Title: Positive representations of quantum groups and higher Teichmuller theory

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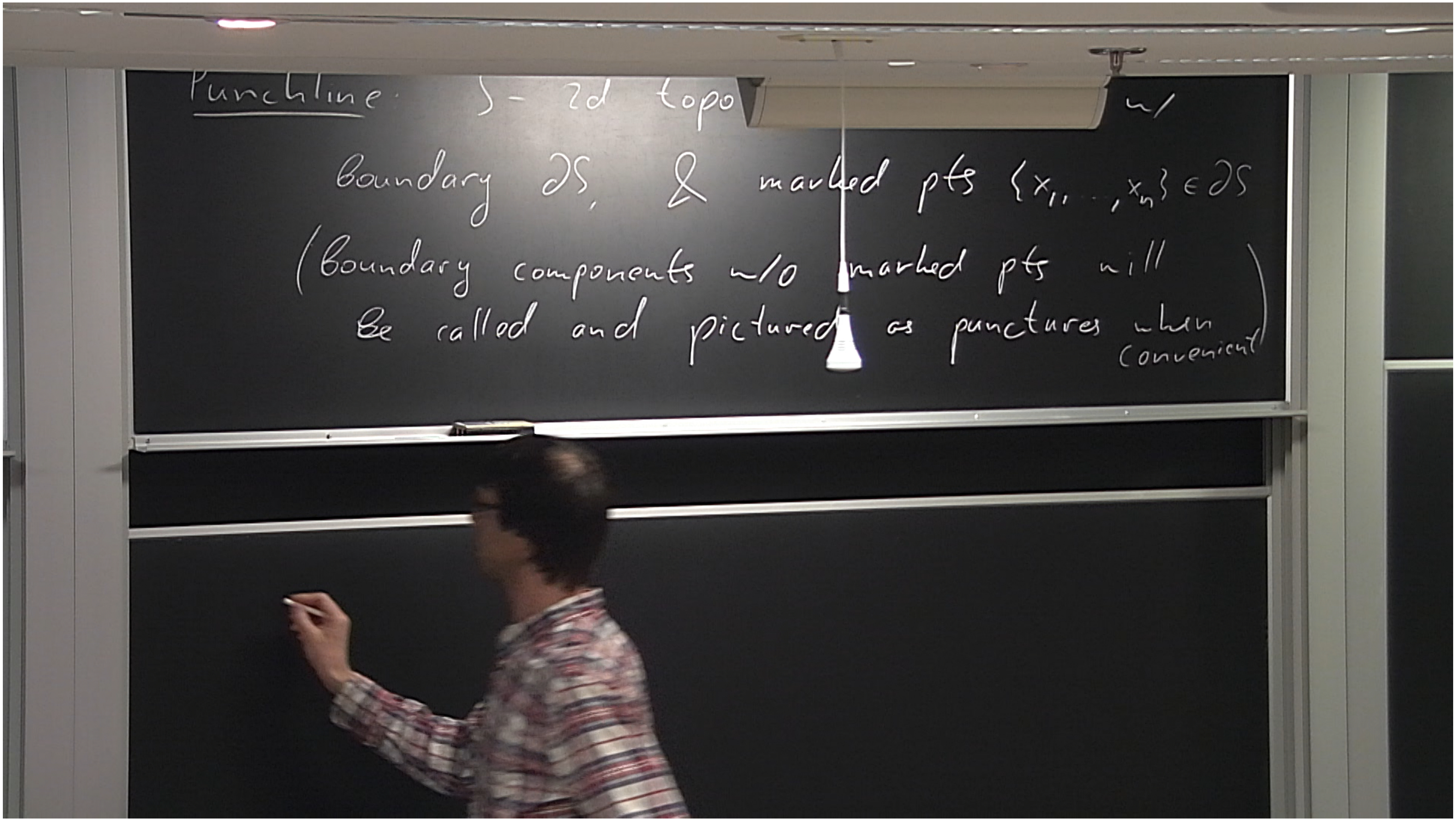
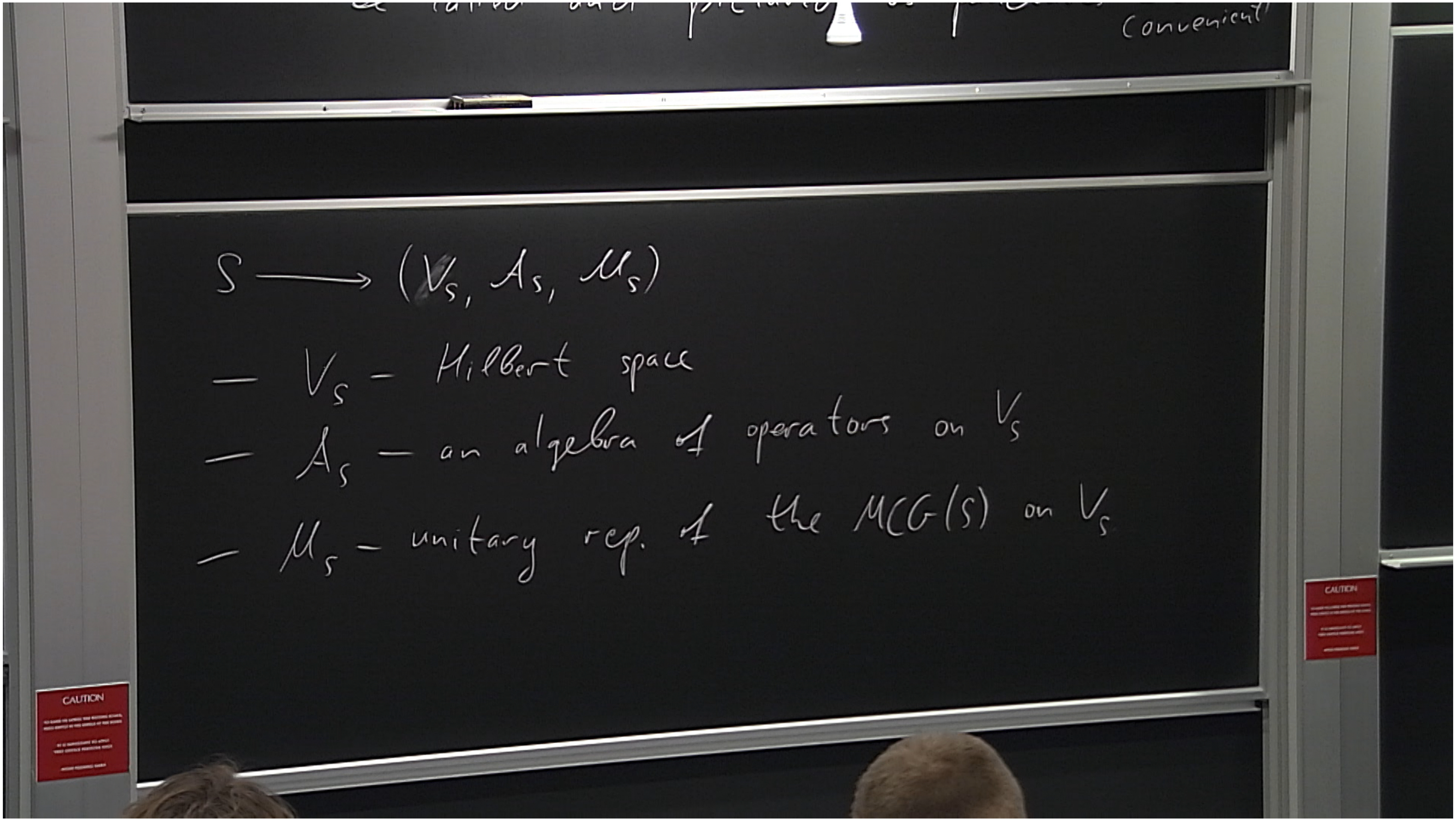
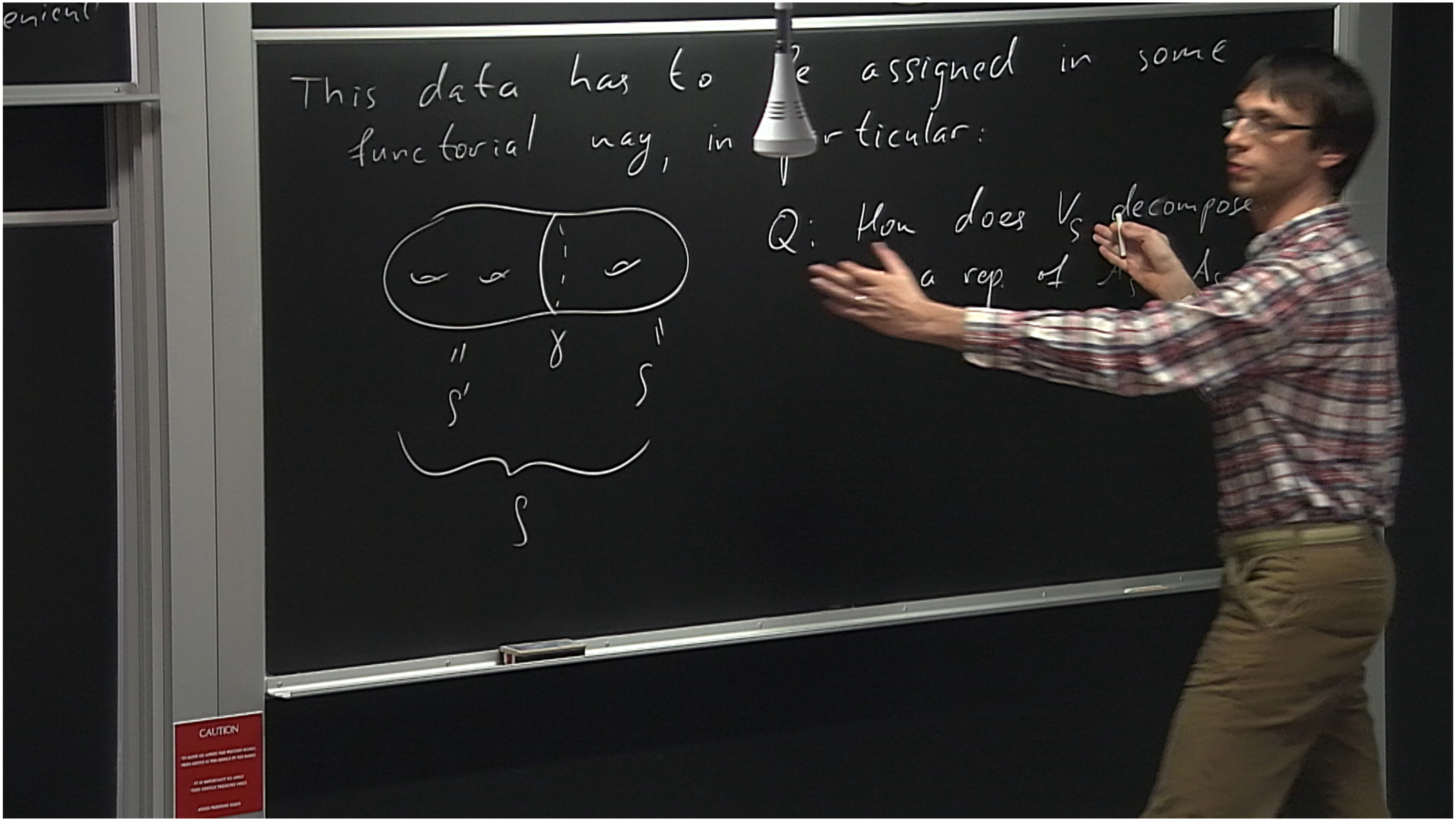
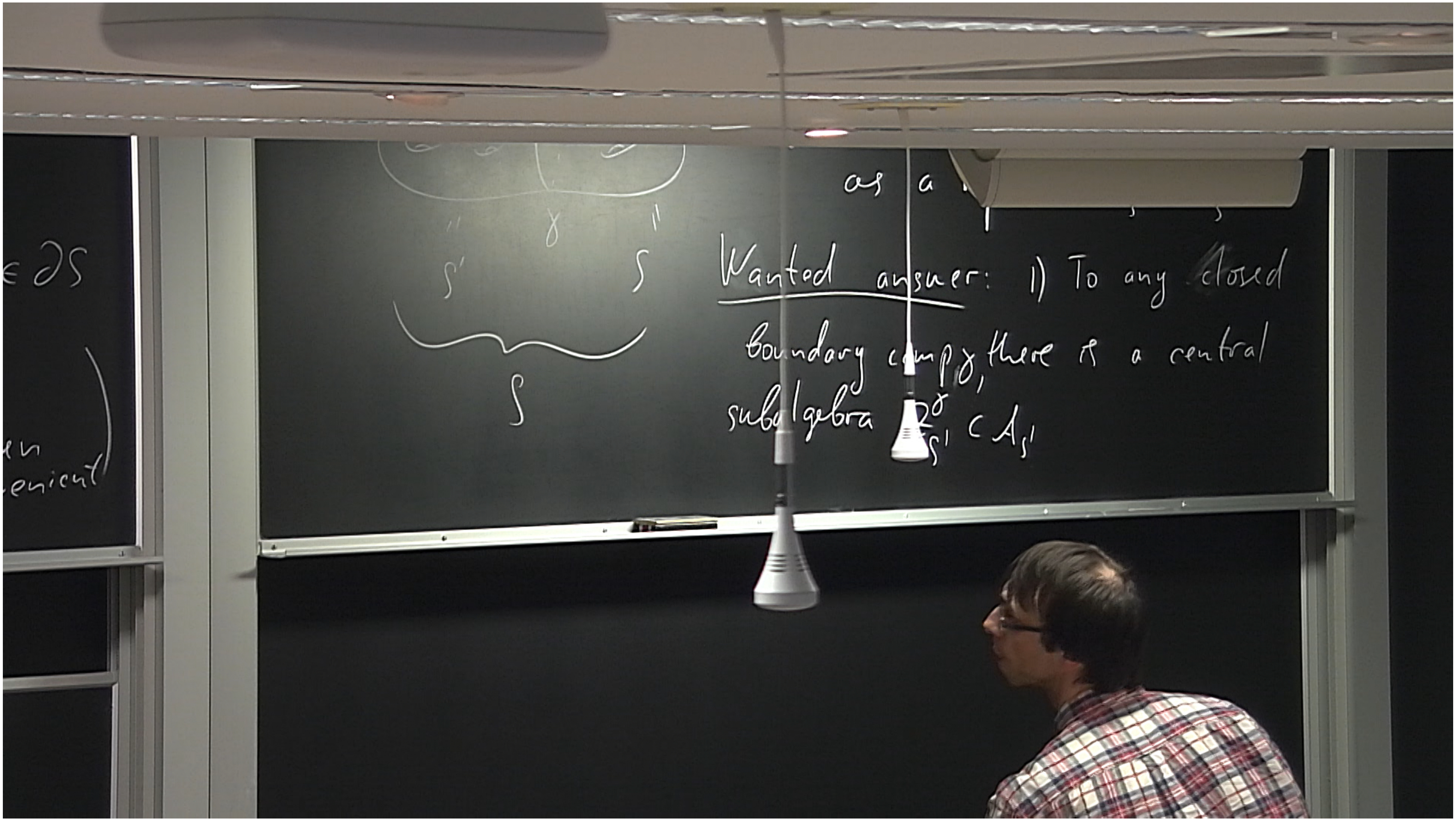
Abstract: <p>Positive representations are infinite-dimensional bimodules for the quantum group and its modular dual where both act by positive essentially self-adjoint operators. Fifteen years ago Ponsot and Teschner showed that positive representations are closed under taking tensor products in the case g = sl(2), however similar conjecture remains open for all other types. I will outline its proof for g = sl(n) based on a joint work in progress with Gus Schrader. I will also argue that this conjecture is the key step towards the proof of the modular functor conjecture for quantized higher Teichmuller theories.</p>

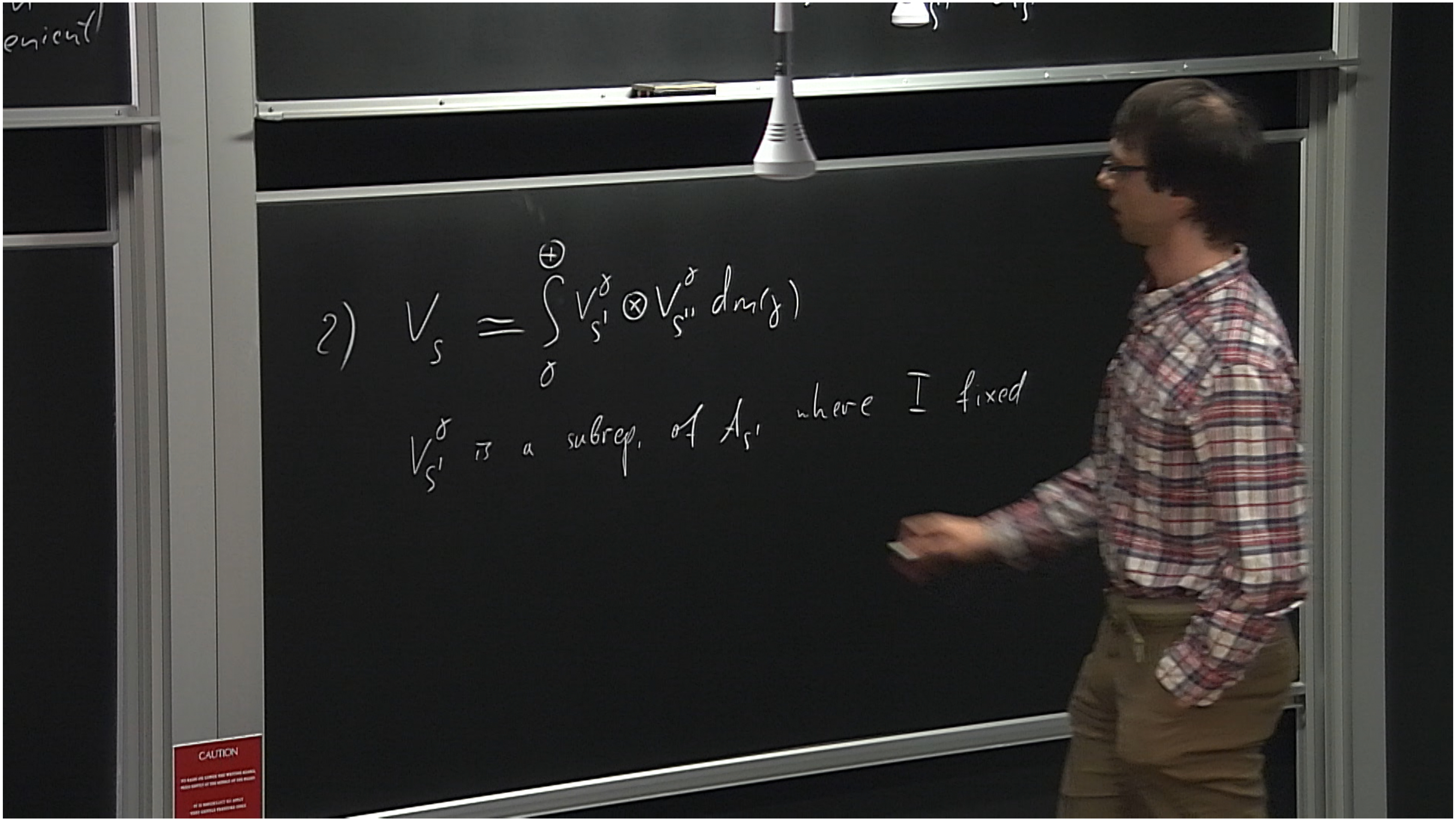

Positive representations of quantum groups
(work in progress, joint with Gus Schrader)
Punchline: S - 2d topological
Boundary ∂S, & marked pts
∈ ∂S


Punchline: S - 2d topo w/
Boundary ∂S, & marked pts {x1, ..., xn} ∈ ∂S
(Boundary components w/o marked pts will
Be called and pictured as punctures when convenient)


(convenient
$S \longrightarrow (V_S, A_S, M_S)$
— V_S – Hilbert space
— A_S – an algebra of operators on V_S
— M_S – unitary rep. of the MCG(S) on V_S


This data has to e assigned in some
functorial way, in rticular:
Q: How does V decompose
a rep. of
CAUTION

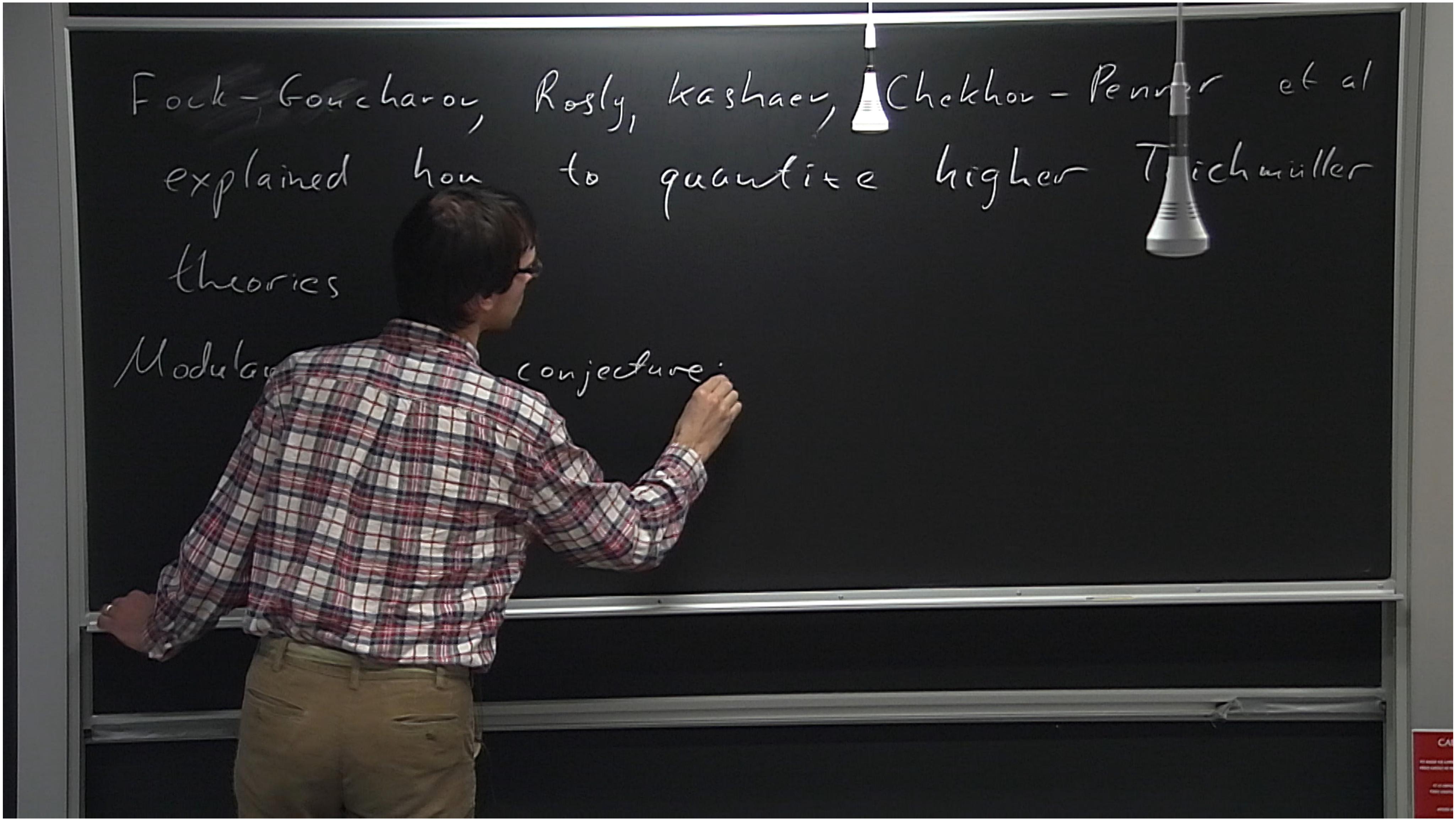

as a
Wanted answer: 1) To any closed
boundary comp γ, there is a central
subalgebra $Z^{\gamma}_{S'} \subset A_{S'}$


2) $V_s \simeq \int_\gamma^\oplus V_{s'}^\gamma \otimes V_{s''}^\gamma \, dm(\gamma)$
$V_{s'}^\gamma$ is a subrep. of $A_{s'}$ where I fixed
CAUTION

2) $V_s \simeq \int_\nu^\oplus V_{s'}^\nu \otimes V_{s''}^\nu \, dm(\nu)$ as $A_{s'}$-modules

$V_{s'}^\nu$ is a subrep. of $A_{s'}$ where I fixed that the cental algebra $Z_{s'}^\gamma$ acts by the weight ν.

$A_{s'}$ acts on $V_{s'}^\nu$ as said, and $A_{s'}$ does not act on $V_{s''}^\nu$ except for $Z_{s'}^\nu$ that acts by ν.


Fock-Goncharov, Rosly, Kashaev, Chekhov-Pen r et al
explained how to quantize higher T ichmüller
theories
Modula conjecture:

provides the needed construction

There is a family of positive rep f quantum groups One way to define th as follows

1) Theorem in progress: (Le-S

$$U_q(\mathfrak{g}) \underset{Z}{\otimes} U_q(\mathfrak{h}) \simeq q.\, G\text{-Te}$$

$$Z = U_q(\mathfrak{h})^W$$

assigns

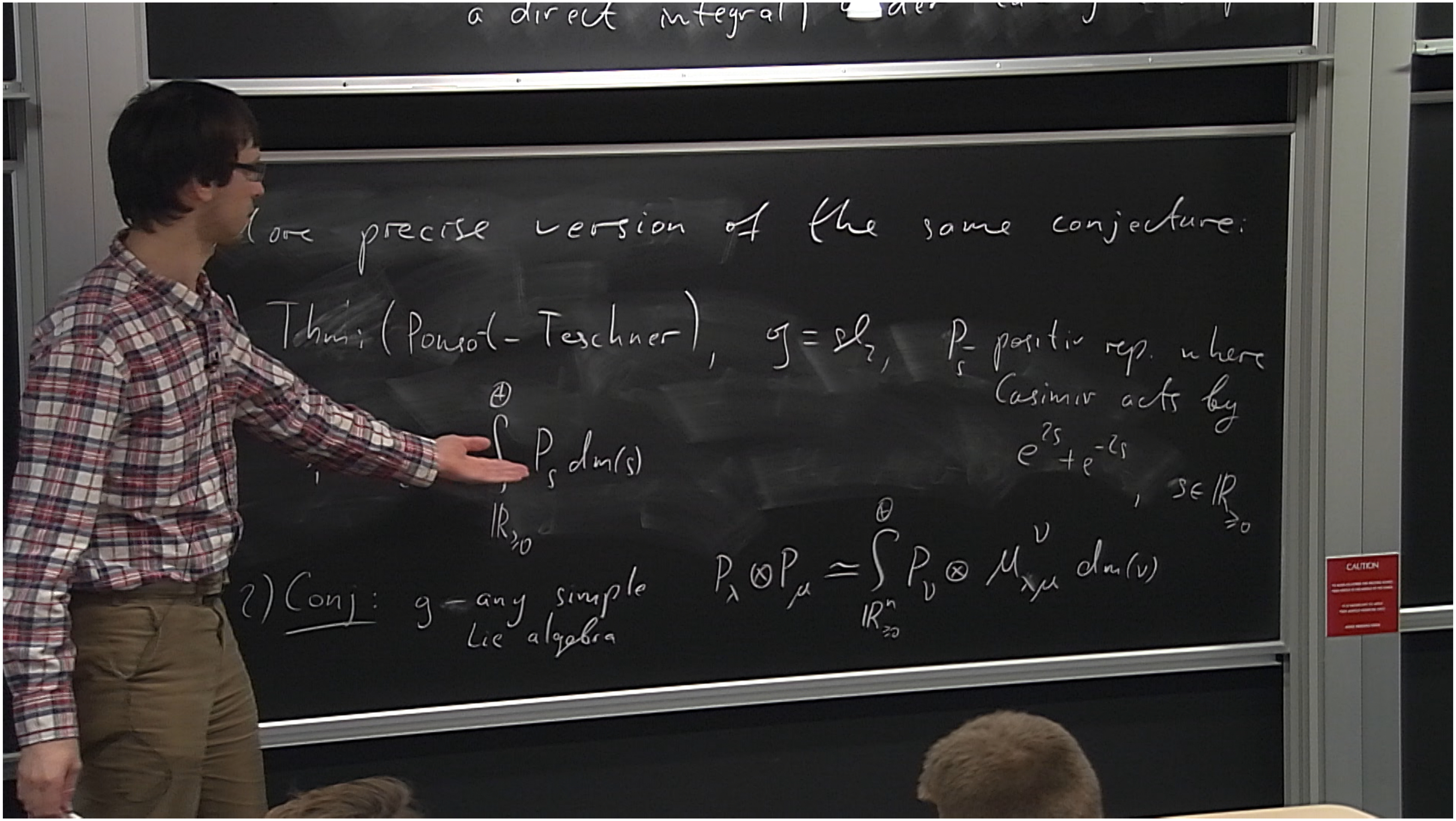
There is a family of positive reps of quantum groups One way to define those is as follows

1) <u>Theorem in progress</u>: (Le-Schrader-S)

$U_q(\mathfrak{g}) \otimes_{Z} U_q(\mathfrak{h}) \simeq$ q. G-Teichmüller theory

$Z = U_q(\mathfrak{h})^{W}$

assigns to a punctured disk w/ 2 marked pts


a direct integral
precise version of the same conjecture:
Thm: (Ponsot – Teschner), $\mathfrak{g} = \mathfrak{sl}_2$, P_s – positiv rep. where Casimir acts by $e^{2s}+e^{-2s}$, $s \in \mathbb{R}_{\geq 0}$
$\int^{\oplus}_{\mathbb{R}_{\geq 0}} P_s \, dm(s)$
2) Conj: $\mathfrak{g}$ – any simple Lie algebra
$P_\lambda \otimes P_\mu \simeq \int^{\oplus}_{\mathbb{R}^n_{\geq 0}} P_\nu \otimes M^\nu_{\lambda\mu} \, dm(\nu)$

a direct integral ...

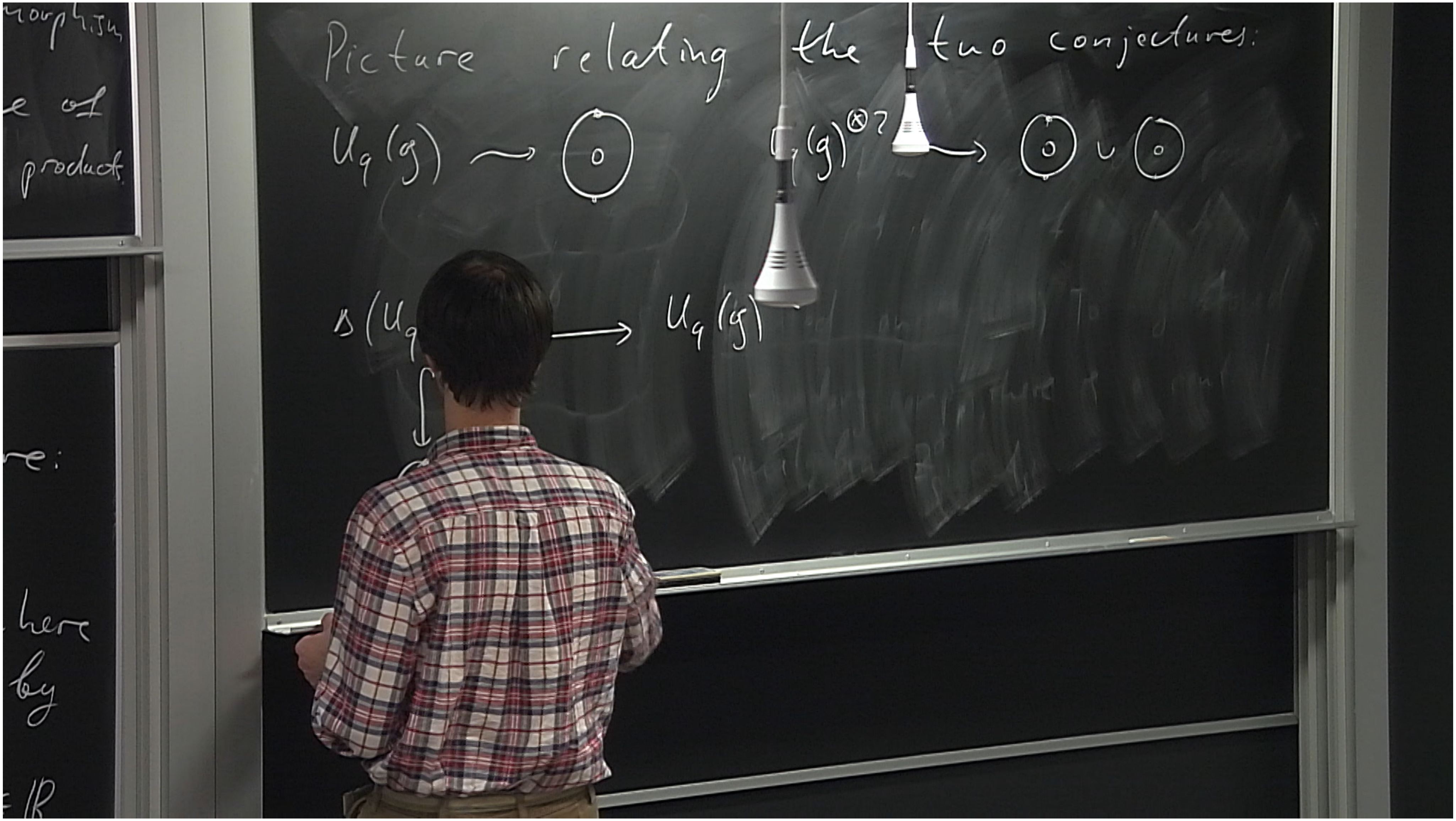
More precise version of the same conjecture:

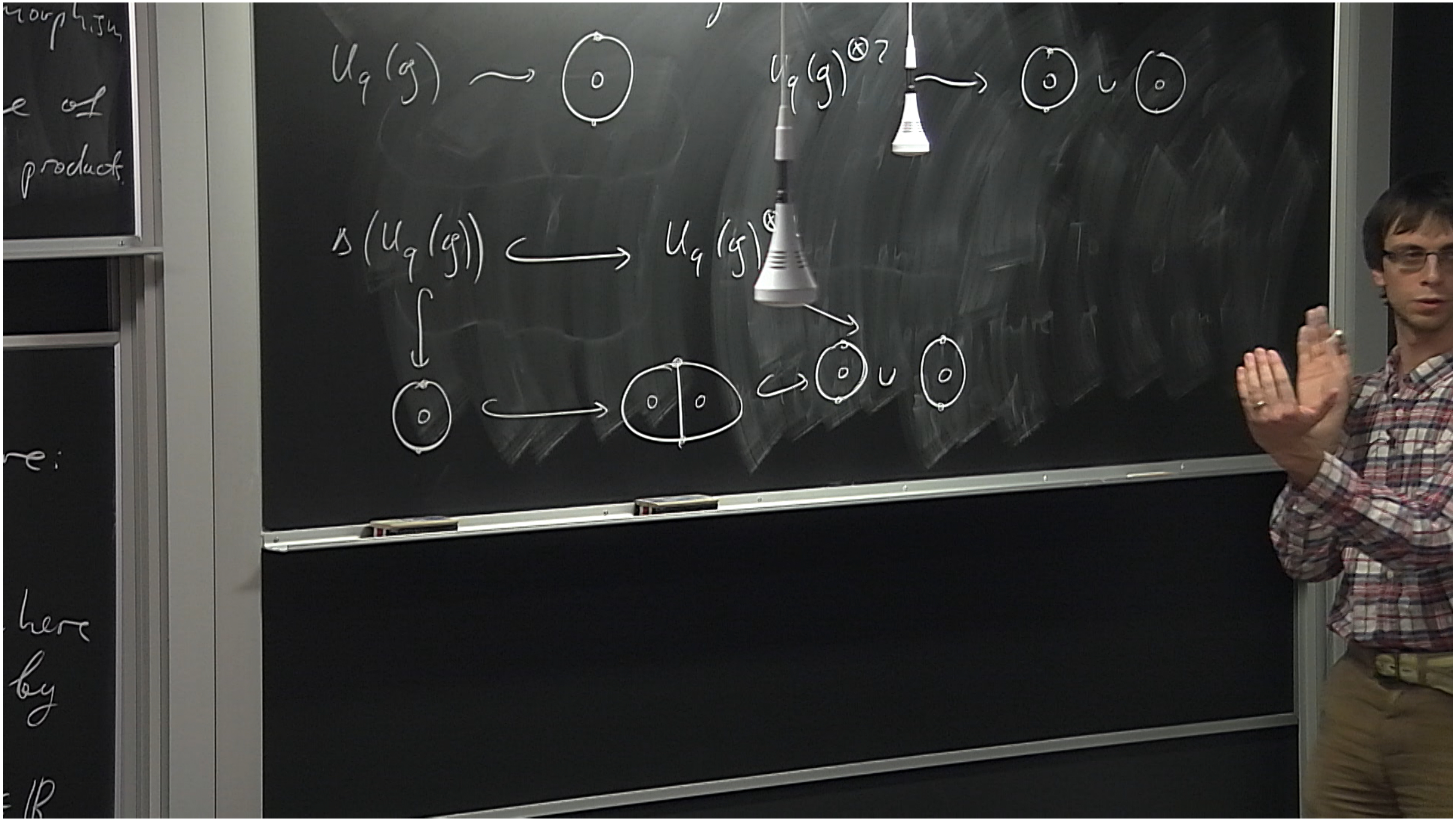
1) Thm: (Ponsot – Teschner), $\mathfrak{g}=\mathfrak{sl}_2$, P_s – positive rep. where Casimir acts by $e^{2s}+e^{-2s}$, $s\in\mathbb{R}_{\geq 0}$

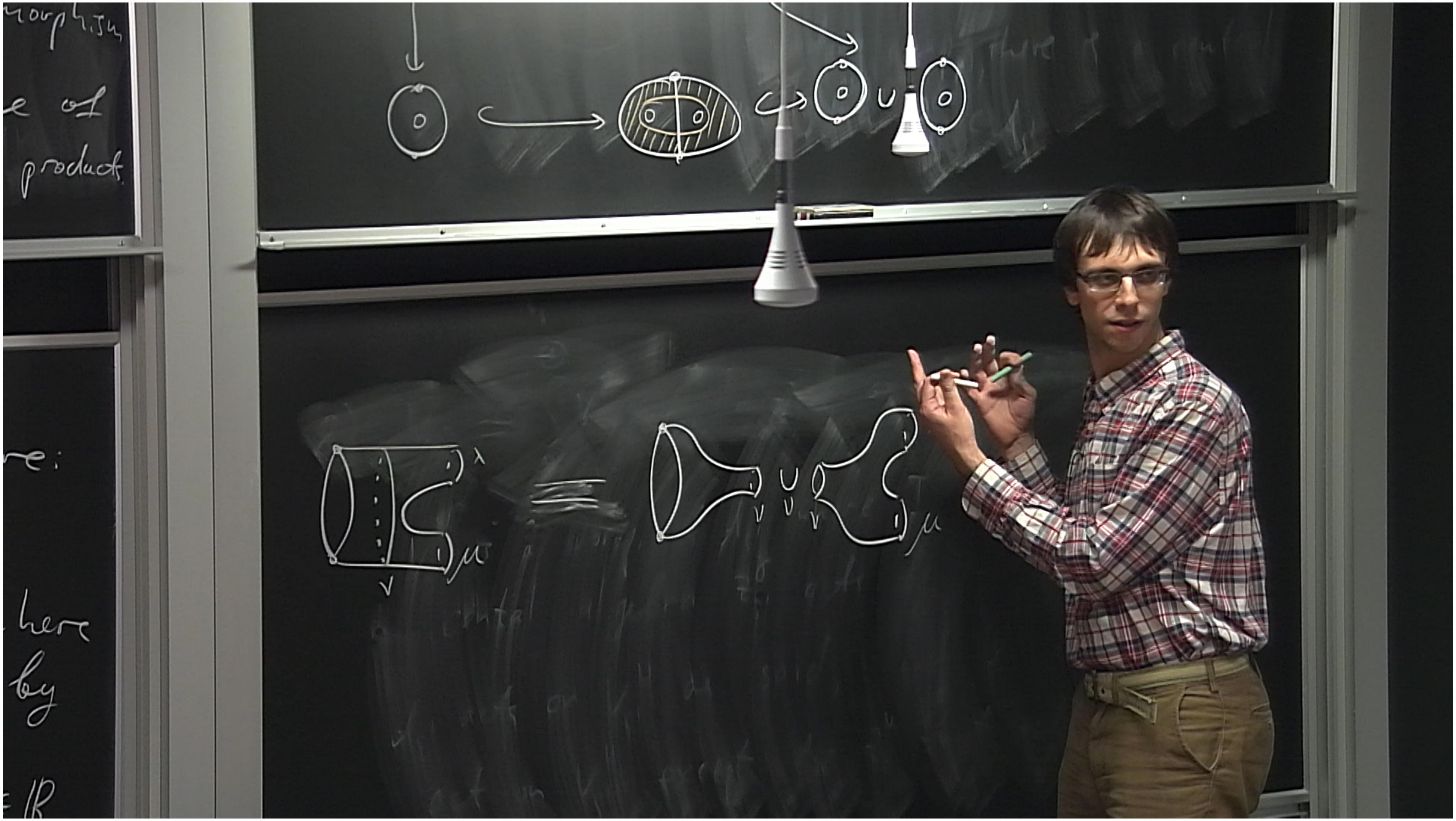
$$P_{s_1}\otimes P_{s_2} = \int^{\oplus}_{\mathbb{R}_{\geq 0}} P_s \, dm(s)$$

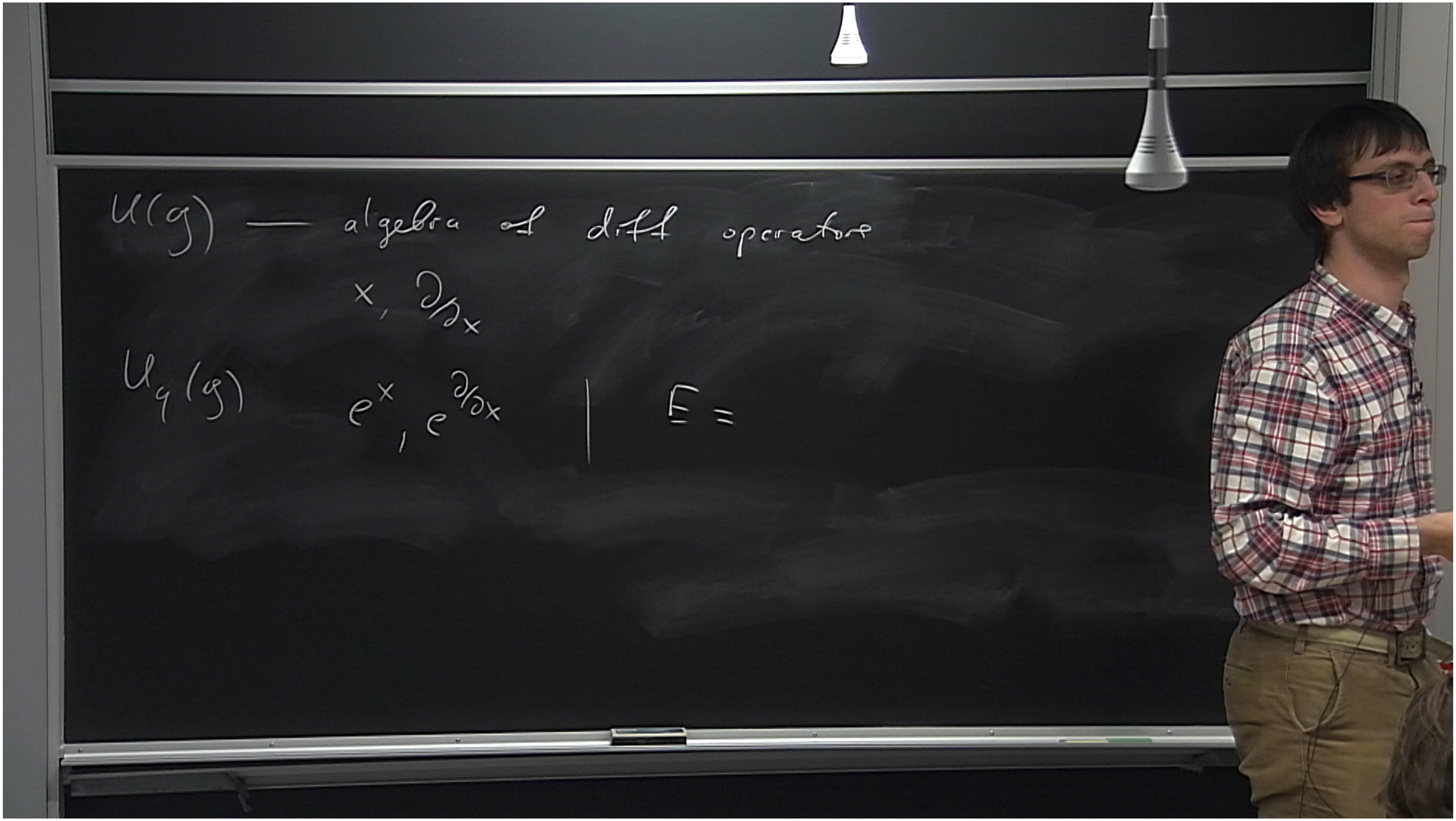
2) <u>Conj</u>: $\mathfrak{g}$ – any simple Lie algebra

$$P_\lambda \otimes P_\mu \simeq \int^{\oplus}_{\mathbb{R}^n_{\geq 0}} P_\nu \otimes M^{\nu}_{\lambda\mu} \, dm(\nu)$$

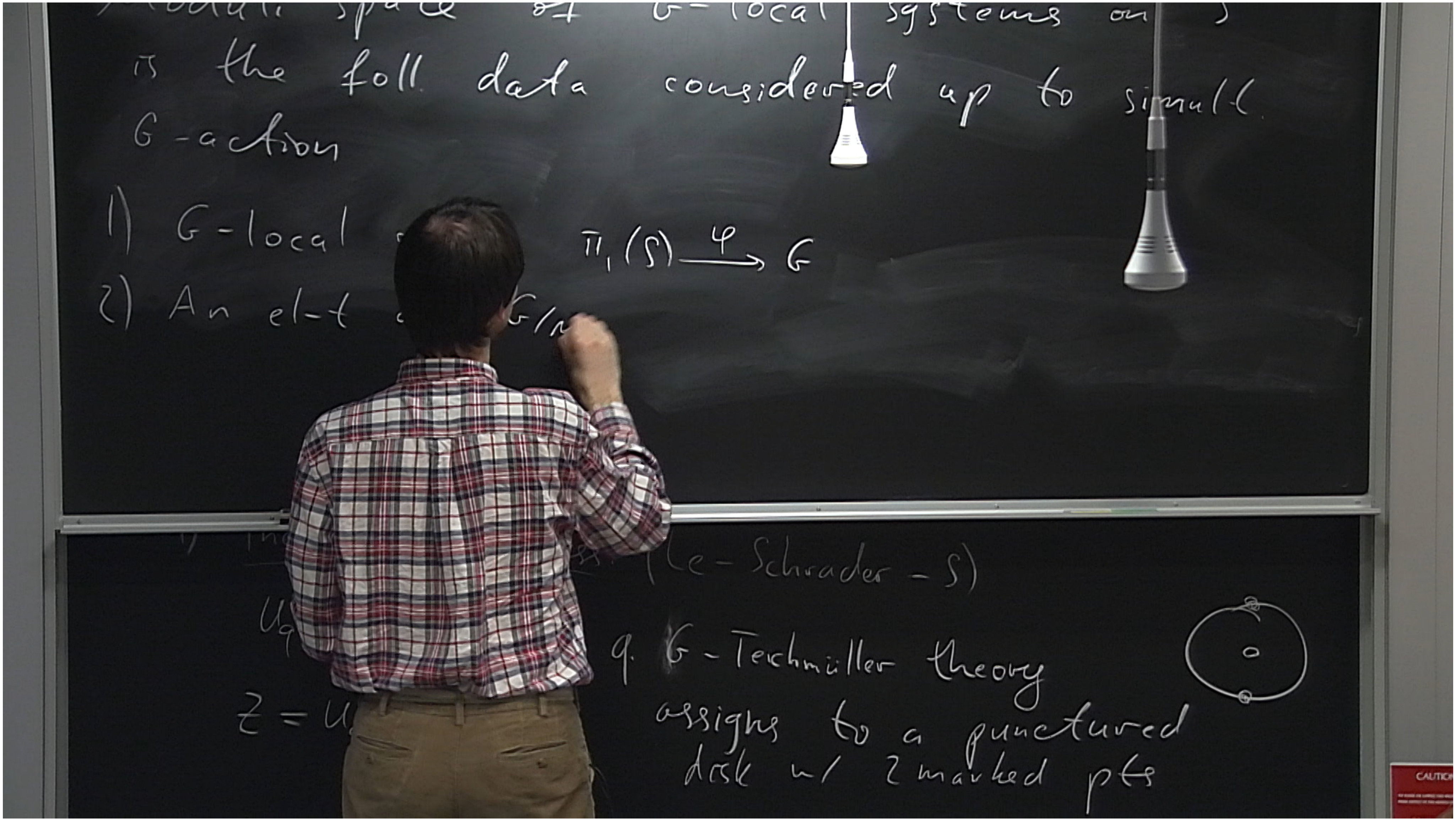

Picture relating the two conjectures:
here
by

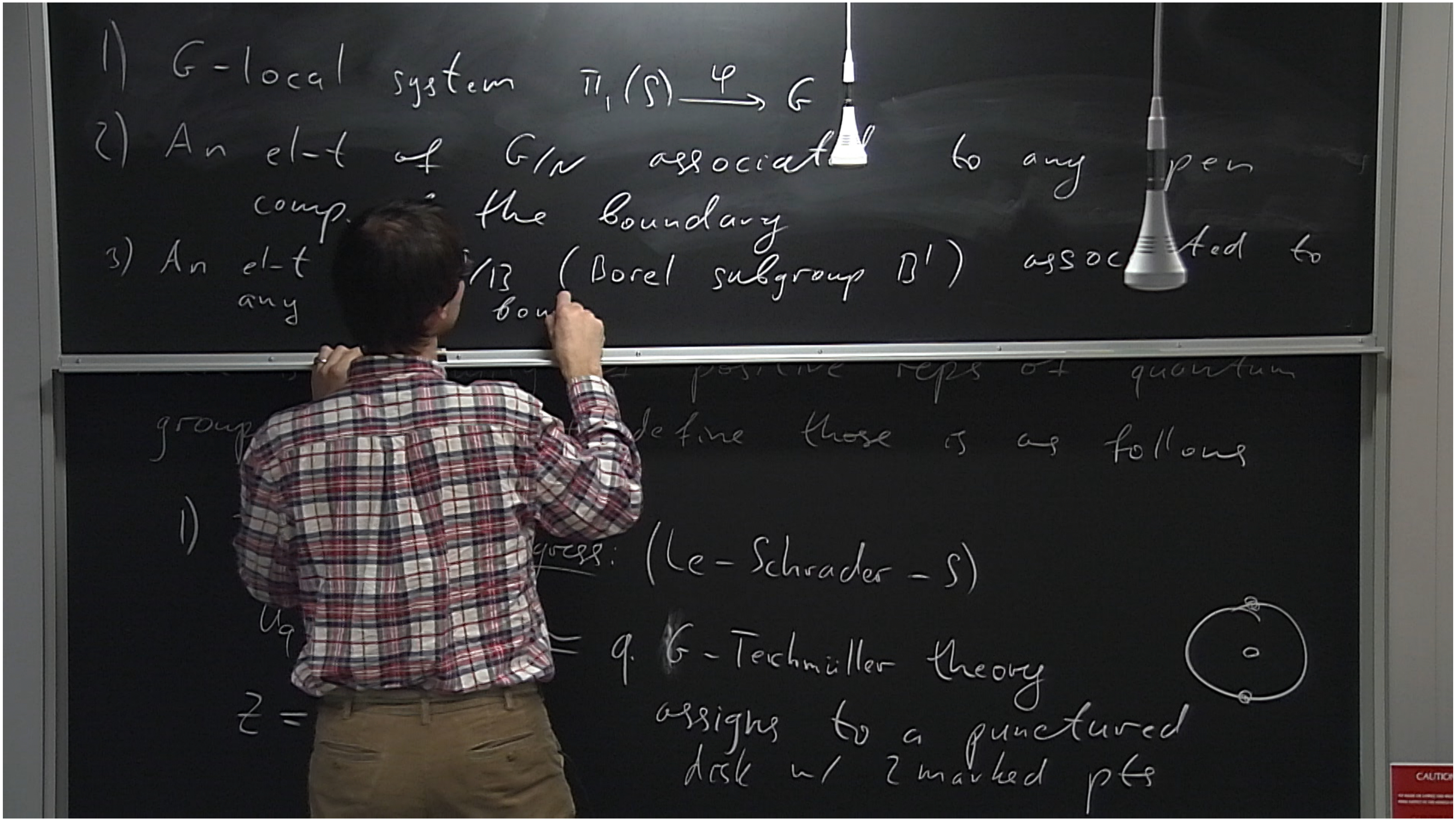

product
here
by
$U_q(g)$
$U_q(g)^{\otimes 2}$
$s(U_q(g))$
$U_q(g)^{\otimes}$


morphism
of
product
here
by



$U(g)$ — algebra of diff operators
$x, \partial/\partial x$
$U_q(g)$
$e^x, e^{\partial/\partial x}$
$E =$


is the foll data considered up to simult.
G-action
1) G-local
$\pi_1(S) \xrightarrow{\varphi} G$
2) An el-t
(Le-Schrader-S)
q. G-Teichmüller theory
assigns to a punctured
disk w/ 2 marked pts


1) G-local system π1(S) φ→ G
2) An el-t of G/N associated to any open
comp. the boundary
3) An el-t /B (Borel subgroup B') associated to
any bou
positive reps of quantum
group define those is as follows
1)
(Le-Schrader-S)
Uq
= q. G-Teichmüller theory
Z =
assigns to a punctured
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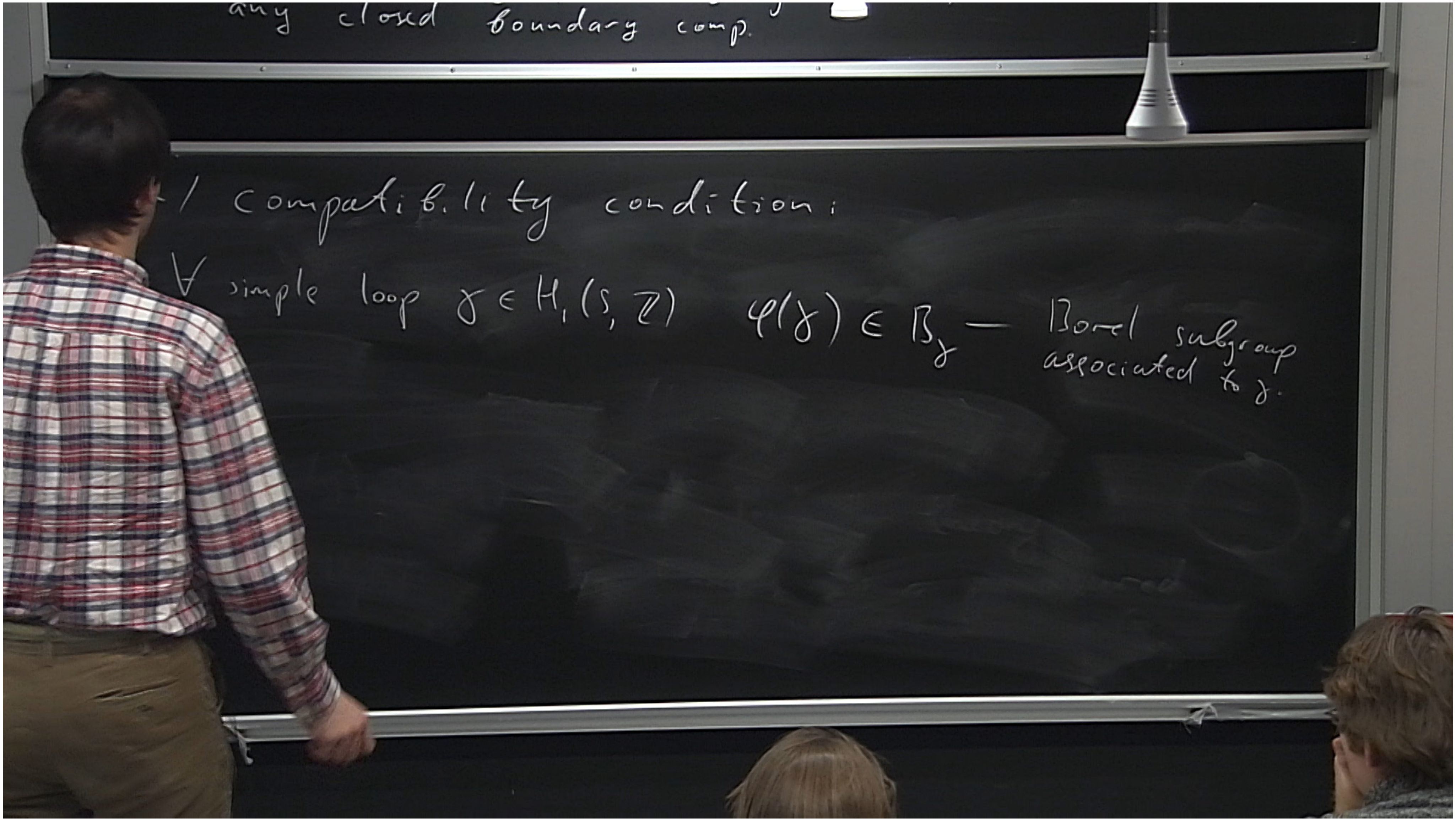
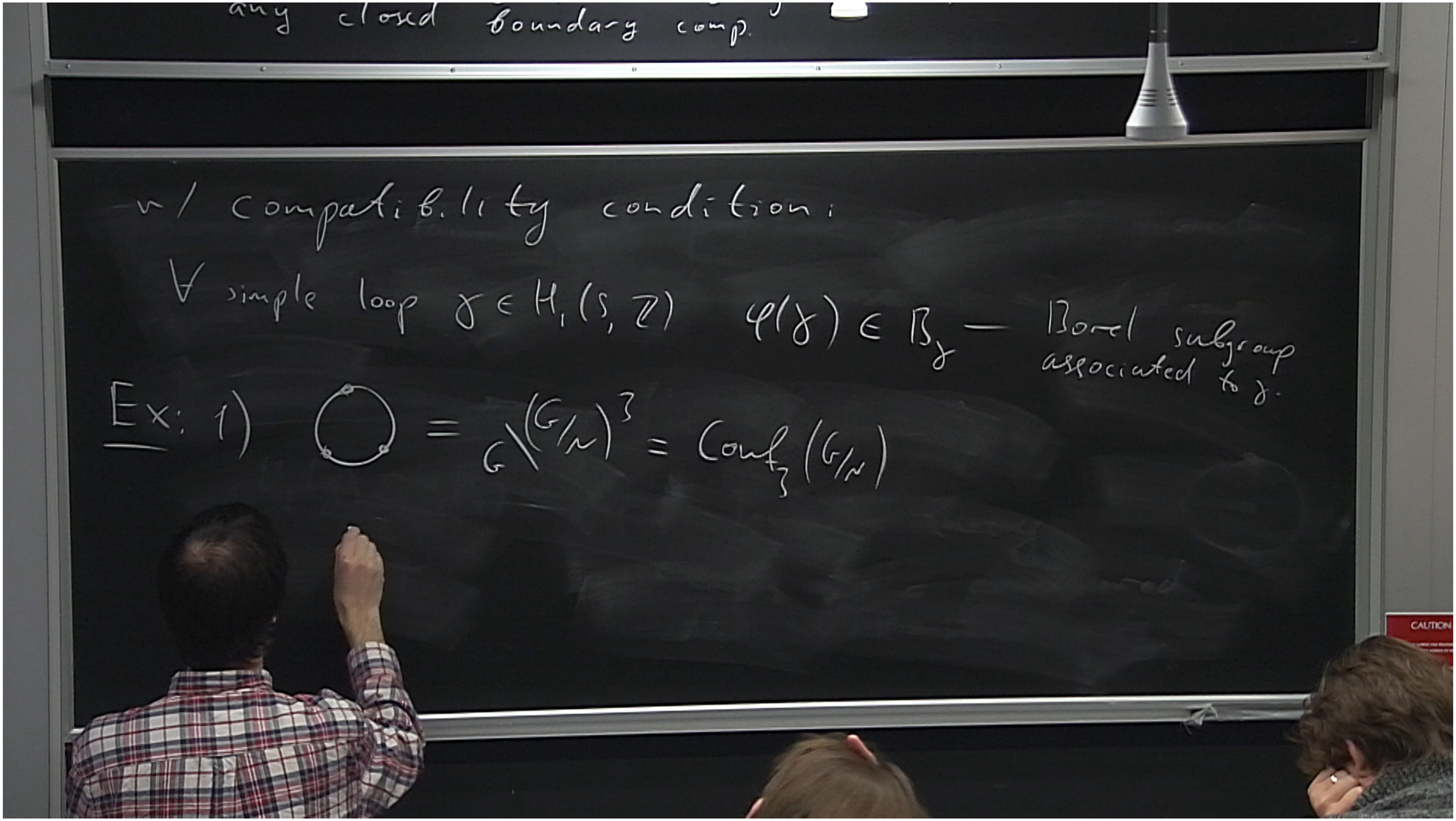

any closed boundary comp.
/ compatibility condition:
∀ simple loop $\gamma \in H_1(S, \mathbb{Z})$ $\varphi(\gamma) \in B_\gamma$ — Borel subgroup associated to γ.


any closed boundary comp.
w/ compatibility condition:
$\forall$ simple loop $\gamma \in H_1(S, \mathbb{Z})$ $\varphi(\gamma) \in B_\gamma$ — Borel subgroup associated to γ.
Ex: 1) $= G\backslash (G/N)^3 = \mathrm{Conf}_3(G/N)$
CAUTION

any closed boundary comp.

w/ compatibility condition:

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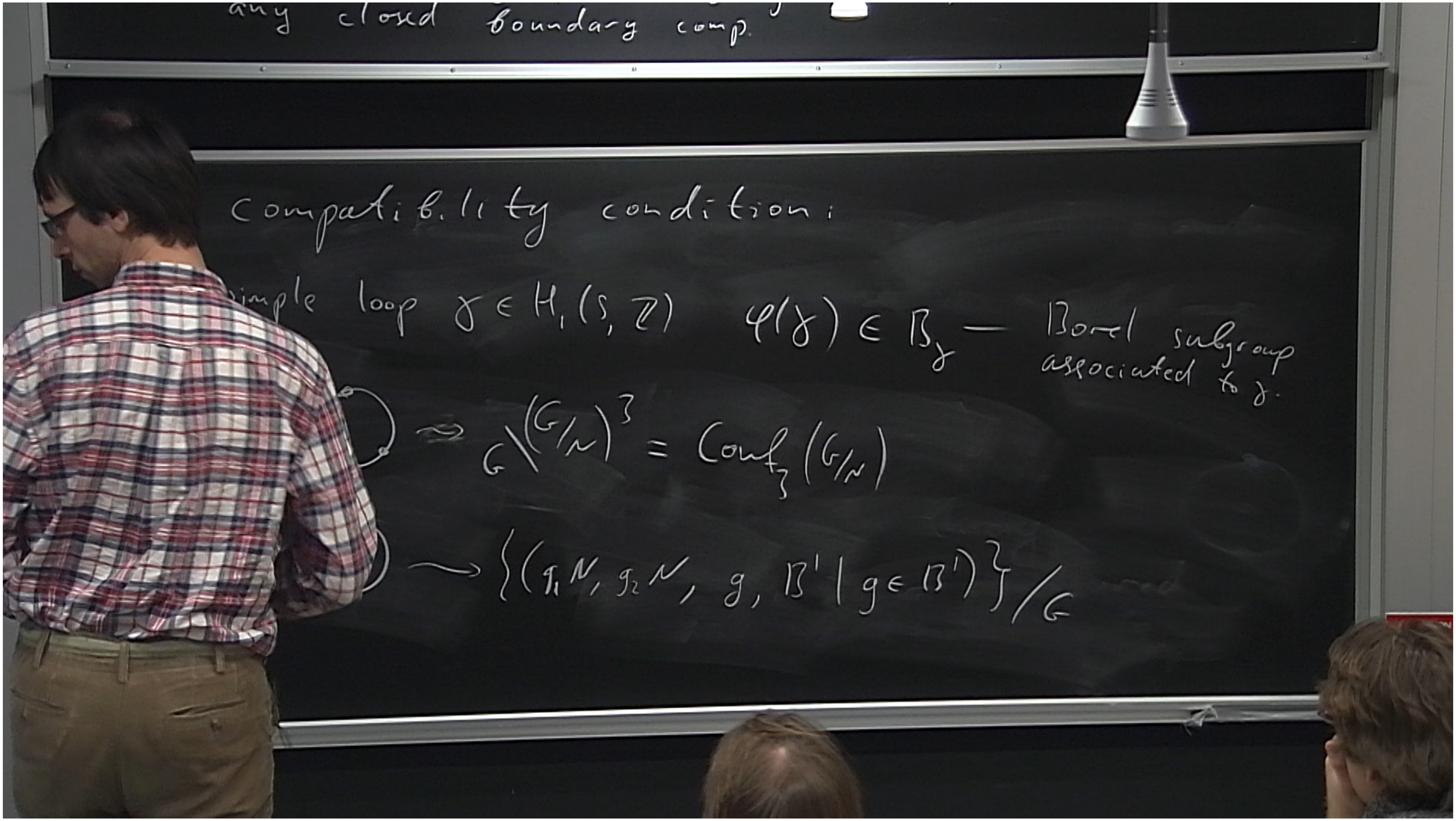
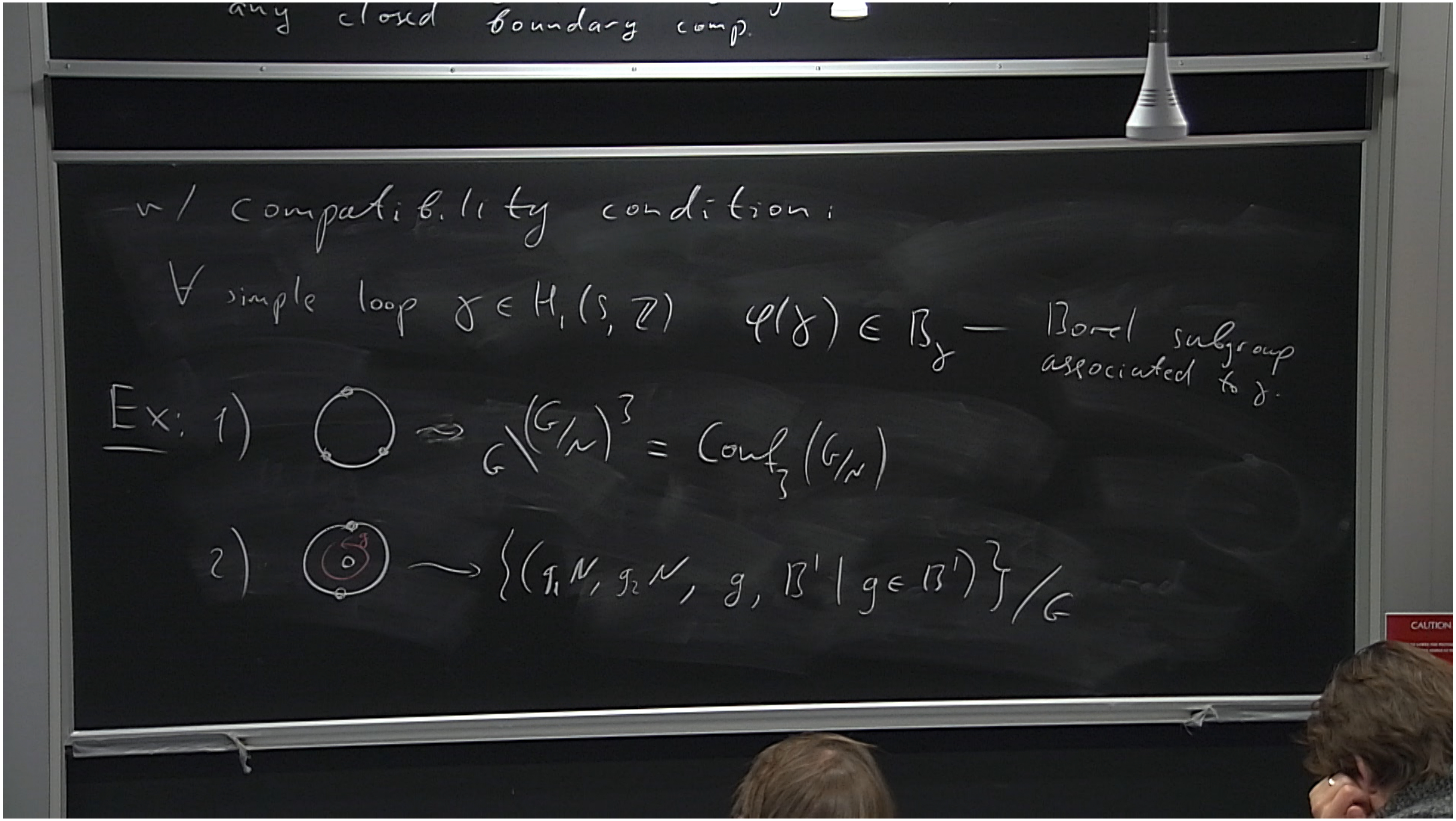
Ex: 1)  $\rightsquigarrow \; G \backslash (G/N)^3 = \mathrm{Conf}_3(G/N)$

2)  $\longrightarrow \{(g_1 N, g_2 N, g$


any closed boundary comp.
compatibility condition:
imple loop γ ∈ H₁(S, ℤ)   φ(γ) ∈ B_γ — Borel subgroup associated to γ.
G\(G/N)³ = Conf₃(G/N)
{(g₁N, g₂N, g, B' | g ∈ B')}/G


any closed boundary comp.
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Ex: 1) $\rightsquigarrow G \backslash (G/N)^3 = \mathrm{Conf}_3(G/N)$
2) $\longrightarrow \{(g_1 N, g_2 N, g, B' \mid g \in B')\}/G$
g

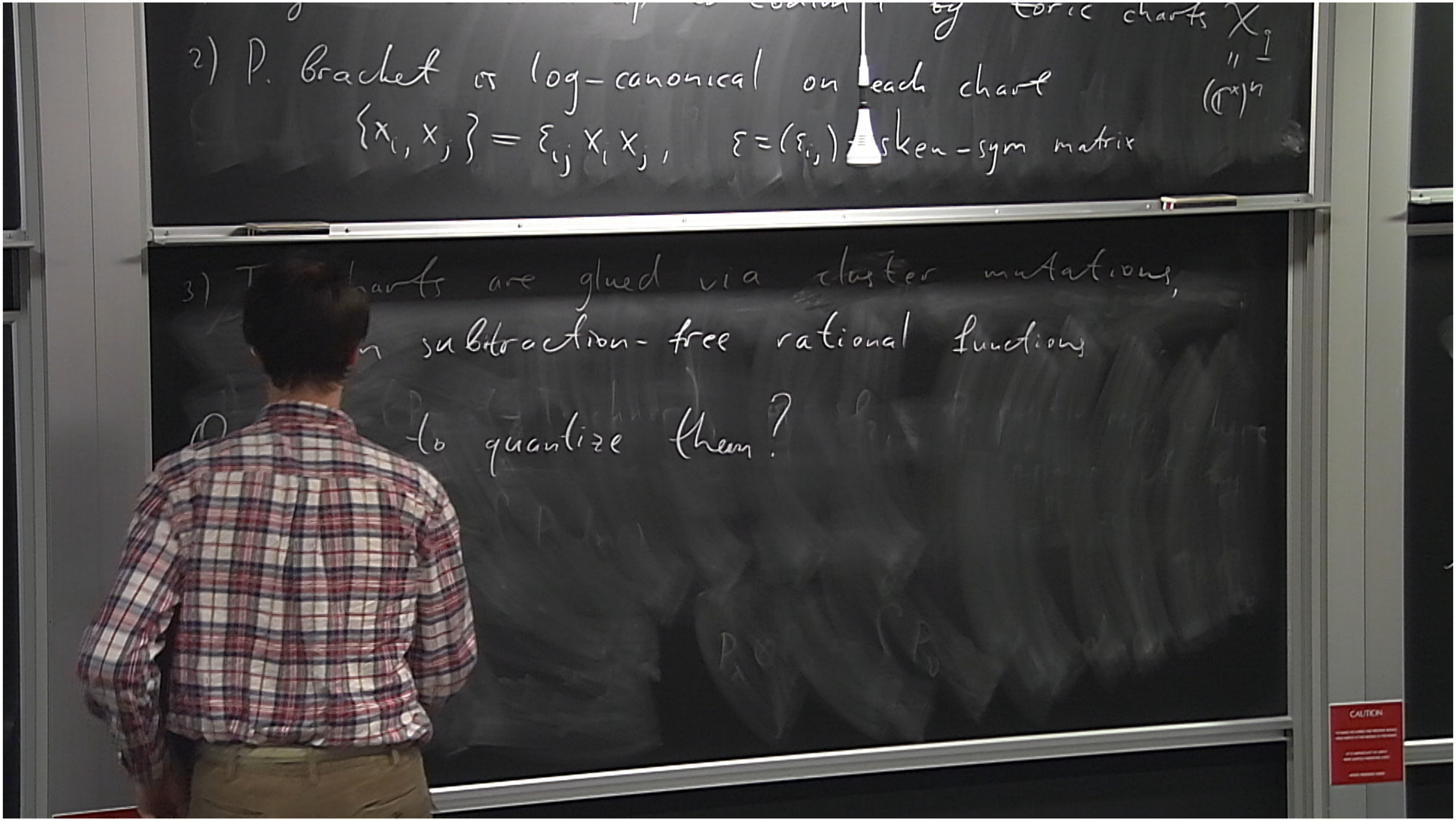
by toric charts X_i

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \wr$

2) P. Bracket is log-canonical on each chart $\quad (\mathbb{C}^\times)^n$

$$\{x_i, x_j\} = \varepsilon_{ij} x_i x_j, \quad \varepsilon = (\varepsilon_{ij}) \text{ skew-sym matrix}$$

3) The charts are glued via cluster mutations, certain subtraction-free rational functions



by toric charts X_i
$= (\mathbb{C}^\times)^n$
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3) charts are glued via cluster mutations,
subtraction-free rational functions
to quantize them?

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3) The charts are glued via cluster mutations, certain subtraction-free rational functions

Q: How to quantize them?

A: Replace $\{x_1, \dots, x_n\}$ by $\{X_1, \dots, X_n\}$ w/ relations

$$X_i X_j = q^{\varepsilon_{ij}} X_j X_i, \quad q = e^{\hbar}$$

Can check:

$$\lim_{\hbar \to 0} \frac{X_i X_j - X_j X_i}{\hbar} = \varepsilon_{ij} X_i X_j$$

... covered by toric charts $X_I \simeq (\mathbb{C}^*)^n$

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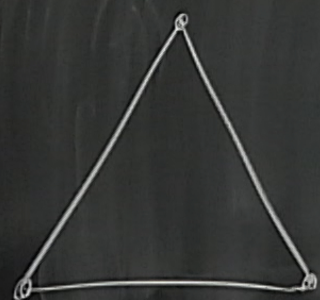
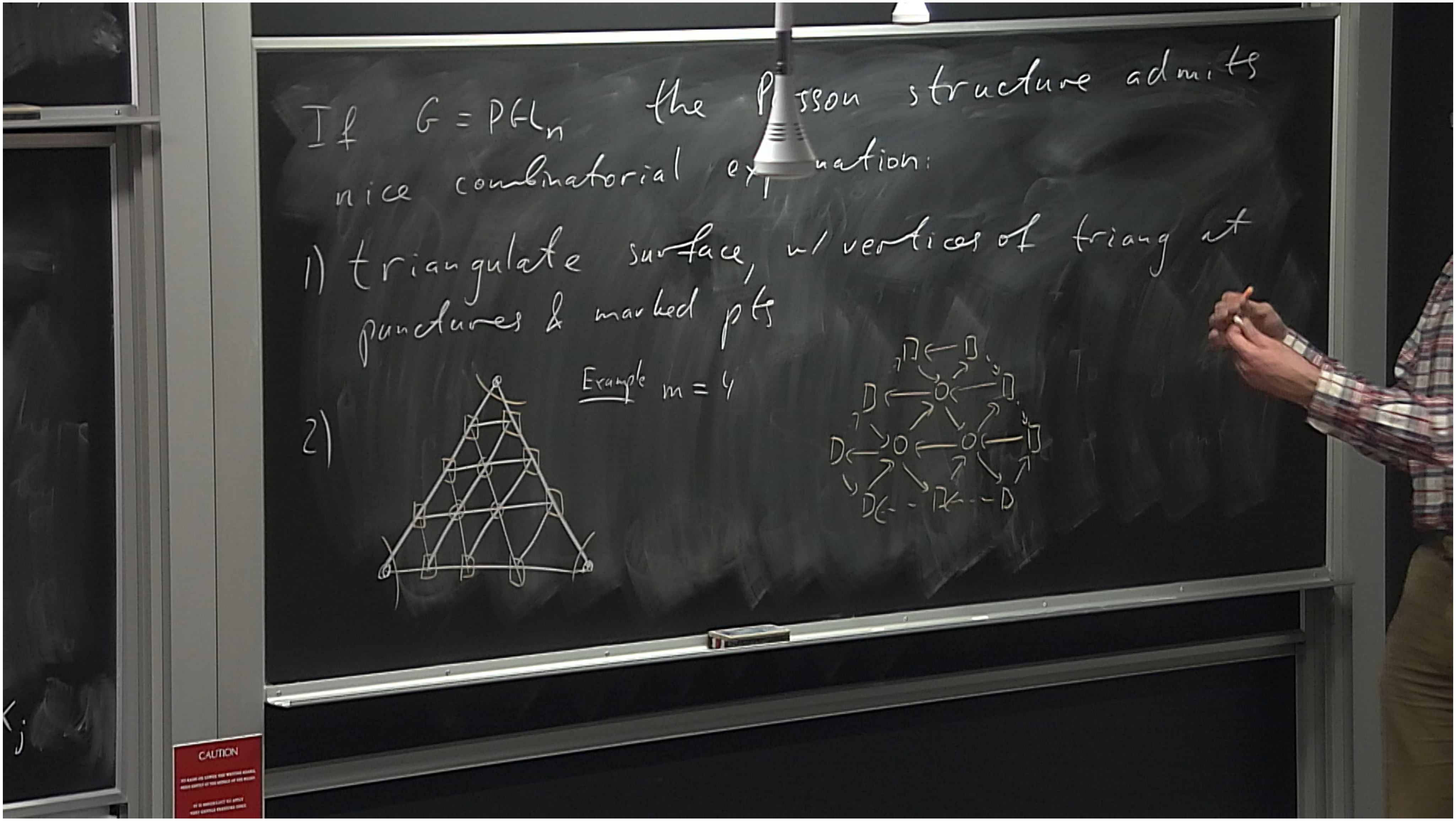
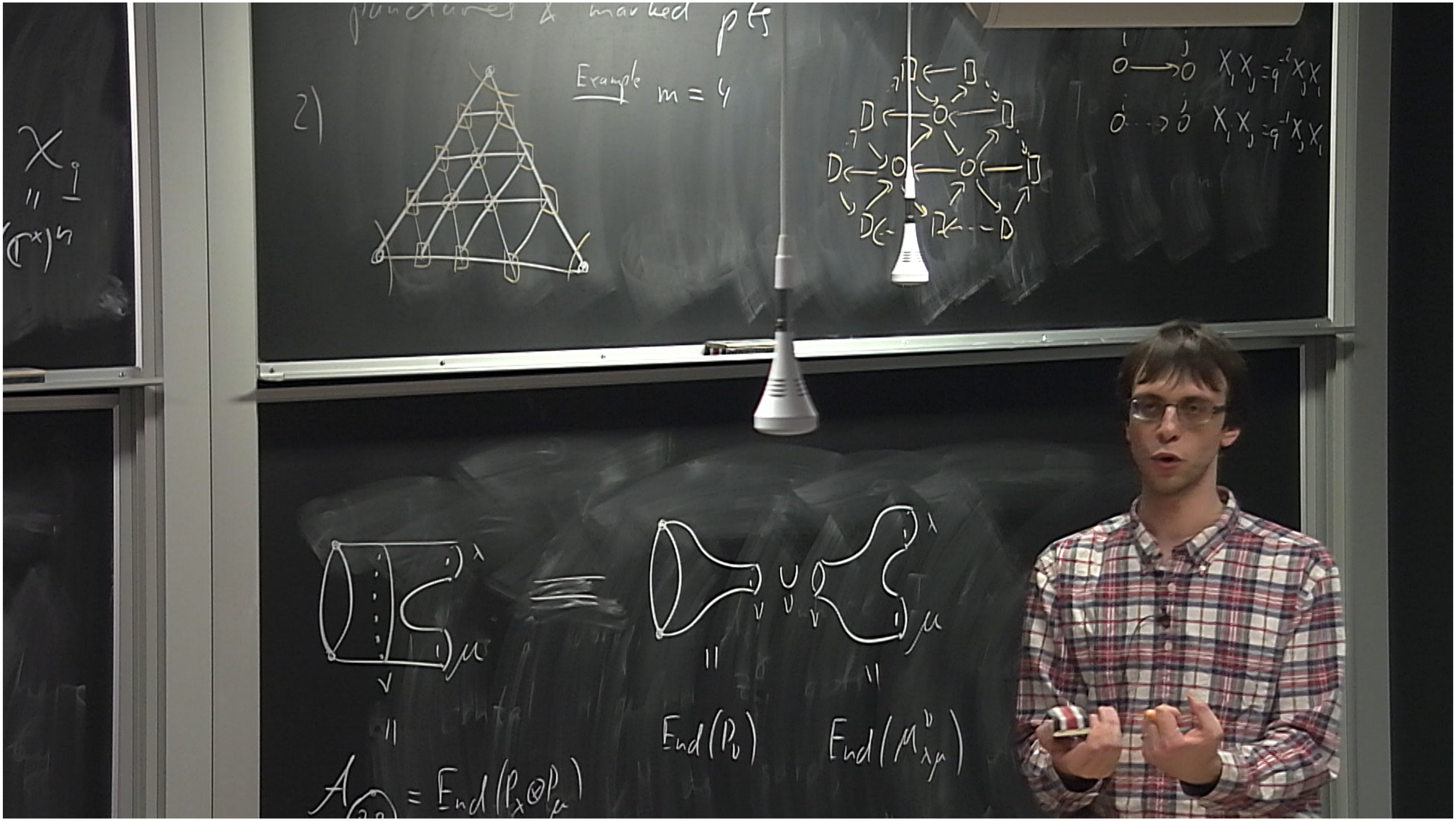
If $G = PGL_n$ the Poisson structure admits nice combinatorial explanation:

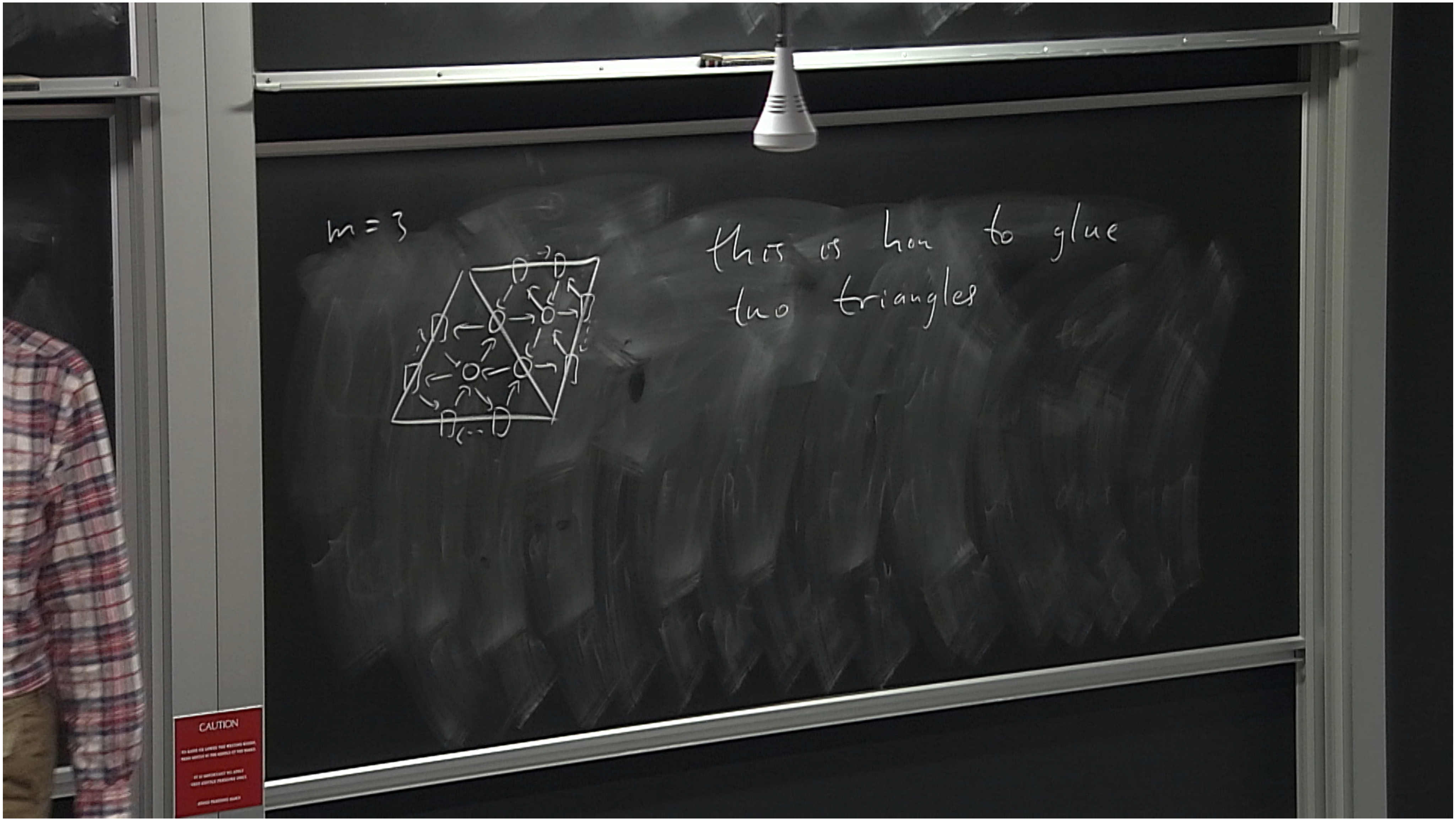
1) triangulate surface, w/ vertices of triang at punctures & marked pts

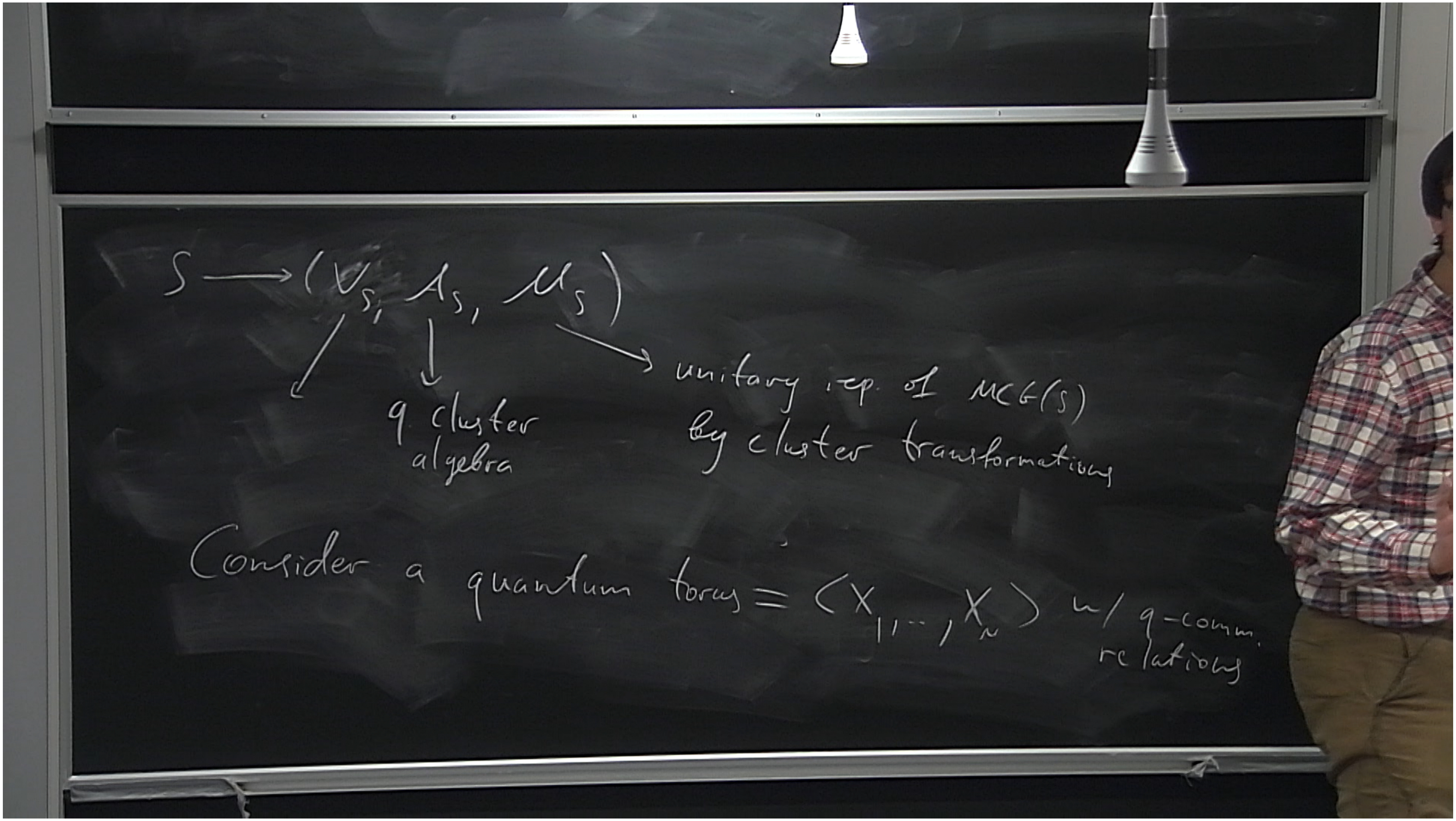
2)




If G = PGL_n the P sson structure admits
nice combinatorial ex nation:
1) triangulate surface, w/ vertices of triang at
punctures & marked pts
2)
Example m = 4




m = 3
this is how to glue
two triangles
CAUTION



$S \longrightarrow (\mathcal{X}_S, \mathcal{A}_S, \mathcal{M}_S)$
q. cluster algebra
unitary rep. of MCG(S) by cluster transformations
Consider a quantum torus $= \langle X_1, \ldots, X_n \rangle$ w/ q-comm. relations

$$S \longrightarrow (\mathcal{X}_S, \mathcal{A}_S, \mathcal{M}_S)$$

$\mathcal{A}_S \to$ q. cluster algebra

$\mathcal{M}_S \to$ unitary rep. of $MCG(S)$ by cluster transformations

Consider a quantum torus $= \langle X_1, \dots, X_n \rangle$ w/ q-comm. relations

$P_i X_j = q^{ij} X_j P_i$

<u>Step back:</u> Consider pair of operators x, $p = 2\pi i \frac{\partial}{\partial x}$

they act on functions of x. Now, consider

$X = e^{\hbar x}$, $P = e^{\hbar p}$, $PX = q^2 XP$ where $q = e^{\pi i \hbar}$

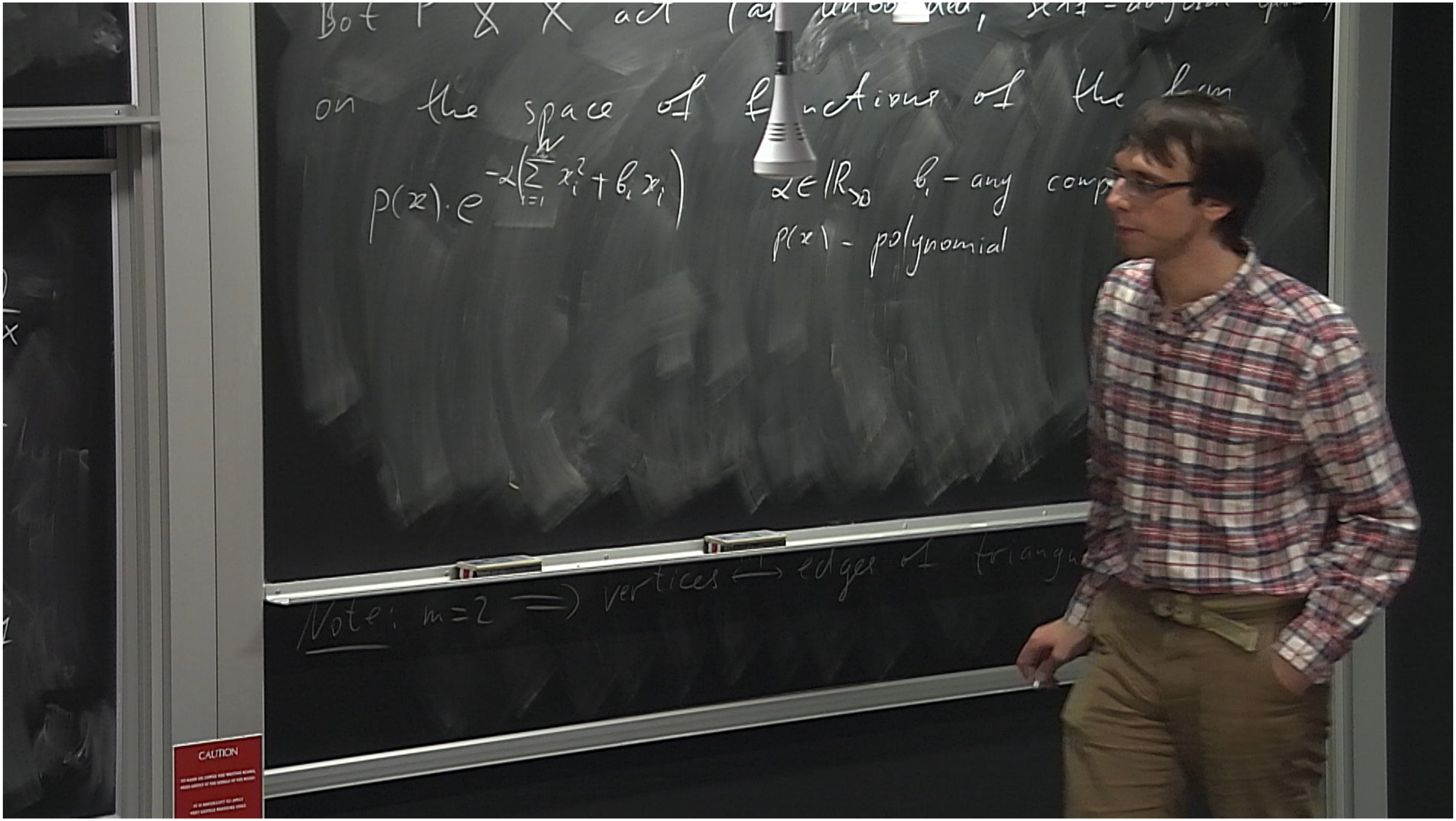
$P_i X_j = q^{\delta_{ij}} X_j P_i$

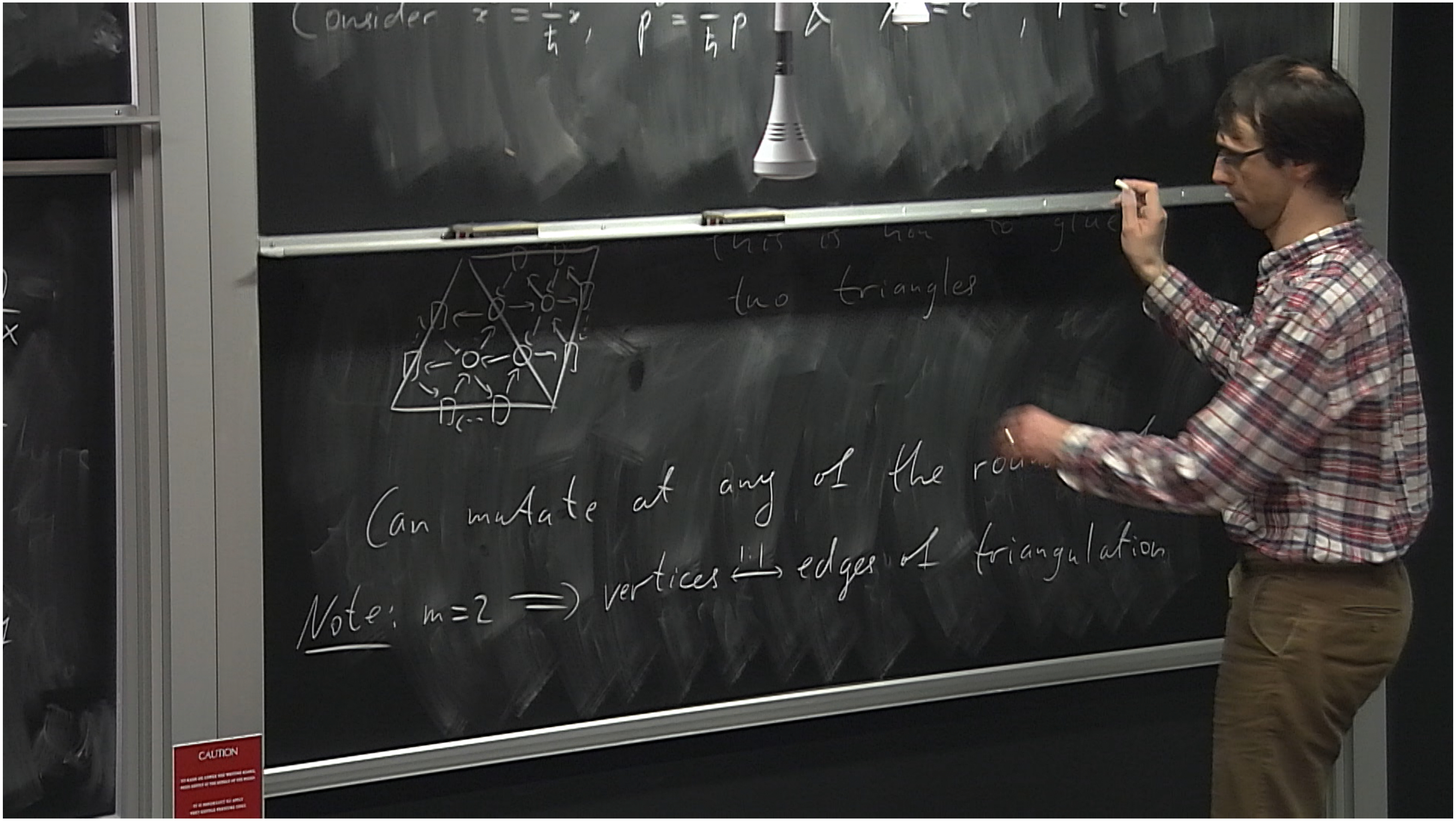
<u>Step back:</u> Consider pair of operators x, $p = 2\pi i \frac{\partial}{\partial x}$

they act on functions of x. Now, consider

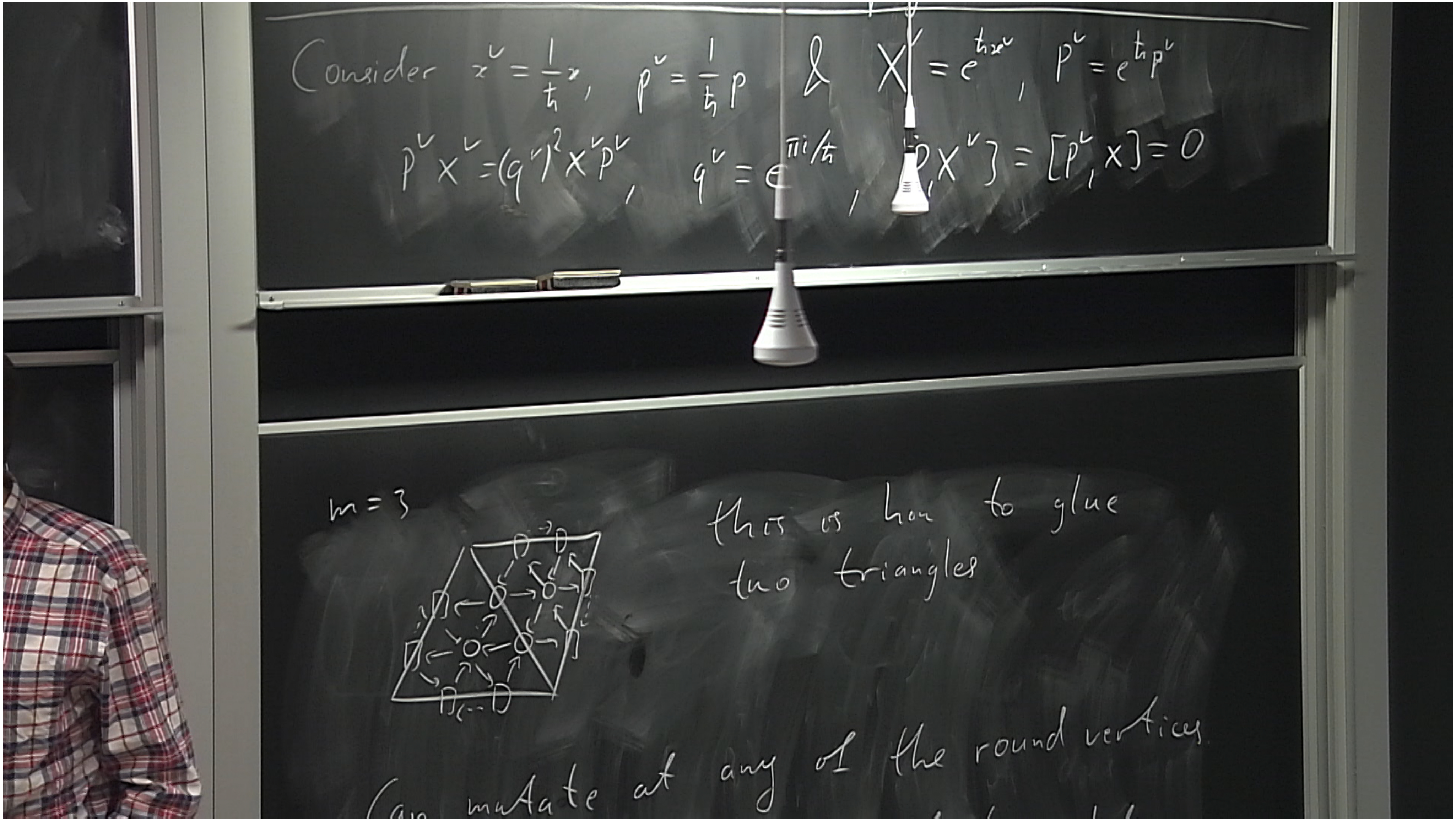
$X = e^{\hbar x}$, $P = e^{\hbar p}$, $PX = q^2 XP$ where $q = e^{\pi i \hbar}$

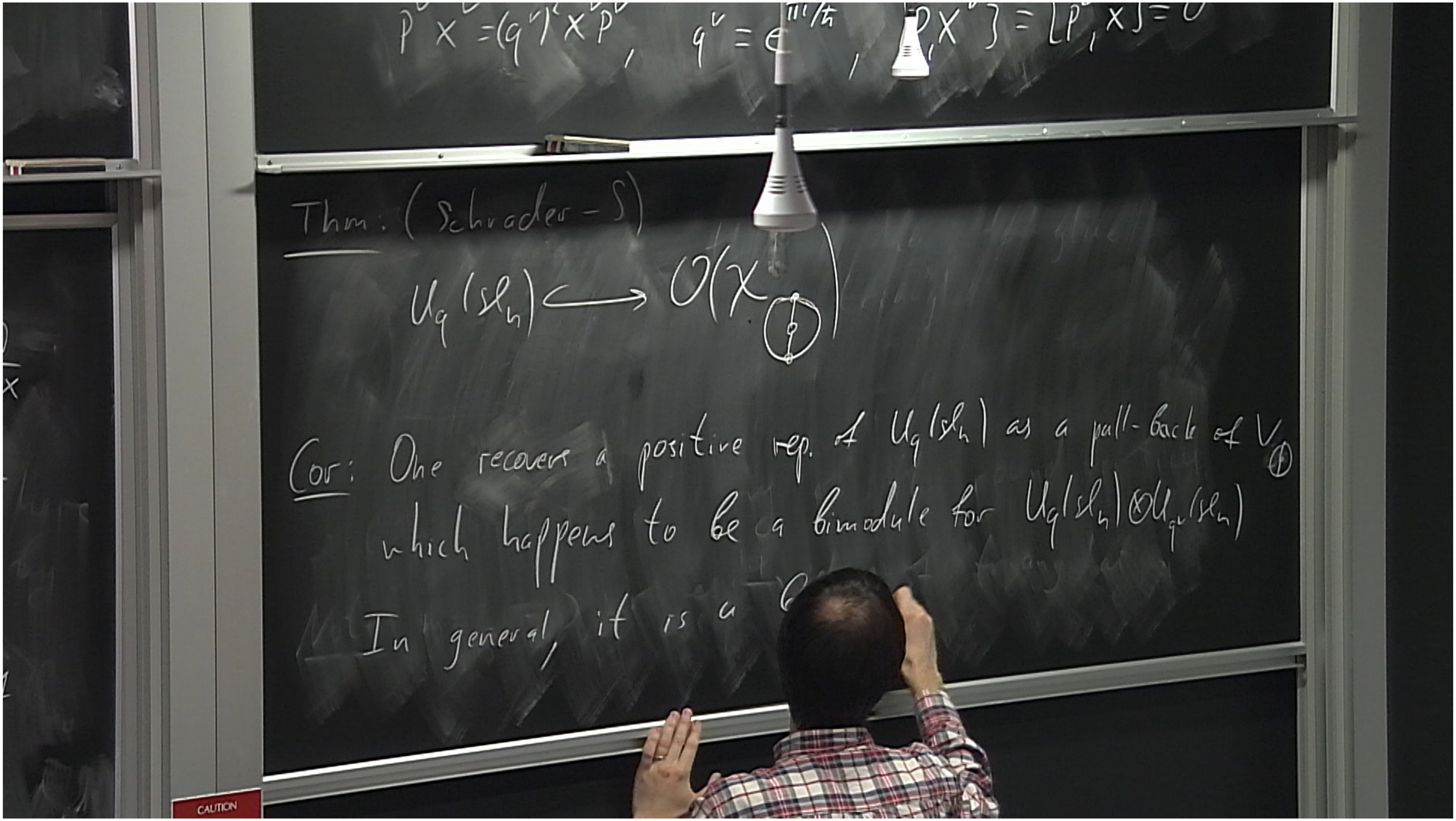
X, P act on functions of x by mult. by $e^{\hbar x}$ & shift by $2\pi i \hbar$, i.e. $P f(x) = f(x + 2\pi i)$


on the space of functions of the
p(x)·e^{-α(∑_{i=1}^{n} x_i^2 + b_i x_i)}
α ∈ ℝ_{>0} b_i - any comp
p(x) - polynomial
Note: n=2 ⟹ vertices ⟷ edges of triangu



this is how to glue
two triangles
Can mutate at any of the rou
Note: m=2 ⟹ vertices ←1:1→ edges of triangulation


Consider $x^\vee = \frac{1}{\hbar}x$, $p^\vee = \frac{1}{\hbar}p$ & $X^\vee = e^{\hbar x^\vee}$, $P^\vee = e^{\hbar p^\vee}$
$P^\vee X^\vee = (q^\vee)^2 X^\vee P^\vee$, $q^\vee = e^{\pi i/\hbar}$, $[P, X^\vee] = [P^\vee, X] = 0$
n = 3
this is how to glue
two triangles
Can mutate at any of the round vertices


Thm: (Schrader-S)
$U_q(sl_n) \hookrightarrow \mathcal{O}(\mathcal{X})$
Cor: One recovers a positive rep. of $U_q(sl_n)$ as a pull-back of V
which happens to be a bimodule for $U_q(sl_n) \otimes U_{q^\vee}(sl_n)$
In general, it is a