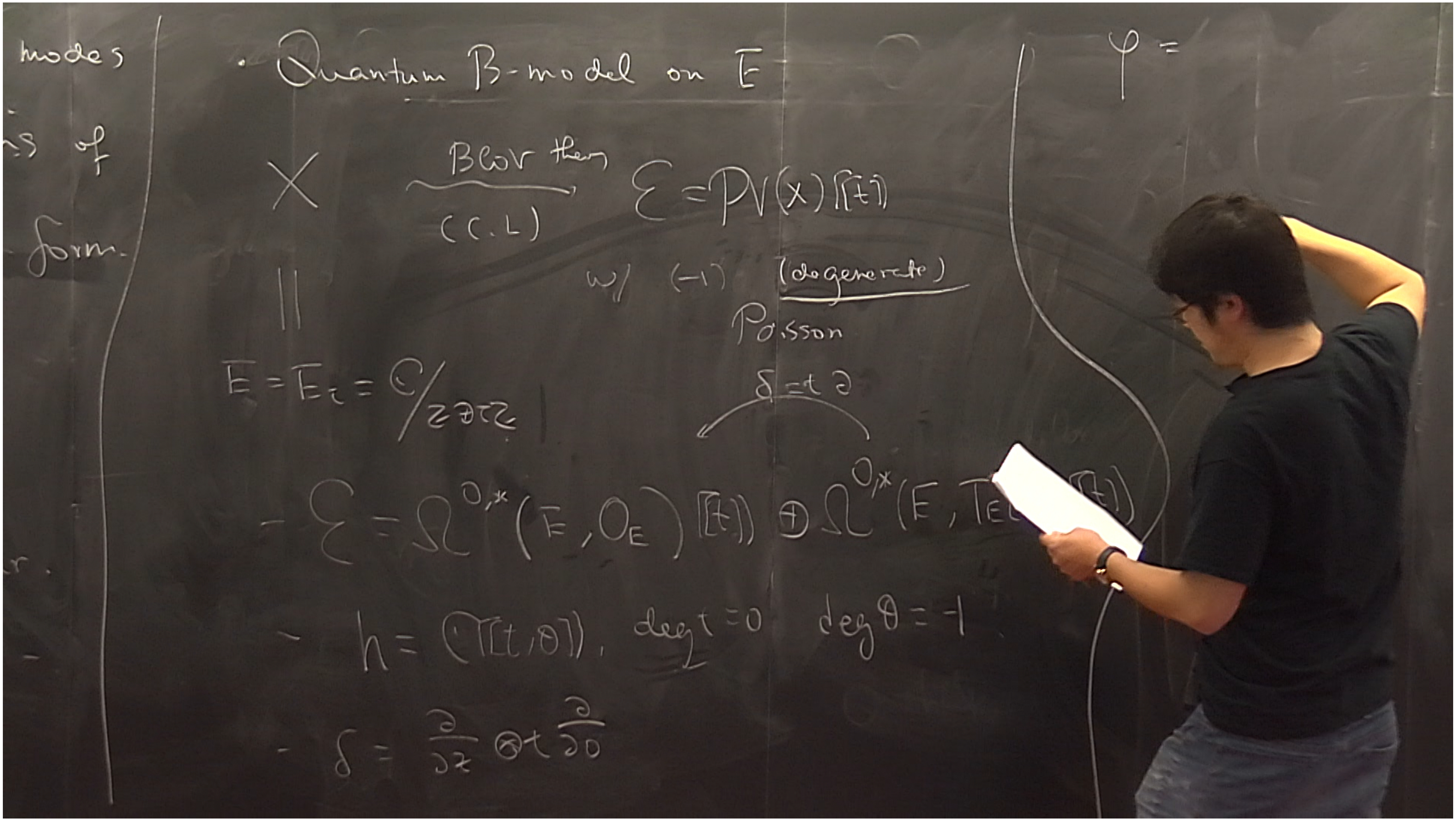
$\delta = t\,\partial$

- $\mathcal{E} = \Omega^{0,*}(E, \mathcal{O}_E)[[t]] \oplus \Omega^{0,*}(E, T_E[1]$

- $h = \mathbb{C}[[t, \theta]]$, $\deg t = 0$ $\deg \theta = -1$

- $\delta = \frac{\partial}{\partial z} \otimes t \frac{\partial}{\partial \theta}$



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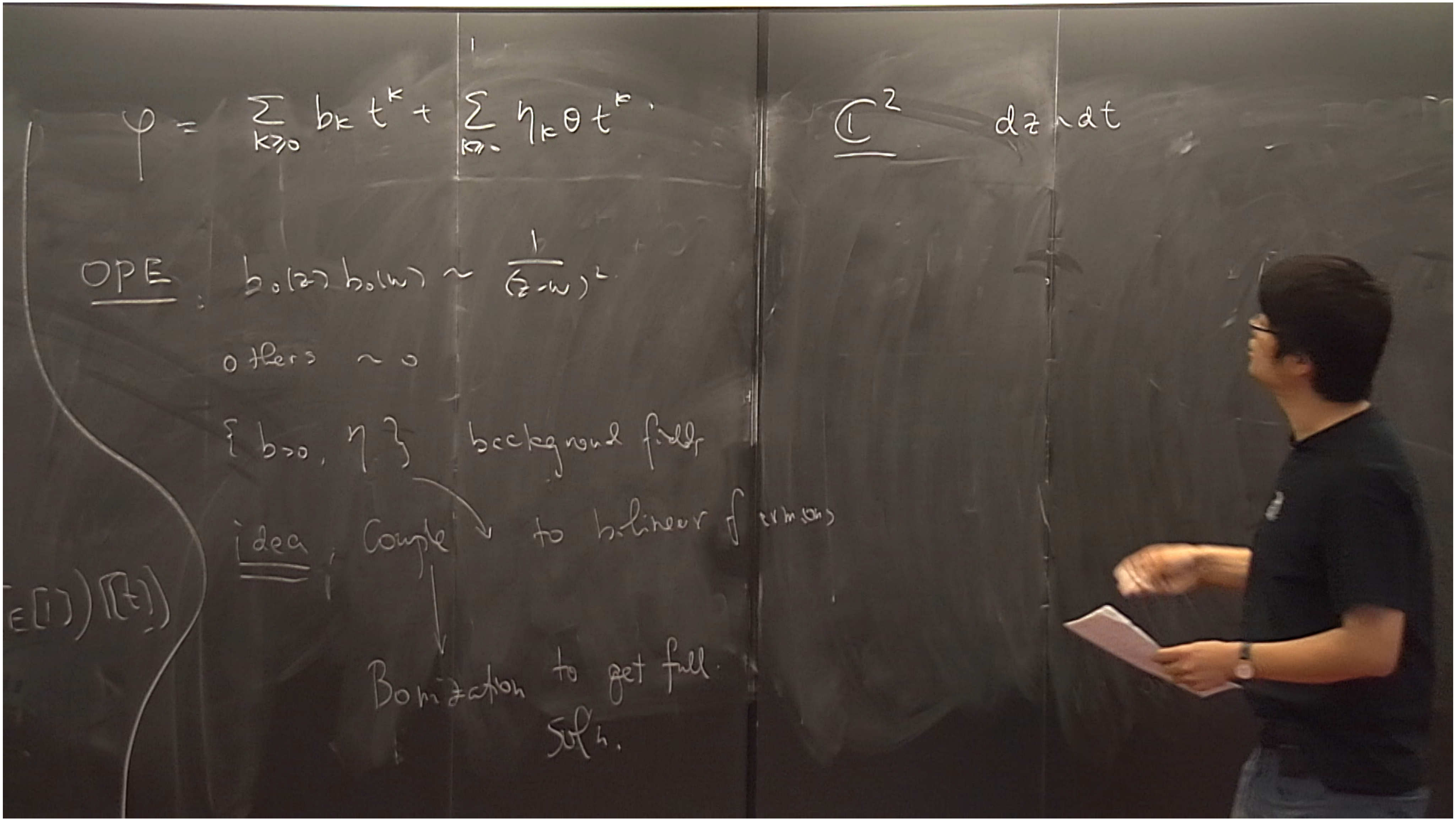
- $\delta = \frac{\partial}{\partial \bar{z}} \oplus t\frac{\partial}{\partial \theta}$

$\varphi =$



$$\varphi = \sum_{k\geq 0} b_k t^k + \sum_{k\geq 0} \eta_k \theta t^k .$$

$\underline{\mathbb{C}^2}$ $dz \wedge dt$

$\underline{\text{OPE}}$: $b_0(z)\, b_0(w) \sim \frac{1}{(z-w)^2}$

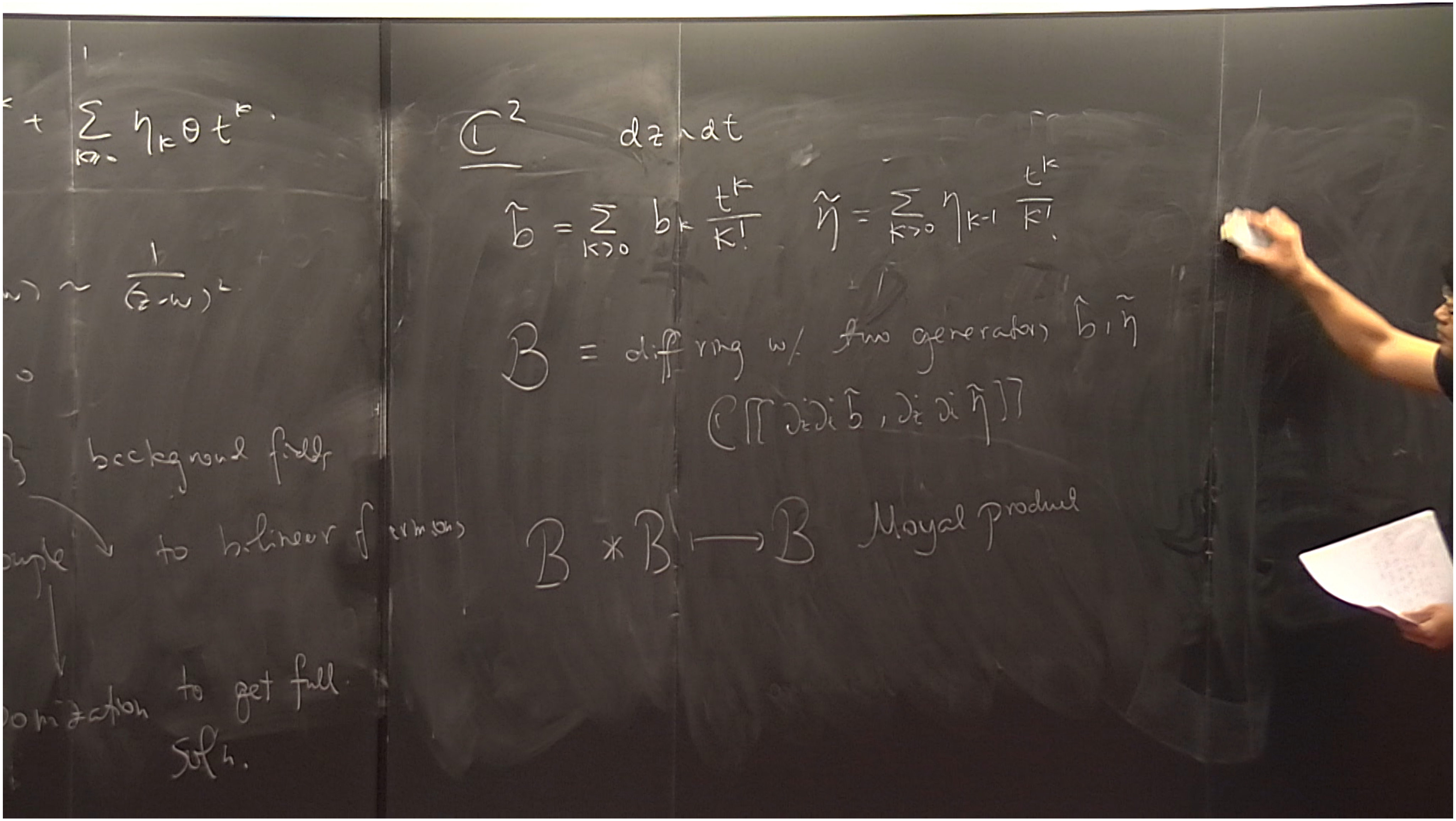
others ~ 0

$\{b_{>0}, \eta\}$ background fields

$\underline{\underline{\text{idea}}}$: Couple to bilinear fermions

Bosonization to get full soln.

$\varepsilon[1])[[t]])$



$+ \sum_{k\geq 0} \eta_k \theta t^k$.

$\sim \frac{1}{(z-w)^2}$

} background fields

ouple ↘ to bilinear fermions

omization to get full sol^n.

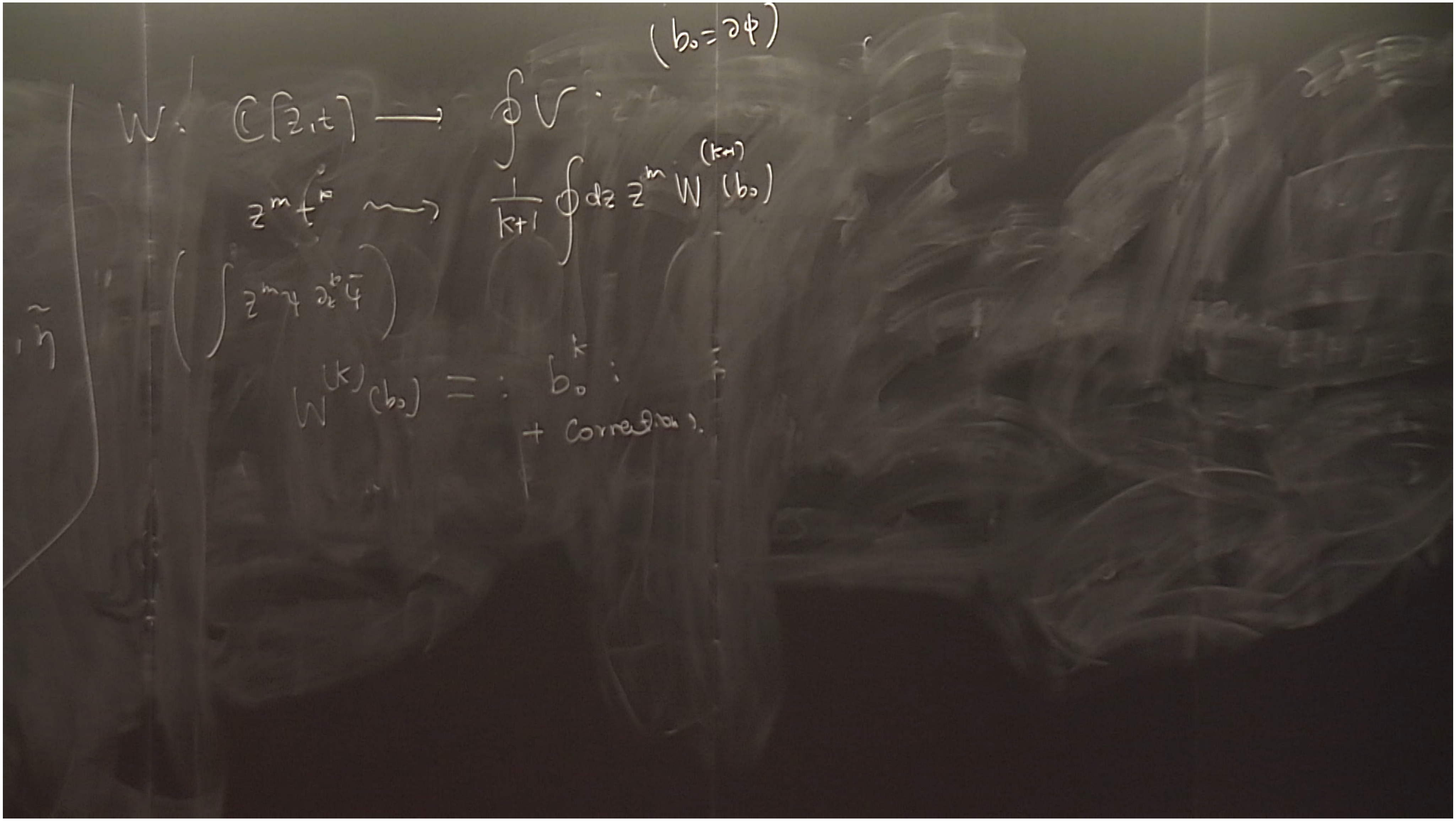
$\underline{\mathbb{C}^2}$ $dz \wedge dt$

$\hat{b} = \sum_{k\geq 0} b_k \frac{t^k}{k!}$ $\tilde{\eta} = \sum_{k>0} \eta_{k-1} \frac{t^k}{k!}$.

$\mathcal{B}$ = diff ring w/ two generators $\hat{b}, \tilde{\eta}$

$\mathbb{C}[[\partial_z \partial_i \hat{b}, \partial_z \partial_i \tilde{\eta}]]$

$\mathcal{B} * \mathcal{B} \longmapsto \mathcal{B}$ Moyal product



$(b_0 = \partial\phi)$

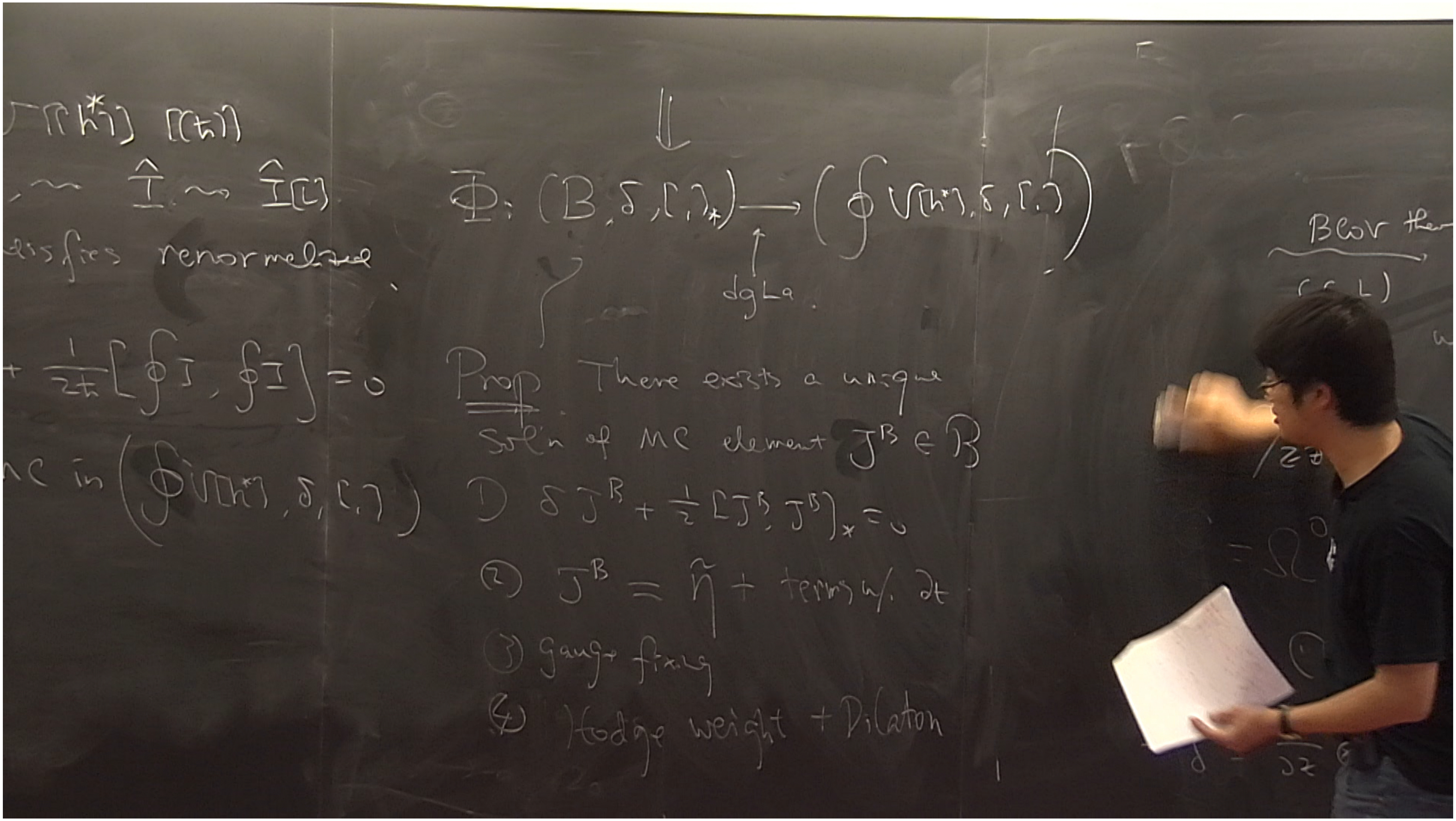
$W : \mathbb{C}[z, t] \longrightarrow \oint V$

$z^m t^k \rightsquigarrow \frac{1}{k+1} \oint dz\, z^m W^{(k+1)}(b_0)$

$\left(\int z^m \psi\, \partial_z^k \bar{\psi} \right)$

$W^{(k)}(b_0) = :b_0^k: + \text{corrections}.$

$\tilde{\eta}$



$[[h^*]]\ [[\hbar]]$

$\rightsquigarrow \hat{I} \rightsquigarrow \hat{I}[L]$

satisfies renormalized

$+ \frac{1}{2\hbar}[\oint I, \oint I] = 0$

MC in $(\oint \mathcal{V}[[h^*]], \delta, [\,,\,])$

$\Downarrow$

$$\Phi : (\mathcal{B}, \delta, [\,,\,]_*) \longrightarrow (\oint \mathcal{V}[[h^*]], \delta, [\,,\,])$$

$\uparrow$ dgLa

Prop There exists a unique soln of MC element $J^B \in \mathcal{B}$

① $\delta J^B + \frac{1}{2}[J^B, J^B]_* = 0$

② $J^B = \tilde{\eta}$ + terms w/. ∂t

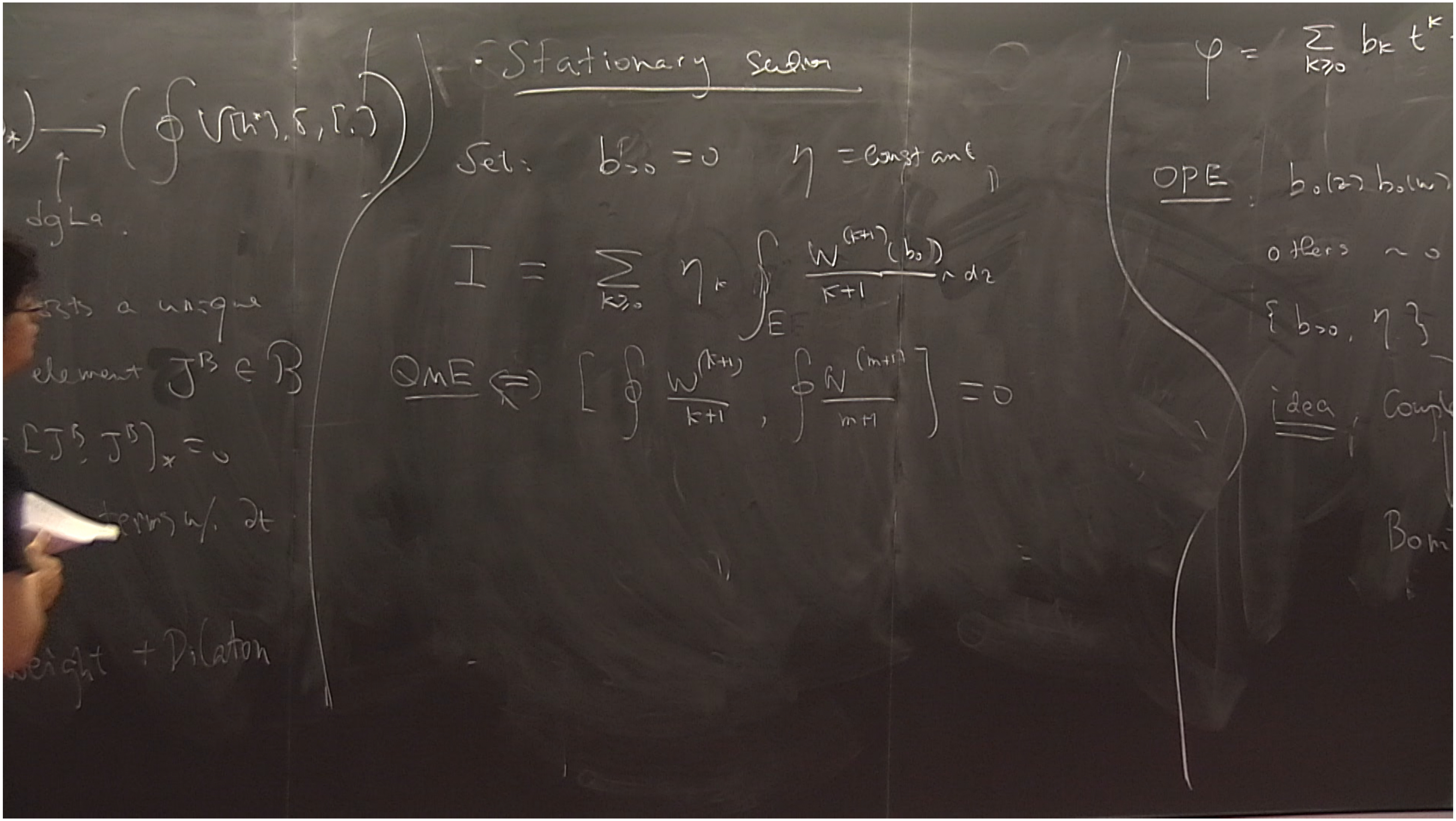
③ gauge fixing

④ Hodge weight + Dilaton

BCOV the

$(\quad L)$

$= \Omega^0$



$$)_* \longrightarrow \left(\oint V[h^*], \delta, [\,,\,] \right)$$

$\uparrow$

dgLa.

ists a unique

element $J^B \in \mathcal{B}$

$[J^B, J^B]_* = 0$

terms w/. ∂t

weight + Dilaton

· Stationary Section

Set: $b_{>0} = 0$ $\quad \eta$ = constant

$$I = \sum_{k \geq 0} \eta_k \int_E \frac{W^{(k+1)}(b_0)}{k+1} \wedge dz$$

QME $\Longleftrightarrow$ $\left[\oint \frac{W^{(k+1)}}{k+1}, \oint \frac{W^{(m+1)}}{m+1} \right] = 0$

$$\varphi = \sum_{k \geq 0} b_k t^k$$

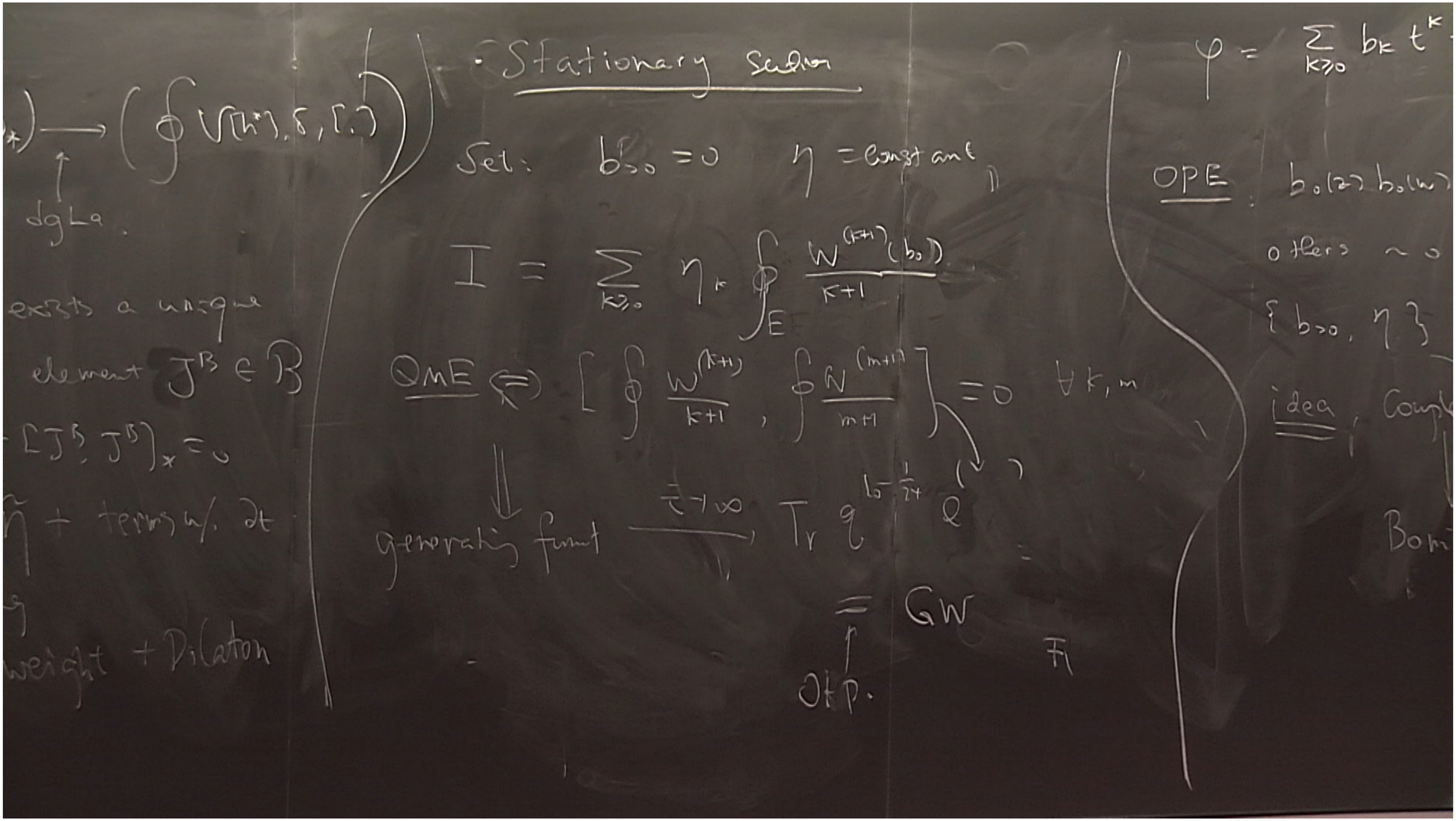
OPE: $b_0(z) b_0(w)$

others ~ 0

$\{ b_{>0}, \eta \}$

idea: Coup

Bom



$$)_* \xrightarrow{} \left(\oint \mathcal{V}[h^*], \delta, [\,,\,] \right)$$

$\uparrow$ dgLa.

exists a unique element $J^B \in B$

$[J^B, J^B]_* = 0$

$\tilde{\eta}$ + terms w/. ∂t

g

weight + Dilaton

• Stationary Section

Set: $b_{>0} = 0$ $\quad \eta$ = constant

$$I = \sum_{k \geq 0} \eta_k \oint_E \frac{W^{(k+1)}(b_0)}{k+1}$$

QME $\Leftrightarrow$ $\left[\oint \frac{W^{(k+1)}}{k+1}, \oint \frac{W^{(m+1)}}{m+1} \right] = 0 \quad \forall k, m$

$\Downarrow$

generating funct $\xrightarrow{\tau \to \infty}$ $\mathrm{Tr}\, q^{L_0 - \frac{1}{24}} e^{(\quad)}$

$= GW$

$\uparrow$ Otp.

$\mathcal{F}_1$

$$\varphi = \sum_{k \geq 0} b_k t^k$$

OPE: $b_0(z) b_0(w)$

others ~ 0

$\{ b_{>0}, \eta \}$

idea: Coup

Bom