Title: From skein theory to presentations of Thompson groups

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Abstract: <p>Motivated by the question about reconstructing a Conformal field theory from the data of a subfactor of finite index, Jones studied the continuous limit of the periodic quantum spin chain which Thompson group acts on. Based on planar algebras, the topological axiomatization of subfactors, we will illustrate the idea of the correspondence between the skein theory of planar algebras and presentation of Thompson groups.</p>

From skein theory to presentations of Thompson groups

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von Neumann algebras

Definition

Let $\mathcal{H}$ be a Hilbert space and $\{A_\alpha\}$ be a net in $\mathcal{B}(\mathcal{H})$.

- $A_\alpha \to A(SOT)$ if and only if $||A_\alpha h - Ah|| \to 0$ for all $h \in \mathcal{H}$.
- $A_\alpha \to A(WOT)$ if and only if $\langle A_\alpha h, k\rangle \to \langle Ah, k\rangle$ for all $h, k \in \mathcal{H}$.

Definition (von Neumann algebras)

Suppose $\mathscr{A}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1. We say $\mathscr{A}$ is a von Neumann algebra if $\mathscr{A}$ is closed in the weak operator topology.

Theorem (Bicommutant theorem, von Neumann)

Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint subalgebra containing 1. Then A'' equals to the weak operator closure of A, where for any subset S of $\mathcal{B}(\mathcal{H})$,

$$S' = \{x \in \mathcal{B}(\mathcal{H}) : xs = sx \ \forall s \in S\}.$$

Subfactors

Definition (Factor)

Suppose M is a von Neumann algebra. We say M is a factor if and only if $M \bigcap M' = \mathbb{C}$. Furthermore, M is a factor of type II_1 if there exists a unique normal tracial functional τ on M, such that

$$\tau(xy) = \tau(yx) \ \forall x, y \in M;$$
$$\tau(1) = 1.$$

- A subfactor of type II_1 is a unital inclusion of II_1 factors $N \subset M$.
- The standard $M - M$ bimodule $L^2(M, \tau)$.

$$\langle x, y \rangle = \tau(y^* x)$$

- The index of a subfactor $N \subset M$ is defined as

$$[M : N] = \dim_N L^2(M, \tau)$$

Theorem (Jones' index theorem)

Suppose $N \subset M$ is a subfactor of type II_1, then

$$[M : N] \in \{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, +\infty]$$

Furthermore, at every possible index, there exists at least one subfactor to realize it.

- The Jones' tower

$$N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \cdots$$

- The standard invariant

$$\begin{array}{ccccc} \mathbb{C} = N' \cap N & \subset N' \cap M & \subset N' \cap M_1 & \subset N' \cap M_2 & \subset N' \cap M_3 \cdots \\ & \cup & \cup & \cup & \cup \\ & \mathbb{C} = M' \cap M & \subset M' \cap M_1 & \subset M' \cap M_2 & \subset M' \cap M_3 \cdots \end{array}$$

Planar algebras

Axiomatizations of standard invariants:

- Ocneanu: Paragroup for finite depth
- Popa: (Standard) $\lambda-$lattice
- Jones: (Subfactor) planar algebra.

Definition (Planar tangles)

A planar tangle T consists of the following data:

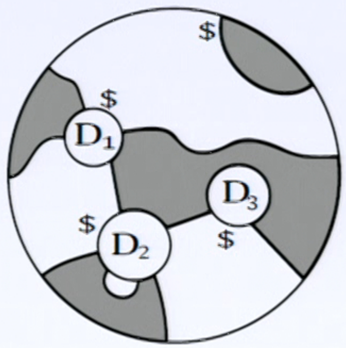
- A smooth disc $D^T \subset \mathbb{R}^2$;
- A certain finite set $\mathfrak{D}_T$ of disjoint smooth discs in the interior of D^T;
- A finite number of disjoint smooth curves in D^T.

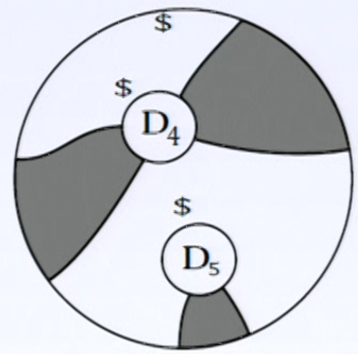
The connected component of the subset of $\mathbb{R}^2$ by taking away from D^T the strands and the discs in $\mathfrak{D}_T$ are called the regions of T. For each disc, there is a distinguished interval indicated by placing a \$ near it.

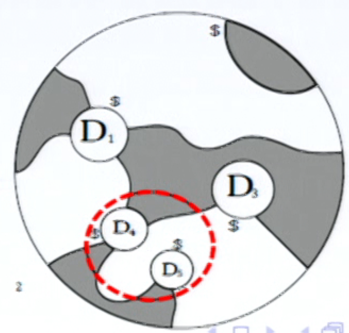
A planar tangle T is said to be ***shaded*** if its regions are shaded with two colors such that if the closure of two regions meet then they have different shadings. We denote the one with \$ sitting in unshaded region by $+$ and shaded by $-$.

Composition of tangles

Let T and S be planar tangles. Suppose that the outer boundary disc D^S of S is the same as as some disc $D_S \in \mathfrak{D}_T$, and the position of \$'s are corresponding to the same interval.

$T =$ 

$S =$ 

$T \circ_{D_2} S =$ 

Planar algebras

Definition (Planar algebras)

A (shaded) planar algebra $\mathscr{P}_\bullet$ is a family of a $\mathbb{Z}_2$-graded vectors spaces $\mathscr{P}_{n,\pm}$ for all $n \in \mathbb{N}$ with the action of planar tangles, which is a multilinear map

$$Z_T : \bigotimes_{D \in \mathfrak{D}_T} \mathcal{P}_{\partial D} \to \mathcal{P}_{\partial(D^T)}$$

for every planar tangle T with $\mathfrak{D}_T$ non-empty with the following axioms:

- If Θ is an orientation preserving diffeomorphism of $\mathbb{R}^2$, then

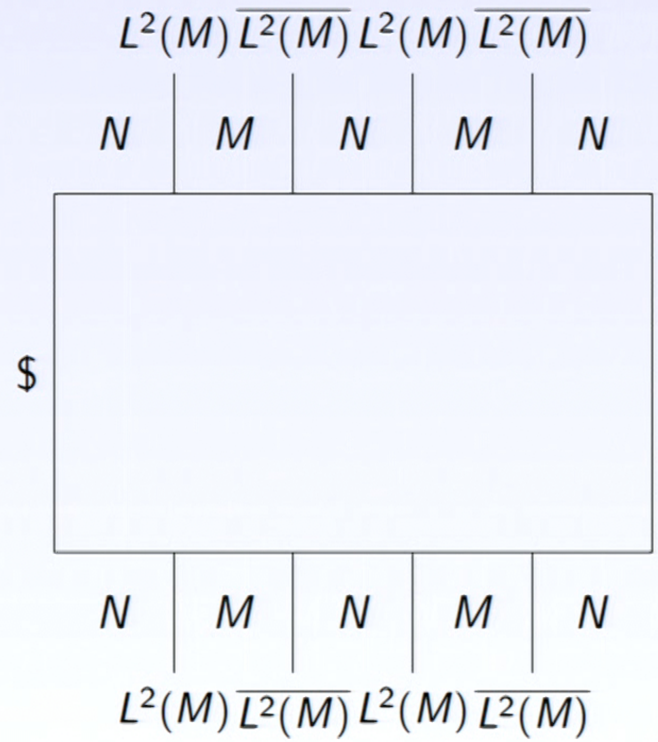
$$Z_{\Theta(T)}(v_1 \otimes v_2 \cdots \otimes v_n) = Z_T(v_1 \otimes v_2 \cdots \otimes v_n)$$

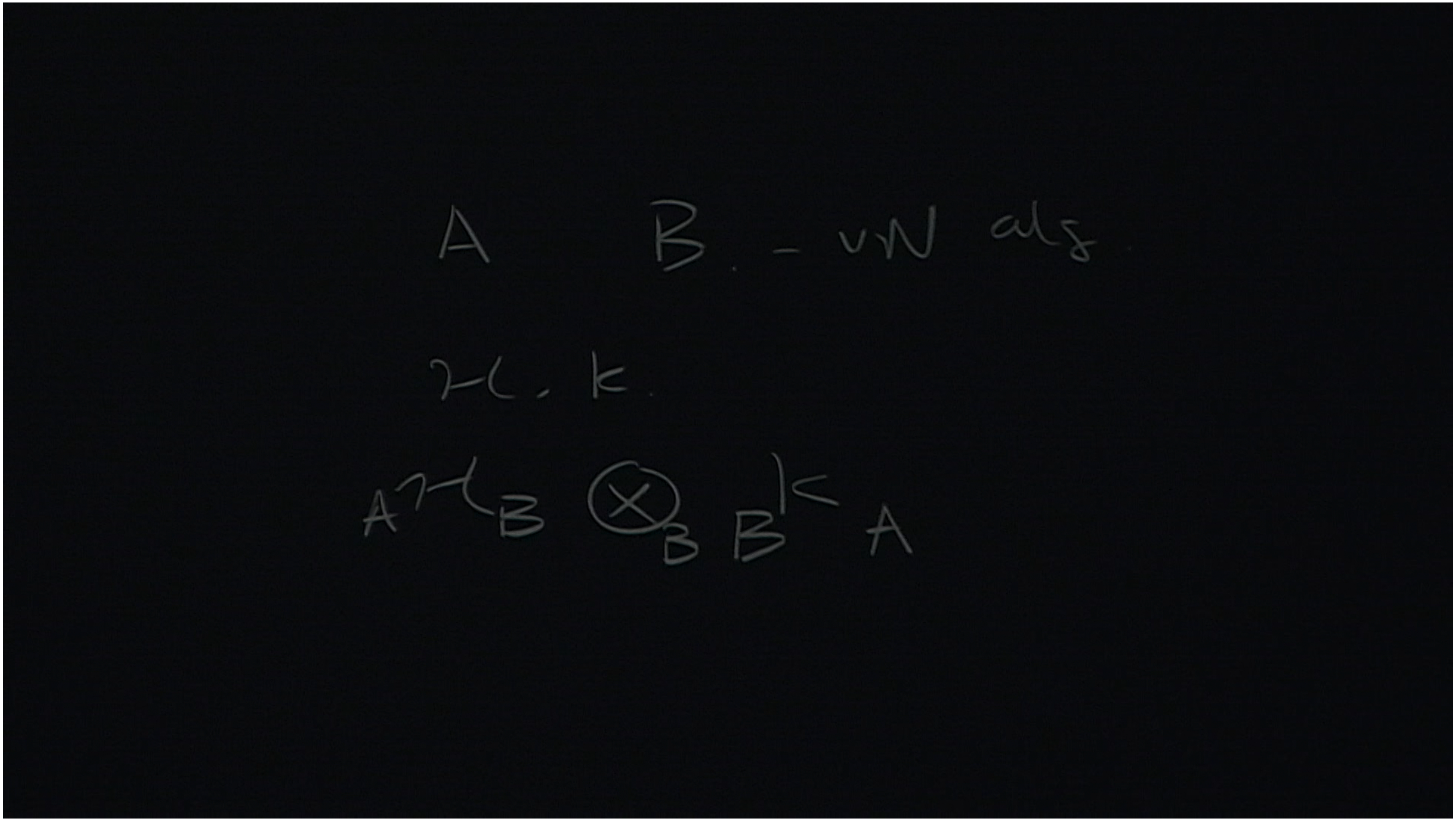
- Naturality:

$$Z_{T \circ S} = Z_T \circ Z_S$$

For a subfactor $N \subset M$,

$\mathscr{P}_{4,+} =$




A B . - vN alg.
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Definition (Conformal nets)

For each interval $\mathcal{I} \subset S^1$, we have a von Neumann algebra $\mathscr{A}(\mathcal{I}) \subset \mathcal{B}(\mathcal{H})$. There exists a continuous projective unitary representation $\alpha \mapsto u_\alpha$ of $DiffS^1$ satisfying the following axioms:

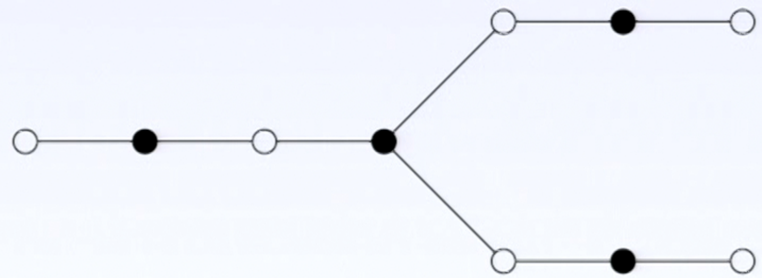
$$(i)\mathscr{A}(\mathcal{I}) \subset \mathscr{A}(\mathcal{J}) \text{ if } \mathcal{I} \subset \mathcal{J}$$
$$(ii)[\mathscr{A}(\mathcal{I}), \mathscr{A}(\mathcal{J})] = 0 \text{ if } \mathcal{I} \cap \mathcal{J} = \emptyset$$
$$(iii)u_\alpha \mathscr{A}(\mathcal{I}) u_\alpha^{-1} = \mathscr{A}(\alpha(\mathcal{I}))$$
$$(iv)\sigma(Rot(S^1)) \subset \mathbb{Z}^+ \cup \{0\}$$

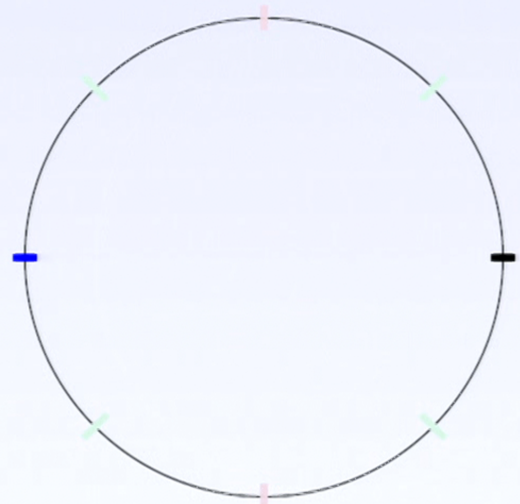
- Jones-Wasserman Subfactors.

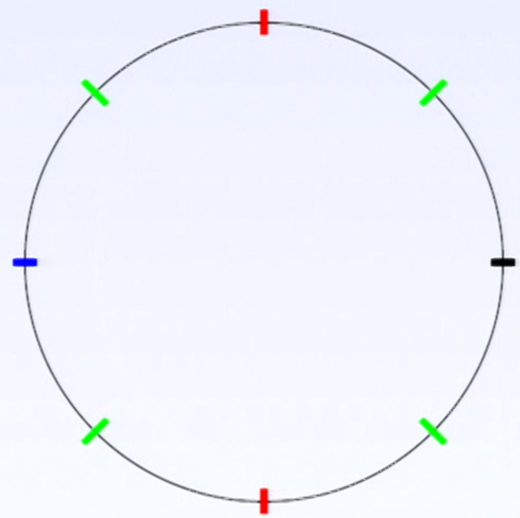
Question

Do all subfactors come from conformal field theory?

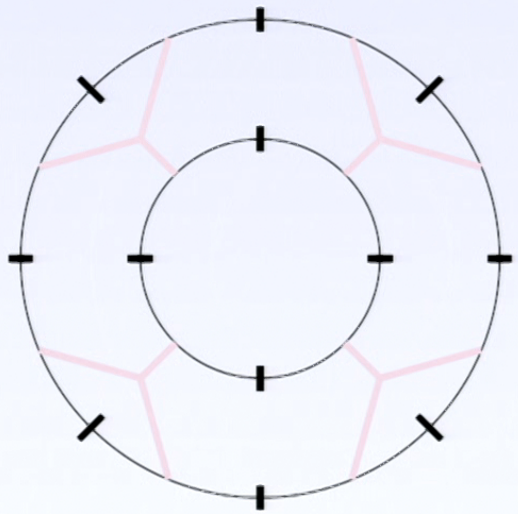
The Haagerup subfactor:





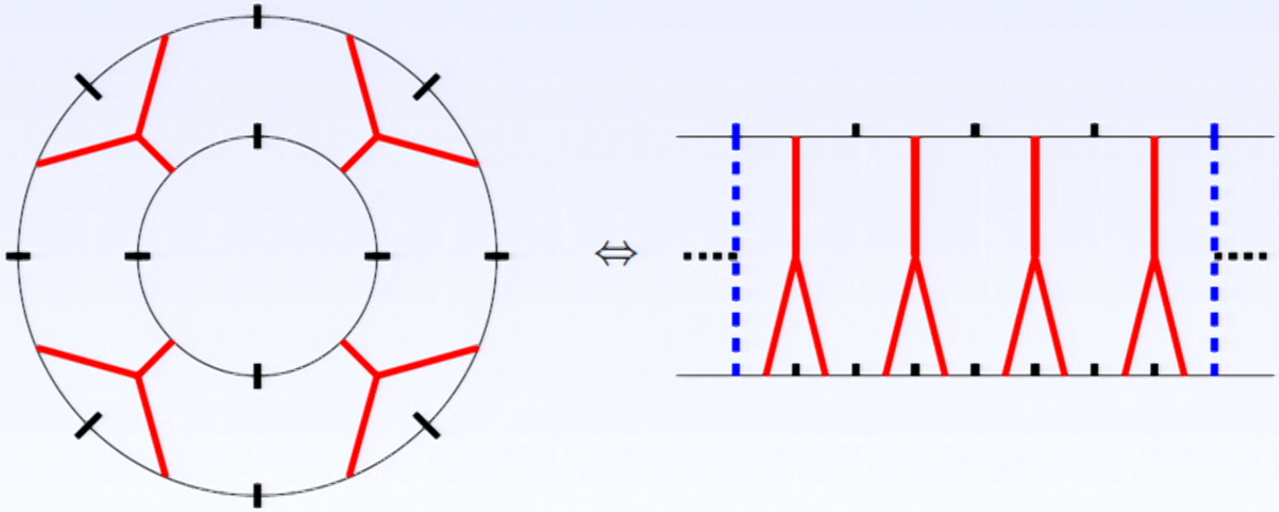


The inclusion from $\mathcal{H}_4$ to $\mathcal{H}_8$:



$\Leftrightarrow$

The inclusion from $\mathcal{H}_4$ to $\mathcal{H}_8$:



Thompson group

Definition (Thompson group F)

Let F be the set of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many dyadic rational numbers such that on intervals of differentiability the derivatives are power of 2.

Proposition (Classical presentation)

The Thompson group F has the following classical presentation:

$$\langle x_1, x_2, x_3, \cdots | x_k^{-1} x_n x_k = x_{n+1}, \ k < n \rangle$$

A "new" definition

Definition

Let $\mathcal{F}$ be a category has the following properties:

- (Unit) There is an element $1 \in Obj(\mathcal{F})$ with $Mor_{\mathcal{F}}(1, a) \neq \emptyset$ for all $a \in Obj(\mathcal{F})$.
- (Stabilisation) Let $\mathcal{D} = \bigcup_{a \in Obj(\mathcal{F})} Mor(1, a)$. Then for each $f, g \in \mathcal{D}$, there exists morphisms p, q such that $pf = qg$.
- (Cancellation) If $pf = qf$ for $f \in \mathcal{D}$, then $p = q$.

Example

The category of binary forests.

Definition (Partial order on $\mathcal{D}$)

For $f, g \in \mathcal{D}$, we say $f \leq g$ if there exists a morphism p such that $g = pf$.

Definition

Let $A_f = Mor(1, target(f))$ together with $\ell_f^g : A_f \to A_g$ when $f \leq g$ and $g = pf$ given by

$$\ell_f^g(v) = pv$$

form a direct system. We consider the direct limit of the system, i.e,

$$A = \lim_{\rightarrow} A_f$$

Proposition (Jones, 2016)

Let $\mathcal{F}$ be the category of binary forests. Then A is isomorphic to Thompson group F given by

$$(f_1, g_1) \circ (f_2, g_2) = (pf_1, qg_2) \quad \text{with} \quad pg_1 = qf_2$$

Remark

Such groups are called the fraction groups, due to

$$\frac{f_1}{g_1}\frac{f_2}{g_2} = \frac{pf_1}{pf_2}\frac{qf_2}{qg_2} = \frac{pf_1}{qg_2}$$

Actions of Thompson group F

Definition

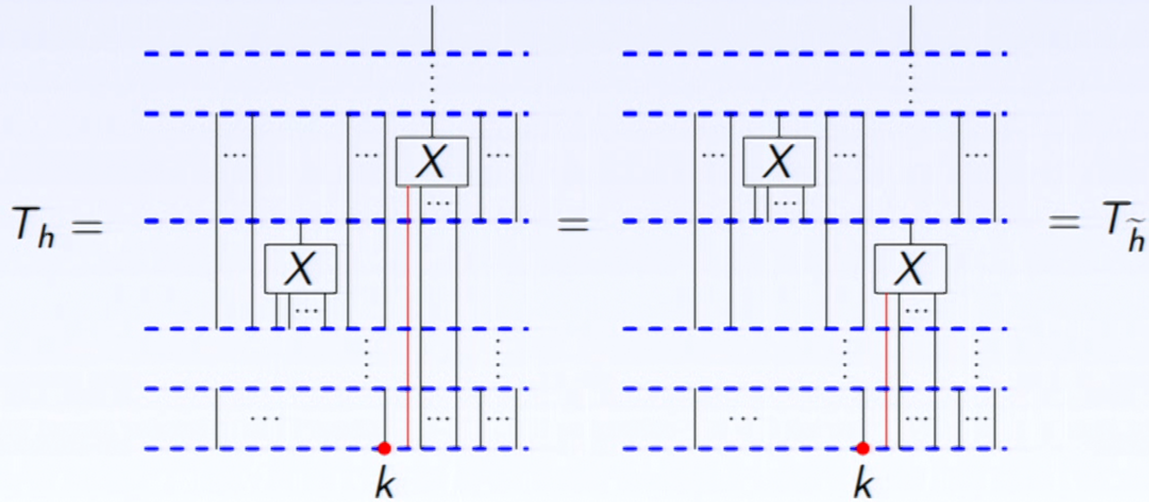
Let $\mathcal{C}$ be a category and Φ is a functor from $\mathcal{F}$ to $\mathcal{C}$.
Direct system: $A(\Phi)_f$, $f \in \mathcal{D}$.
Direct limit: $A(\Phi) = \varinjlim A(\Phi)_f$
Thompson group action:

$$(f_1, g_1) \circ (f_2, g_2) = (pf_1, \Phi(q)g_2) \quad \textit{with} \quad pg_1 = qf_2$$

Remark

Let $\mathcal{C}$ be the category of linear spaces, we obtain a representation of Thompson group F.

Then there exists $T_{\tilde{h}}$ such that



Proposition (Positive elements)

Let $P_X = \{(T, S_n) : T \in Alg(X)_n, \ \forall n \in \mathbb{N}\}$. *Then* P_X *is a semigroup under* $\circ$. *Furthermore, it generates the group* G_X.

Definition

Suppose $n \in \mathbb{N}$, there exists $a(n), b(n) \in \mathbb{N}$ such that $a(n)$ is the largest integer satisfying $(N-1)a(n) + b(n) = n$ with $0 \leq b(n) < N$. We define

$$x_n = (S_{a(n)} \cdot X_n, S_{a(n)+1})$$

Theorem

The group G_X *has a classical presentation*

$$G_X \cong \langle x_n, n \in \mathbb{N} | x_k^{-1} x_n x_k = x_{n+N-1}, \ \forall k < N \rangle. \tag{1}$$

i.e, the group G_X *is isomorphic to* F_N.